Codazzi algebras associated to pairs of Levi-Civita connections on the tangent space of fibered manifolds

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Abstract. We develop the properties of the Codazzi algebras associated to pairs of Levi-Civita connections on the tangent space of fibered manifolds, and of the the corresponding J_1-planar characteristic algebras.

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1 Fibered manifolds

A *fibered manifold* is a triple (Y, π, X), where Y and X are finite dimensional differentiable manifolds and $\pi : Y \to X$ is a surjective submersion with dim $X = n$ and dim$Y = m+n$. At every point $y \in Y$, the following two equivalent conditions defining submersions are satisfied:

(1) the tangent mapping $T_y\pi : T_yY \to T_{\pi(y)}X$ is surjective,

(2) there exists a chart (V, ψ), $\psi = (u^i, y^\sigma)$ at $y \in Y$, where $1 \leq i \leq n$, $1 \leq \sigma \leq m$, and a chart (U, φ), $\varphi = (x^i)$ at $x = \pi(y) \in X$, such that $U = \pi(V)$ and $x^i\pi = u^i$.

We further write x^i instead of u^i, and call (V, ψ), $\psi = (x^i, y^\sigma)$, a *fibered chart*. The chart (U, φ), $\varphi = (x^i)$, on X is unique, and is said to be *associated with* (V, ψ), $\psi = (x^i, y^\sigma)$. A *section* of a fibered manifold (Y, π, X), is a mapping $\gamma : U \to Y$, where $U \subset X$ is an open set, such that

$$\pi \circ \gamma = id_U. \tag{1.1}$$

A vector field τ on Y is said to be π-*projectable*, or simply *projectable*, if there exists a vector field ξ on X such that

$$T\pi \cdot \tau = \xi \circ \pi. \tag{1.2}$$

If ξ exists, then it is unique, and it is called *the* π-projection *of* τ. In a fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, a π-projectable vector field τ is expressed as $\tau = \xi^i \frac{\partial}{\partial x^i} + \tau^\sigma \frac{\partial}{\partial y^\sigma}$, where $\xi^i = \xi^i(x^j)$ and $\tau^\sigma = \tau^\sigma(x^j, y^\sigma)$.

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2 Jet prolongations of a fibered manifold

Let $y \in Y$, $x = \pi(y)$, and let $\Gamma^r_{x,y}$ be the set of smooth sections γ of Y defined at x such that $\gamma(x) = y$. Let $r > 0$ be a positive integer. We define the binary equivalence relation $\gamma_1 \sim \gamma_2$ by requiring that there exists a fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$ at y, such that

$$D_{i_1} D_{i_2} \dots D_{i_k}(y^\sigma \gamma_1 \varphi^{-1})(\varphi(x)) = D_{i_1} D_{i_2} \dots D_{i_k}(y^\sigma \gamma_2 \varphi^{-1})(\varphi(x))$$

for all $k = 1, 2, \dots, r$ and for all $i_1, i_2, \dots, i_k$ such that $1 < i_1 < i_2 < \dots < i_k < n$ is an equivalence on the set $\Gamma^r_{x,y}$. The equivalence class containing a section γ is called an r-jet with *source* x and *target* y or the r-jet of y at x, and is denoted by $J^r_x\gamma$. We denote by J^rY the set of r-jets with source in X and target in Y. The *canonical jet projections* are the mappings $\pi^{r,s}$ (respectively π^r) of J^rY onto J^sY, where $0 < s < r$ (respectively on X), defined by $\pi^{r,s}(J^r_x\gamma) = (J^s_x\gamma)$ (respectively by $\pi^r(J^r_x\gamma) = x$).

The *smooth structure* of J^rY associated with the smooth structure of Y is defined as follows. Let (V, ψ), $\psi = (x^i, y^\sigma)$, where $1 \leq i \leq n$, $1 \leq \sigma \leq m$, be a fibered chart on Y, (U, φ), $\varphi = (x^i)$, the associated chart on X. Then the *associated fibered chart* (V^r, ψ^r), $\psi^r = (x^i, y^\sigma, y^{\sigma j_1}, \dots, y^{\sigma j_1 j_2 \cdots j_r})$ on J^rY is defined by the following two conditions:

$$V^r = (\pi^{r,0})^{-1}(V),$$

and for any $J^r_x\gamma \in V^r$,

$$y^{\sigma j_1 j_2 \cdots j_k}(J^r_x\gamma) = D_{j_1} D_{j_2} \dots D_{j_k}(y^\sigma \gamma \varphi^{-1})(\varphi(x)),$$

where $k = 1, 2, \dots, r$ and $1 < j_1 < j_2 < \dots < j_k < n$. If (V', ψ'), $\psi' = (x'^i, y'\sigma)$ is another fibered chart such that $V \cap V' \neq \emptyset$, then by writing

$$y'^\sigma \gamma \varphi'^{-1} = y' \psi^{-1} \circ \psi \gamma \varphi^{-1} \circ \varphi \varphi'^{-1}$$

and by using the chain rule, we get the *transformation formula* in recurrent form

$$y'^{\sigma j_1 j_2 \cdots j_k} = D_{j_1} D_{j_2} \dots D_{j_k}(y'^\sigma \psi^{-1} \circ \psi \gamma \varphi^{-1} \circ \varphi \varphi'^{-1})(\varphi'(x)).$$

As well, we note that $\dim J^rY = n + m \begin{pmatrix} n+r \\ n \end{pmatrix}$.

3 The horizontalization of tangent vectors

A vector bundle morphism acts on the tangent spaces to jet prolongations of a given fibered manifold. Similarly, as in the case of differential forms, this vector bundle morphism is induced by the structure of jet prolongations.

Let $r > 0$ be an integer. One can assign to every tangent vector $\xi \in TJ^{r+1}Y$ at a point $J^{r+1}_x\gamma \in J^{r+1}Y$ a tangent vector $h\xi \in TJ^rY$ at $J^r_x\gamma = \pi^{r+1,r}(J^{r+1}_x\gamma) \in J^rY$ by

$$h\xi = T_x J^r\gamma \circ T\pi^{r+1} \cdot \xi.$$

The mapping $h : TJ^{r+1}Y \to TJ^rY$ defined by this formula is a vector bundle morphism over the jet projection $\pi^{r+1,r}$; we call h the *π-horizontalization*, or simply the *horizontalization* and we have

$$h\xi = \xi^j \left(\frac{\partial}{\partial x^i} + \sum_{k=0}^{r} \sum_{j_1 \le j_2 \le \ldots \le j_k} y^{\sigma}_{j_1 j_2 \ldots j_k j} \frac{\partial}{\partial y^{\sigma}}_{j_1 j_2 \ldots j_k} \right).$$

Lemma 3.1. *[5] Let τ be a π-projectable vector field on Y, let (V, ψ), $\psi = (x^i, y^\sigma)$ be a fibered chart on Y, and let τ be expressed by $\tau = \xi^i \frac{\partial}{\partial x^i} + \tau^\sigma \frac{\partial}{\partial y^\sigma}$. Then $J^r\tau$ is expressed with respect to the associated chart (V^r, ψ^r) by*

$$J^r\tau = \xi^i \frac{\partial}{\partial x^i} + \left(\sum_{k=0}^{r} \sum_{j_1 \le j_2 \le \ldots \le j_k} \tau^{\sigma}_{j_1 j_2 \ldots j_k} \frac{\partial}{\partial y^{\alpha}_{j_1 j_2 \ldots j_k}} \right),$$

where the components $\tau^{\sigma}_{j_1 j_2 \ldots j_k}$ are determined by the recurrent formula

$$\tau^{\sigma}_{j_1 j_2 \ldots j_k} = d_{j_k} \tau^{\sigma}_{j_1 j_2 \ldots j_{k-1}} - y^{\sigma}_{j_1 j_2 \ldots j_{k-1} i} \frac{\partial \xi^i}{\partial x^{j_k}}.$$

Lemma 3.2. *[5] Let τ_1 and τ_2 be two π-projectable vector fields on Y. Then the Lie bracket $[\tau_1, \tau_2]$ is also a π-projectable vector field on Y, and $J^r[\tau_1, \tau_2] = [J^r\tau_1, J^r\tau_2]$.*

Let $A(\tau_1, \tau_2) = \nabla_{\tau_1}\tau_2 - \nabla'_{\tau_1}\tau_2$ be a tensor field of type $(1, 2)$ which defines the deformation algebra associated to the pair of linear connections (∇, ∇') on TJ^rY, which we denote by $U(J^rY, A)$. For details, we refer the reader to [4, Prop. 3 p. 226].

4 Codazzi algebras and applications

Definition 4.1. [7] Let g and g' be two Riemannian metrics on the C^∞-differentiable manifold J^rY, and let ∇, ∇' be the Levi-Civita connections associated to the Riemannian metrics g and g'. The algebra defined by the $(1, 2)$ tensor field

$$A(\tau_1, \tau_2) = \nabla_{\tau_1}\tau_2 - \nabla'_{\tau_1}\tau_2 \tag{4.1}$$

is called *the Codazzi algebra associated to the pair of linear conections* (∇, ∇') if ∇g is a $(0, 3)$-symmetric tensor field on TJ^rY.

Proposition 4.1. *An algebra $U(J^rY, A)$ is a Codazzi algebra if and only if $\nabla' gA$ is a symmetric tensor field.*

Proof. Let $\nabla'' g = \nabla' gA$ be a linear connection. Then $\nabla'' g + \nabla' g = 2\nabla g = 0$, which proves the claim. □

It is known that ∇'' is not a metric connection ([7]).

Proposition 4.2. *An algebra $U(J^rY, A)$ given by* (4.1) *is a Codazzi algebra if and only if*

$$(\nabla_X J_1)Y = (\nabla_Y J_1)X, \tag{4.2}$$

which means ∇_{J_1} is a symmetric field in the covariant index and J_1 is the $(1,1)$-tensor field associated to the pair of metrics g and g' given by

$$g'(X,Y) = g(J_1X,Y), \text{ for all } X,Y \in T^1_0 J^rY \tag{4.3}$$

Proof. Let J_1 be the $(1,1)$-tensor field associated to the pair of metrics g and g' given by (4.3). Then there exists a contraction C such that

$$g' = C(g \otimes J)$$

and we get $\nabla g = C(g \otimes \nabla J_1)$, which infers (4.2). □

Examples. a) Let the metrics g' and g be conformal equivalent, i.e.,

$$g' = e^\sigma g$$

and let $\nabla J_1 = d\sigma J_1$. The algebra $U(J^rY, \nabla - \nabla')$ is Codazzi if and only if $d\sigma = 0$, and then g' and g are homothetic and $\nabla = \nabla'$.

b) Let g be the metric induced by a hypersurface J^rY of a space $R^{n+m\binom{n+r}{n}+1}$ and let g' be the second fundamental form. If we suppose that g is non-degenerate, then the the Codazzi equation proves that there exists a Codazzi algebra on J^rY and J_1 is the Weingarten mapping.

Let J_1 be a fixed tensor field of type $(1,1)$ on J^rY and let ξ be a fixed vector field on J^rY. We shall give some properties of the vectors and of jet prolongations of J_1-planar characteristic vector fields and of the Codazzi algebras.

Definition 4.2. [7] a) A tangent vector v of $T_{J^r_x\gamma}J^rY$ is *called characteristic vector of the deformation algebra* $U(J^r_x\gamma, A)$, if the vector subspace $\langle v\rangle$ generated by v is a subalgebra of the deformation algebra $U(J^r_x\gamma, A)$.

b) Let V' be a vector subspace of $T_{J^r_x\gamma}J^rY$. We say that the deformation algebra $U(J^r_x\gamma, A)$ *deviates to the direction of* J_1 in the vector subspace V', if we have the condition

$$A_{J^r_x\gamma}(V', V') \subseteq \langle V', J_{J^r_x\gamma}(V')\rangle,$$

which means that for all $v, w \in V'$, $A_{J^r_x\gamma}(v, w)$ is in the vector subspace generated by $V' \cup J_{J^r_x\gamma}(V')$.

c) A tangent vector v of $T_{J^r_x\gamma}J^rY$ is called *J_1-planar characteristic vector of the deformation algebra* $U(J^r_x\gamma, A)$, if the deformation algebra $U(J^r_x\gamma, A)$ deviates to the direction of J_1 in the vector subspace $\langle v\rangle$ generated by v.

The following results are known from [7]:

a) The set of the characteristic vectors is a cone.
b) The set of the J_1-planar characteristic vectors is also a cone.
c) Any eigenvector is J_1-planar characteristic. The 0-planar characteristic vectors are the eigenvectors of the deformation algebra.
d) An eigenvector of $J_{1J^r_x\gamma}$ is J_1-planar characteristic if and only if it is characteristic.

Theorem 4.3. *Let $U(J^rY, A)$ be a Codazzi algebra given by (4.1) and let $dimJ^rY \geq 3$. Then the following statements are equivalent:*

(1) The algebra $U(J^rY, A)$ is J_1-planar characteristic.
(2) $A = 0$, i.e., $\nabla = \nabla'$.

Proof. The implication (2) $\Rightarrow$ (1) is straightforward. For the proof of (1) $\Rightarrow$ (2), we remark that $A_p \neq 0$. We further denote

$$L = \{p \in J^r Y \mid \text{ the algebra } U(J^r Y, A) \text{ is } J_1\text{-planar characteristic}\}.$$

Assume that $L \neq \varnothing$ and that there exists $p \in J^r Y$ such that g' and g are not proportional at p. Then there exist both an open set U, $p \in U$ and a frame $(e_1, \ldots, e_N)$, where $\mathrm{N} = n + m\binom{n+r}{n} + 1$ on U, and the differentiable functions f_i, $i = 1, \ldots, N$, such that

$$J_1 e_i = f_i e_i, \text{ for } i = 1, \ldots, N,$$

and

$$f_1(q) \neq fn(q), \quad A_q \neq 0, \tag{4.4}$$

for all $q \in J^r Y$. We note that there exist the 1-forms ω and η on $J^r Y$; from the fact that the algebra $U(J^r Y, A)$ is J_1-planar characteristic and using the unity partition, we have

$$A = \omega \otimes I + I \otimes \omega + \eta \otimes J_1 + J_1 \otimes \eta. \tag{4.5}$$

The derivation A_Y of a vector field Y verifies the relation:

$$A_Y g = \nabla_Y g = C(g \otimes \nabla_Y J_1), \tag{4.6}$$

and from (4.2), we get

$$A_Y g'(Z, X) = A_X g'(Z, Y), \text{ for all } X, Y, Z \in T_0^1 J^r Y.$$

Using that the algebra $U(J^r Y, A)$ is commutative, we obtain

$$g'(A(X, Z), Y) = g'(A(Y, Z), X),$$

which infers

$$g(A(X, Z), J_1 Y) = g(A(Y, Z), J_1 X). \tag{4.7}$$

By using (4.5) in (4.8), we get

$$\omega + f_i \eta = 0, \text{ for all } i = 1, \ldots, N. \tag{4.8}$$

From (4.4) and (4.8), we obtain that $\omega = \eta = 0$. Then $A = 0$, which is in contradiction with (4.7). Thus g' and g are proportional at every point. If $L = \varnothing$, then using that the algebra $U(J^r Y, A)$ is Codazzi and that the conformal metrics g' and g are homothetic, it results $A = 0$. □

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