On some geometric flows

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Abstract. One of the most interesting questions in Riemannian geometry is that of deciding whether a manifold admits curvatures of certain kinds. In this paper we will be interested in a specific method, using some geometric flows. This direction of study arise from the concept of Ricci flow, introduced by Hamilton. We introduce the notion of Kähler-Riemann flow, generalizing the Kähler-Ricci and Riemann flows, [13], [22], respectively. Our approach leads to a new perspective and provides h-projectively equivalent metrics on Kähler manifolds. Also gradient Kähler-Riemann solitons, with vanishing Bochner curvature tensor, are studied, giving a classification theorem.

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1 Introduction

The Ricci flow is a powerful analytic method for studying the geometry and the topology of manifolds. The main idea is to start with an initial metric on a given manifold and deform it along its Ricci tensor. In simple situations, the flow can be used to deform g into a metric distinguished by its curvature. For example, if M is two-dimensional, the Ricci flow, once suitably renormalised, deforms a metric conformally to one of constant curvature, and thus gives a proof of the two-dimensional uniformisation theorem. More generally, the topology of M may preclude the existence of such distinguished metrics, and the Ricci flow can then be expected to develop a singularity in finite time. Nevertheless, the behaviour of the flow may still serve to tell us much about the topology of the underlying manifold.

There have been many developmens for Riemann and Kähler manifolds from different perspectives [7], [10], [23], [24].

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2 Ricci flow. Ricci wave

Let (M, g) be a Riemannian manifold. The Ricci flow is the evolution equation

$$\frac{\partial g_{ij}}{\partial t} = -2S_{ij},$$

where S is the Ricci tensor field.

Note that, due to the minus sign on the right hand side of the equation, a solution to this parabolic type equation, whose prototype was the heat equation, shrinks in the directions of the positive Ricci curvature and expands in directions of negative Ricci curature.

Generally speaking, it compresses all the positive curvature parts of the manifold into nothingness, while expanding the negative curvature parts of the manifold until they become very homogeneous.

The Ricci flow is mainly used in proving generic theorems of the form [11]:

Theorem A. *Let M be a compact 3-manifold which admits a Riemann metric with strictly positive curvature. Then M also admits a metric of constant positive curvature.*

Theorem B. *Let M be a compact 3-dimensional manifold with nonnegative Ricci curvature. Then M is diffeomorphic to a quotient of one of the spaces S^3 or $S^2 \times \mathbb{R}^1$ or $\mathbb{R}^3$.*

The original goal in studying Ricci flow was related to the classification problem of 3-manifolds, but generalizations in higher dimensions are still to be made.

The hyperbolic geometric flow is the hyperbolic version of Ricci flow. Kong and Liu introduced the Ricci wave:

$$\frac{\partial^2 g_{ij}}{\partial t^2} = -2S_{ij}.$$

These PDE's play a significant role in general relativity and modern theoretical physics. For instance, let us consider the initial Einstein metric

$$ds^2 = \frac{1}{1 - kr^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta \, d\varphi^2,$$

where k is a constant taking the values $-1, 0, 1$. The metric

$$ds^2 = (-2kt^2 + c_1 t + c_2) \left\{ \frac{1}{1 - kr^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta \, d\varphi^2 \right\}$$

is a solution of the hyperbolic geometric flow, where c_1 and c_2 are two constants related by some conditions and it plays an important role in cosmology.

3 Riemann flow. Riemann wave

The idea of Ricci flow and Ricci wave was generalized by C. Udrişte [21], [22] to the concept of Riemann flow (respectively Riemann wave), which is a PDE that evolves the metric tensor G:

$$\frac{\partial G_{ijkl}}{\partial t} = -2R_{ijkl},$$

(respectively
$$\frac{\partial^2 G_{ijkl}}{\partial t^2} = -2R_{ijkl})$$
where $G = \frac{1}{2} g \wedge g$, R is the Riemann curvature tensor associated to the metric g and " $\wedge$ " is the Kulkarni-Nomizu product. For $(0,2)$-tensors a and b, their *Kulkarni-Nomizu product* [14] $a \wedge b$ is given by
$$(a \wedge b)(X_1, X_2; X, Y) = a(X_1, X)b(X_2, Y) + a(X_2, Y)b(X_1, X)$$
$$-a(X_1, Y)b(X_2, X) - a(X_2, X)b(X_1, Y).$$
These extensions are natural, since some results in the Riemann flow and the Riemann wave resemble the case of Ricci flow and Ricci wave. For instance, the Riemann flow satisfies the short time existence and the uniqueness [13]. Also [21]:

Theorem C. *If (M, g_0) is a Riemann manifold $(n \geq 2)$ of constant negative sectional curvature, then an evolution metric of the Riemann flow is*
$$g_t = (1 + (n-1)t)\, g_0.$$
The manifold expands homothetically for all time.

Theorem D. *For the round unit sphere (S^n, g_0), $n \geq 2$, an evolution metric of the Riemann flow is $g_t = (1 - (n-1)t)\, g_0$ and the sphere collapses to a point in finite time.*

4 The complex case

The natural question arises: *How much of the above can be developed for the complex case?*

In particular, if M is a Kähler manifold with an initial metric g_0, what sort of information can we obtain by deforming the metric by certain geometric flow?

The Kähler Ricci flow:
$$\frac{\partial g_{i\bar{j}}}{\partial t} = -2S_{i\bar{j}},$$
becomes strictly parabolic and it is easy to prove the short time existence. One of the most powerful theorems, concerning convergence results is

Theorem D. [4] *Let M be a compact Kähler manifold with the first Chern class $c_1(M)$.*

a) If $c_1(M) = 0$, then for any initial Kähler metric g_0, the solution to the Kähler-Ricci flow exists for all time and converges to a Ricci flat metric as $t \mapsto \infty$.

b) If $c_1(M) < 0$ and the initial metric g_0 is chosen to represent the first Chern class, then the solution to the Kähler-Ricci flow exists for all time and converges to an Einstein metric of negative scalar curvature as $t \mapsto \infty$.

c) If $c_1(M) > 0$ and the initial metric g_0 is chosen to represent the first Chern class, then the solution to the Kähler-Ricci flow exists for all time.

In the case when $c_1(M) > 0$, the convergence problem is still open, even when M has positive holomorphic bisectional curvature.

In the non-compact case, the major conjecture where the Kähler-Ricci flow provides some insight is the following:

Conjecture (Greene-Wu and Yau). *Suppose M is a complete non-compact Kähler manifold with positive holomorphic bisectional curvature. Then M is biholomorphic to $\mathbf{C}^n$.*

The corresponding conjecture for the compact case has already been proven, the manifold being biholomorphic to $\mathbf{P}_n(\mathbf{C})$.

5 Kähler-Riemann flow. Kähler-Riemann solitons.

We extend the notion of Riemann flow for a Riemann space and the notion of Kähler-Ricci flow on complex case to the concept of Kähler-Riemann flow on a Kähler manifold (M, g, J) :

$$\frac{\partial G_{i\bar{j}k\bar{l}}}{\partial t} = -2R_{i\bar{j}k\bar{l}},$$

where R is the Riemann curvature tensor field and $G = \frac{1}{2} g \wedge g$.

Conjecture (Short time existence and uniqueness). *Let (M, g_0, J) be a complex n-dimensional ($n \geq 2$) Kähler manifold. Then there exists $\epsilon > 0$ such that the initial value problem*

$$\frac{\partial G_{i\bar{j}k\bar{l}}(x,t)}{\partial t} = -2R_{i\bar{j}k\bar{l}}(x,t), G(x,0) = G_0$$

has unique solution $G(x,t)$ on $M \times [0, \epsilon]$.

Propositon 5.1. *Let (M, g, J) be a complex n-dimensional ($n \geq 2$) Kähler manifold. The Kähler-Ricci type flow*

$$\frac{\partial g_{i\bar{j}}}{\partial t} = \alpha S_{i\bar{j}} + \beta g_{i\bar{j}}$$

leads to the following Kähler-Riemann type flow:

$$\frac{\partial G_{i\bar{j}k\bar{l}}}{\partial t} = \alpha E_{i\bar{j}k\bar{l}} + 2\beta G_{i\bar{j}k\bar{l}},$$

where α and β are certain smooth functions on M and E is a piece of the semi-traceless part of the Riemann curvature, $E_{i\bar{j}k\bar{l}} = S_{i\bar{l}} g_{k\bar{j}} + S_{k\bar{l}} g_{i\bar{l}}$.

Proof. Since

$$\frac{\partial G_{i\bar{j}k\bar{l}}}{\partial t} = -\frac{\partial g_{i\bar{l}}}{\partial t} g_{k\bar{j}} - \frac{\partial g_{k\bar{j}}}{\partial t} g_{i\bar{l}},$$

by a direct computation one gets

$$\frac{\partial G_{i\bar{j}k\bar{l}}}{\partial t} = \alpha E_{i\bar{j}k\bar{l}} + 2\beta G_{i\bar{j}k\bar{l}},$$

which concludes the proof. □

We generalize the notion of Ricci soliton, acording to the Kähler-Riemann flow, in the following manner:

A solution of the Kähler-Riemann flow is said to be a Kähler-Riemann soliton if it moves along under a one parameter family of automorphisms of M generated by some holomorphic vector field X i.e.

$$R_{i\bar{j}k\bar{l}} = g_{i\bar{l}}(\nabla_{\bar{j}}X_k + \nabla_{\bar{k}}X_j) + g_{k\bar{j}}(\nabla_{\bar{l}}X_i + \nabla_i X_{\bar{l}}).$$

In the case that X is gradient of some potential function$f$, the metric $g_{i\bar{j}}$ is said to be a gradient Kähler-Riemann soliton and one has

$$R_{i\bar{j}k\bar{l}} + \lambda G_{i\bar{j}k\bar{l}} = g_{i\bar{l}}\nabla_{\bar{j}}\nabla_k f + g_{k\bar{j}}\nabla_{\bar{l}}\nabla_i f;$$

$$\nabla_i \nabla_j f = 0.$$

A solution is said to be expanding (shrinking, respectively steady) gradient Kähler-Riemann soliton if there exists a potential function f and a constant λ such that $\lambda < 0$ ($\lambda > 0$, respectively $\lambda = 0$).

Proposition 5.2. *Let* $(\mathbb{R}^2, g_\Sigma)$ *be a manifold with*

$$g_\Sigma = \frac{dz d\bar{z}}{1+|z|^2}.$$

Letting function $f = \frac{1}{2}\log(1+|z|^2)$, *then the metric is Kähler on* $\mathbf{C}$ *and is a gradient steady Kähler-Riemann soliton, with potential function* f.

Proof. Indeed, $R_{1\bar{1}1\bar{1}} = 2g_{1\bar{1}}f_{,1\bar{1}} = \frac{1}{(1+|z|^2)^3}$. □

In complex local coordinates, the Bochner curvature tensor on a complex n-dimensional Kähler manifold (M, J, g) is given by

$$B_{i\bar{j}k\bar{l}} = R_{i\bar{j}k\bar{l}} - \frac{1}{n+2}(g_{i\bar{j}}S_{k\bar{l}} + g_{i\bar{l}}S_{k\bar{j}} + g_{k\bar{l}}S_{i\bar{j}} +$$

$$+ g_{k\bar{j}}S_{i\bar{l}} + \frac{\tau}{2(n+1)(n+2)}(g_{k\bar{l}}g_{i\bar{j}} - g_{k\bar{j}}g_{i\bar{l}}),$$

where τ is the scalar curvature.

Theorem 5.1. *Let* (M, g, f) *be a complex n-dimensional gradient Kähler-Riemann soliton with vanishing Bochner curvature tensor. Then the manifold has constant holomorphic sectional curvature. Consequently,* M *is* $\mathbf{C}^n$ *(steady case),* $\mathbf{CP}^n$ *(shrinking case),* B^n *(expanding case), or their quotients.*

Proof. The computations of [5] and [24] can be adapted in our context. One has

$$\tau + 2 \mid \nabla f \mid^2 + 4\lambda f = const.,$$

where τ is the scalar curvature.

One considers the Cotton tensor field associated to the Bochner tensor ([24]):

$$C_{i\bar{j}k} = \frac{(n+2)^2}{n} g^{p\bar{q}} \nabla_p B_{\bar{q}i\bar{j}k} = (n+2)\nabla_i S_{k\bar{j}} - \frac{n+2}{2(n+1)}(\nabla_i \tau g_{k\bar{j}} + \nabla_k \tau g_{i\bar{j}}).$$

Since the eigenvalues of the Ricci tensor are all real numbers, we can take holomorphic coordinates $z^1, \dots, z^n$ such that $\frac{\partial}{\partial z^i}$ is the eigenvector of the Ricci tensor. Therefore $S_{i\bar{j}} = \nu_i \delta_{i\bar{j}}$.

Using the the gradient Kähler-Riemann soliton equation and the fact that the Bochner tensor is vanishing, we obtain

$$0 = \mid C_{i\bar{j}k} \mid = 16^2 \sum_l \mid \nabla_l f \mid^2 (\sum_i \nu_i^2 + \nu_l^2 - 2\frac{\nu_l \sum_i \nu_i}{n+1} - \frac{\nu_l^2}{n+1} - \frac{(\sum_i \nu_i)^2}{n+1}).$$

If $\mid \nabla f \mid \neq 0$, then the previous formula implies that $\nu_1 = \cdots = \nu_n = \nu$ i.e. the Ricci tensor has a unique eigenvalue ν. So, $S_{i\bar{j}} = \nu g_{i\bar{j}}$. At the points where $\mid \nabla f \mid = 0$ one has $S_{i\bar{j}} = \lambda g_{i\bar{j}}$. Hence the manifold M is Kähler-Einstein. Since the Bochner tensor is vanishing, then M has constant holomorphic sectional curvature and the theorem is proven. □

6 h-projective equivalent metrics

Let (M, J) be a complex manifold. If the Kähler metrics g and $\bar{g}$ are projective equivalent [15] (i.e. if their unparametrised geodesic coincide), then the associated Levi-Civita connections coincide i.e. $\nabla = \overline{\nabla}$ and there are only trivial examples of projective Kähler metrics.

Otsuki and Tashiro introduced another notion in the complex case. Therefore, in h-projective geometry, the unparametrized geodesics are replaced by the "generalized complex geodesics", known as h-planar curves.

Let (M, J, g) be a Kähler manifold and ∇ the Levi-Civita connection. A regular curve $\gamma : I \mapsto M$ is called h-planar with respect to g if satisfies $\nabla_{\dot{\gamma}}$ where $\dot{\gamma} = \alpha\gamma + \beta J(\dot{\gamma})$, for some functions $\alpha, \beta : I \mapsto \mathbb{R}$. Let g and $\bar{g}$ be two Kähler metrics on the complex manifold (M, g). We call g and $\bar{g}$ h-projectively equivalent if each h-planar curve of g is h-planar with respect to $\bar{g}$ and viceversa. ∇ and $\overline{\nabla}$ are h-projectively equivalent iff there exists a (real) 1-form θ such that

$$\overline{\nabla}_X Y - \nabla_X Y = \theta(X)Y + \theta(Y)X - \theta(JX)JY - \theta(JY)JX.$$

A bi-holomorphic mapping $f : M \mapsto M$ is called h-projective transformation if f^*g is h-projectively equivalent to g. Equivalently, we can require that f preserves the set of h-planar curves.

The Bochner curvature tensor is invariant under h-projective transformations. h-projectively equivalent metrics on Kähler manifolds are given by the following:

Theorem 6.1. Let (M, J, g_0) be a Kähler manifold. The class of h-projectively equivalent Kähler metrics given by the Bochner-Kähler-Riemann flow

$$\frac{\partial G_{i\bar{j}k\bar{l}}}{\partial t} = -2B_{i\bar{j}k\bar{l}},$$

verifies

$$G(x,t) = -2B(g_0(x))t + G_0(x).$$

Proof. Implicit solution of a Cauchy problem associated to the Bochner-Kähler-Riemann flow. □

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