On a metric holomorphic connection in complex Lie groups

C. Ida and A. Ionescu

Abstract. In this paper we obtain an unique torsion-free holomorphic connection on a complex Lie group G. Next, the existence of a metric holomorphic tensor on G whose local components have vanishing covariant derivatives with respect the mentioned holomorphic connection is studied. Finally, the holomorphic curvature 2–form of this connection is considered and some properties induced by it are obtained. The case of the complex linear general group $Gl(n;\mathbb{C})$ is considered to ilustrate an example of our discussion.

M.S.C. 2010: 22E10, 32M05, 53B20, 53B21, 53C25, 53C56.
Key words: Complex Lie group; holomorphic connection; holomorphic Riemannian metric; anti-Kählerian metric.

1 Introduction

Complex Riemannian manifolds with analytic (holomorphic) metrics have been investigated by Penrose in connection with the description of theory of non-linear gravitons [14]. A natural step in the construction of Penrose's twistor correspondence is the complexification of the real analytic Riemannian geometry which leads to the notion of complex analytic (holomorphic) Riemannian geometry [12]. In [10], LeBrun proved fundamental relationship between the local complex analytic Riemannian geometry and the global complex analysis. Also, in some papers of Ganchev, Borisov, Ivanov and Mihova, [4, 5, 6, 7] a study of connections and curvatures on complex (holomorphic) Riemannian manifolds is given. Another studies concerning real geometry of holomorphic Riemannian manifolds can be found for instance in [3, 9, 13, 15, 16, 19]. We also notice that holomorphic Riemannian geometry possesses an underlying real geometry consisting of a pseudo-Riemannian metric of neutral signature for which the (integrable) almost complex structure tensor is anti-orthogonal. This leads to the notion of an anti-Kählerian manifold that is a complex manifold with an anti-Hermitian metric and a paralel almost complex structure, namely $g(JX,JY) = -g(X,Y)$ and

BSG Proceedings 21. The International Conference "Differential Geometry - Dynamical Systems" DGDS-2013, October 10-13, 2013, Bucharest-Romania, pp. 74-83.

$\nabla J = 0$. It is shown by Borowiec, Francaviglia, Volovich [2] that a metric on such a manifold must be the real part of a holomorphic metric and also, every complex parallelisable manifold (in particular the factor space G/D of a complex Lie group over the discrete subgroup D) is an anti-Kählerian manifold.

Using the tools of complex (holomorphic) Riemannian geometry, in this paper we present a local description of some holomorphic geometric structures of a complex Lie group G, the latter being regarded as a complex manifold, giving a holomorphic analogue of some results of Rund [17] concerning real Lie groups.

The paper is organized as follows: After a briefly review of some basic properties of the derivatives of the holomorphic composition function of a complex Lie group G we obtain an unique torsion-free holomorphic connection on G. Next, the existence of a metric holomorphic tensor on G whose local components have vanishing covariant derivatives with respect to the mentioned holomorphic connection is analyzed. In contrast with holomorphic Riemanian geometry, the obtained holomorphic metric tensor is only symmetric but not necessarily nondegenerated (even when G is a semi-simple complex Lie group). Finally, as in the case of holomorphic Riemanian (anti-Kählerian) geometry the holomorphic curvature 2–form of this connection is considered and if the complex Lie group is semi-simple then relative to our special holomorphic metric, the complex Lie group G is locally an anti-Kählerian Einstein space. Also, the holomorphic sectional curvature (which is defined as in the holomorphic Riemannian geometry) is constant on G when is evaluated for a pair of right-invariant holomorphic vector fields. Moreover, the covariant derivatives of the components of the holomorphic curvature tensor vanish identically. The case of the complex linear general group $Gl(n;\mathbb{C})$ is considered to ilustrate an example of our discussion. The notions are introduced here in a similar manner with the corresponding results for real Lie groups, [17], using the tools of holomorphic Riemannian (anti-Kählerian) geometry on complex Lie groups. For these reasons the most proofs are omitted here.

2 Preliminaries

By a *complex analytic group*, we mean a group G, which is also a complex analytic manifold, such that the group multiplication $\phi : G \times G \to G$, $\phi(u, v) = u \cdot v$ and the inversion $u \in G \mapsto u^{-1} \in G$ are holomorphic, see for instance [11, 18].

In the following, our considerations will be restricted to a coordinate neighborhood U of the identity e of an r–parameter complex Lie group G. The coordinates of e are identified with $\{0, \dots, 0\} \in \mathbb{C}^r$, while the coordinates of elements of z, u, v, w of U will be denoted by $\{z^\alpha\}$, $\{u^\alpha\}$, $\{v^\alpha\}$, $\{w^\alpha\}$, respectively. Throughout the paper $\alpha, \beta, \gamma, \dots \in \{1, \dots, r\}$ and the summation convetion is also used.

Let us recall some basic properties of the holomorphic composition function on a complex Lie group. The map $\phi : G \times G \to G$ given by $w = \phi(u, v)$ is represented analytically by r equations $w^\alpha = \phi^\alpha(u, v)$, in which $\{\phi^\alpha\}$ denotes a set of r complex-valued holomorphic functions on $G \times G$, where $\phi^\alpha(u, v)$ is an abbreviated notation for $\phi^\alpha(u^1, \dots, u^r, v^1, \dots, v^r)$. Since $u = e \cdot u = u \cdot e$ for all $u \in G$, it follows that up to and including second order terms

$$w^\alpha = \phi^\alpha(u, v) = u^\alpha + v^\alpha + A^{\ \alpha}_{\beta\ \gamma} u^\beta v^\gamma + \dots, \tag{2.1}$$

where the 3–index symbols $A_{\beta\,\gamma}^{\,\alpha}$ are complex constants (in a given coordinate system) in terms of which the structure constants of G are defined as

$$C_{\beta\,\gamma}^{\,\alpha} = A_{\beta\,\gamma}^{\,\alpha} - A_{\gamma\,\beta}^{\,\alpha}. \tag{2.2}$$

The associativity of the group composition implies the Jacobi identities

$$C_{\beta\,\sigma}^{\,\alpha} C_{\varepsilon\,\gamma}^{\,\sigma} + C_{\varepsilon\,\sigma}^{\,\alpha} C_{\gamma\,\beta}^{\,\sigma} + C_{\gamma\,\sigma}^{\,\alpha} C_{\beta\,\varepsilon}^{\,\sigma} = 0 \tag{2.3}$$

which also can be obtained from holomorphic Maurer-Cartan equations of G (see the relations (2.9) below). Let us denote

$$\Phi_\beta^\alpha(u,v) = \frac{\partial \phi^\alpha(u,v)}{\partial u^\beta}, \quad \Psi_\beta^\alpha(u,v) = \frac{\partial \phi^\alpha(u,v)}{\partial v^\beta}, \tag{2.4}$$

so that by (2.1)

$$\Phi_\beta^\alpha(u,0) = \delta_\beta^\alpha, \quad \Psi_\beta^\alpha(0,v) = \delta_\beta^\alpha. \tag{2.5}$$

The derivatives (2.4) give rise to the definitions of the following holomorphic functions on G:
(2.6)

$$\overset{*}{\chi}{}_\beta^\alpha(u) = \Phi_\beta^\alpha(0,u),\ \chi_\beta^\alpha(u) = \Psi_\beta^\alpha(u,0),\ \lambda_\beta^\alpha(u) = \Phi_\beta^\alpha(u,u^{-1}),\ \overset{*}{\lambda}{}_\beta^\alpha(u) = \Psi_\beta^\alpha(u,u^{-1}),$$

it being noted as a direct consequence of (2.5)

$$\overset{*}{\chi}{}_\beta^\alpha(0) = \delta_\beta^\alpha, \quad \chi_\beta^\alpha(0) = \delta_\beta^\alpha, \quad \lambda_\beta^\alpha(0) = \delta_\beta^\alpha, \quad \overset{*}{\lambda}{}_\beta^\alpha(0) = \delta_\beta^\alpha. \tag{2.7}$$

Using the same technique as in the real case, [17], we obtain that $\overset{*}{\chi}{}_\beta^\alpha(u) = \widehat{\lambda}_\beta^\alpha(u)$, where $\widehat{\lambda}_\beta^\alpha(u)$ denotes the elements of the holomorphic matrix that is inverse to $(\lambda_\beta^\alpha(u))$ and $\overset{*}{\lambda}{}_\beta^\alpha(u) = \widehat{\chi}_\beta^\alpha(u^{-1})$, where $\widehat{\chi}_\beta^\alpha(u)$ denotes the elements of the holomorphic matrix that is inverse to $(\chi_\beta^\alpha(u))$.

Also by considering the left and right invariant holomorphic 1–forms on the complex Lie group G defined by

$$\widehat{\chi}^\alpha = \widehat{\chi}_\beta^\alpha(u) du^\beta\ ,\ \lambda^\alpha = \lambda_\beta^\alpha(u) du^\beta \tag{2.8}$$

we obtain the holomorphic Maurer-Cartan equations of the complex Lie group G:

$$\partial\widehat{\chi}^\alpha + \frac{1}{2} C_{\varepsilon\,\beta}^{\,\alpha} \widehat{\chi}^\varepsilon \wedge \widehat{\chi}^\beta = 0\ ,\ \partial\lambda^\alpha - \frac{1}{2} C_{\varepsilon\,\beta}^{\,\alpha} \lambda^\varepsilon \wedge \lambda^\beta = 0, \tag{2.9}$$

where $d = \partial + \overline{\partial}$ is the decomposition of exterior derivative.

3 Metric holomorphic connections on complex Lie groups

3.1 Holomorphic connection 1–forms on complex Lie groups

As G is a complex manifold the complex coordinate transformation can be expressed as

$$\widetilde{u}^\alpha = f^\alpha(u^1,\ldots,u^r) = f^\alpha(u),\ \widetilde{v}^\alpha = f^\alpha(v^1,\ldots,v^r) = f^\alpha(v) \tag{3.1}$$

in terms of a given set of r holomorphic functions of r independent complex variables. Also, since the indentity $e \in G$ must again have $\{0, \ldots, 0\} \in \mathbb{C}^r$ as its coordinates, these holomorphic functions must admit the representation

$$f^{\alpha}(u) = \xi^{\alpha}_{\beta} u^{\beta} + \frac{1}{2}\xi_{\beta}{}^{\alpha}{}_{\gamma} u^{\beta} u^{\gamma} + \ldots, \tag{3.2}$$

with constant complex coefficients. It is supposed that these transformations are invertible, so that

$$\det(f^{\alpha}_{\beta}) \neq 0, \tag{3.3}$$

where we have put

$$f^{\alpha}_{\beta}(u) = \frac{\partial \widetilde{u}^{\alpha}(u)}{\partial u^{\beta}} = \frac{\partial f^{\alpha}(u)}{\partial u^{\beta}}. \tag{3.4}$$

Also, according to (3.2) we shall write

$$\xi^{\alpha}_{\beta} = f^{\alpha}_{\beta}(0) = \frac{\partial \widetilde{u}^{\alpha}(0)}{\partial u^{\beta}}. \tag{3.5}$$

Let us consider now a set of r^2 holomorphic 1–forms $\{\omega^{\alpha}_{\beta} : \alpha, \beta = 1, \ldots, r\}$ on G whose representatives in the holomorphic tangent space $T^{1,0}_u G$ are denoted by $\omega^{\alpha}_{\beta}(u)$. These 1–forms are said to be *holomorphic connection* 1*–forms* on G, [1], if at local complex coordinate changes as in (3.1), we have the following change rule for $\omega^{\alpha}_{\beta}(u)$:

$$\partial f^{\alpha}_{\beta}(u) = f^{\alpha}_{\varepsilon}(u)\omega^{\varepsilon}_{\beta}(u) - f^{\varepsilon}_{\beta}(u)\widetilde{\omega}^{\alpha}_{\varepsilon}(\widetilde{u}). \tag{3.6}$$

Following step by step the construction from the real case, [17] (see also [8]), we can prove

Theorem 3.1. *Let G be a complex Lie group. Then there is an unique torsion-free holomorphic connection on G whose holomorphic connection* 1*–forms $\omega^{\alpha}_{\beta}(u) = \Gamma_{\beta}{}^{\alpha}{}_{\gamma}(u)du^{\gamma}$ has local coefficients given by*

$$\Gamma_{\beta}{}^{\alpha}{}_{\gamma}(u) = \widehat{\lambda}^{\alpha}_{\varepsilon}(u)\left(\frac{\partial \lambda^{\varepsilon}_{\beta}(u)}{\partial u^{\gamma}} + \frac{1}{2}C_{\nu}{}^{\varepsilon}{}_{\mu}\lambda^{\nu}_{\beta}(u)\lambda^{\mu}_{\gamma}(u)\right), \tag{3.7}$$

which, can also be expressed as

$$\Gamma_{\beta}{}^{\alpha}{}_{\gamma}(u) = \frac{1}{2}\widehat{\lambda}^{\alpha}_{\varepsilon}(u)\left(\frac{\partial \lambda^{\varepsilon}_{\beta}(u)}{\partial u^{\gamma}} + \frac{\partial \lambda^{\varepsilon}_{\gamma}(u)}{\partial u^{\beta}}\right). \tag{3.8}$$

3.2 Metric holomorphic connection on complex Lie groups

A holomorphic Riemannian manifold is a complex manifold M, together with a holomorphic tensor field g that is a complex scalar product (i.e., nondegenerate, symmetric, $\mathbb{C}$–bilinear form) on each holomorphic tangent space $T^{1,0}_z M$, $z \in M$. Holomorphic Riemannian geometry has many formal analogies with pseudo-Riemannian geometry and so has not required special study to elaborate its basic features.

In this subsection, by analogy with holomorphic Riemannian geometry we shall now seek holomorphic tensor fields $g \in \left(T^{1,0}G\right)^* \otimes \left(T^{1,0}G\right)^*$ which (if they exist) are such that their covariant derivatives vanish identically for the holomorphic connection specified in the Theorem 3.1.

However, in contrast to holomorphic Riemannian geometry it is not stipulated that such holomorphic tensor fields are symmetric and/or nondegenerate. We shall write locally $g = g_{\alpha\beta} du^\alpha \otimes du^\beta$ in our complex coordinates on G. The covariant derivatives of the components of g with respect to holomorphic connection from the Theorem 3.1 are usually given by

$$g_{\sigma\tau|\gamma} = \frac{\partial g_{\sigma\tau}}{\partial u^\gamma} - g_{\varepsilon\tau}\Gamma^\varepsilon_{\sigma\gamma} - g_{\sigma\varepsilon}\Gamma^\varepsilon_{\tau\gamma}. \tag{3.9}$$

The holomorphic connection from the Theorem 3.1 is said to be *metric with respect to g* if $g_{\alpha\beta|\gamma} = 0$ identically.

The following theorem can be proved step by step in accordance with similar construction from the real case, [17].

Theorem 3.2. *The torsion-free holomorphic connection from the Theorem 3.1 is always metric with respect to the holomorphic tensor field $g \in \left(T^{1,0}G\right)^* \otimes \left(T^{1,0}G\right)^*$ whose local components are given by*

$$g_{\alpha\beta}(u) = C_{\varepsilon\gamma}\lambda^\varepsilon_\alpha(u)\lambda^\gamma_\beta(u), \tag{3.10}$$

where $C_{\alpha\beta} = C^\gamma_{\alpha\delta}C^\delta_{\beta\gamma}$ are the complex Cartan-Killing elements of the complex Lie group G. Moreover, if the complex Lie group G is semi-simple, that is $\det g_{\alpha\beta} \neq 0$, it is the only symmetric holomorphic connection for which this is the case.

Remark 3.1. For the case of holomorphic metric tensor g from (3.10) its symmetry is guaranted from the expression of complex Cartan-Killing elements $C_{\alpha\beta}$. If G is semi-simple then the holomorphic connection coefficients from (3.8) admit a representation in terms of the Christoffel symbols of (3.10).

Remark 3.2. The holomorphic metric tensor $g_{\alpha\beta}$ from (3.10) is not in general unique such that the torsion-free holomorphic connection from Theorem 3.1 is metric with respect to it.

As usual, the holomorphic curvature 2–forms associated with the holomorphic connection 1-forms $\omega^\alpha_\beta(u) = \Gamma^{\ \alpha}_{\beta\ \gamma}(u)du^\gamma$ are specified by the equations of structure, namely

$$\Omega^\alpha_\beta = \partial\omega^\alpha_\beta + \omega^\alpha_\gamma \wedge \omega^\gamma_\beta. \tag{3.11}$$

As in the real case, an explicit evaluation of these holomorphic 2–forms can be carried and thus we obtain the explicit expression for these forms, namely

$$\Omega^\alpha_\beta = \frac{1}{8}\widehat{\lambda}^\alpha_\varepsilon C^\varepsilon_{\nu\mu}C^\mu_{\sigma\tau}\lambda^\nu_\beta(\lambda^\sigma \wedge \lambda^\tau) \tag{3.12}$$

which in view of second relation from (2.9) can be also expressed as

$$\Omega^\alpha_\beta = \frac{1}{4}\widehat{\lambda}^\alpha_\varepsilon C^\varepsilon_{\nu\mu}\lambda^\nu_\beta\partial\lambda^\mu. \tag{3.13}$$

Let us consider now $\mathcal{X}^{1,0}(G)$ be the Lie algebra of holomorphic vector fields on G. As usual, we can asociate with the holomorphic 2–forms (3.12) the curvature operator $R(Z,W)(U)$ for any triplet of holomorphic vector fields Z, W, U as a map $R : \mathcal{X}^{1,0}(G) \times \mathcal{X}^{1,0}(G) \times \mathcal{X}^{1,0}(G) \to \mathcal{X}^{1,0}(G)$ that is represented by

$$\frac{1}{2}R(Z,W)(U) = \left(\Omega^{\alpha}_{\beta}(Z,W)U^{\beta}\right)\frac{\partial}{\partial u^{\alpha}}. \tag{3.14}$$

If the holomorphic curvature 2–forms are locally given by

$$\Omega^{\alpha}_{\beta} = -\frac{1}{2}R^{\alpha}_{\beta\gamma\delta}du^{\gamma} \wedge du^{\delta}, \tag{3.15}$$

the local components of the resulting holomorphic curvature tensor must be

$$R^{\alpha}_{\beta\gamma\delta} = -\frac{1}{4}\widehat{\lambda}^{\alpha}_{\varepsilon}C^{\varepsilon}_{\nu\mu}C^{\mu}_{\sigma\tau}\lambda^{\nu}_{\beta}\lambda^{\sigma}_{\gamma}\lambda^{\tau}_{\delta}. \tag{3.16}$$

The holomorphic Ricci tensor is obtained by contraction over the indices α and δ in (3.16), which yields

$$R_{\beta\gamma} = -\frac{1}{4}C^{\varepsilon}_{\nu\mu}C^{\mu}_{\sigma\varepsilon}\lambda^{\nu}_{\beta}\lambda^{\sigma}_{\gamma}, \tag{3.17}$$

or, in terms of complex Cartan-Killing elements

$$R_{\beta\gamma} = -\frac{1}{4}C_{\nu\sigma}\lambda^{\nu}_{\beta}\lambda^{\sigma}_{\gamma}.$$

By comaparing this holmorphic tensor with the holomorphic metric tensor from (3.10) it is seen that

$$R_{\alpha\beta} = -\frac{1}{4}g_{\alpha\beta}, \tag{3.18}$$

which implies that every complex Lie group is locally *holomorphic Einsteinian.*

Now, as well as we noted, if G is a semi-simple complex Lie group the holomorphic metric tensor from (3.10) is symmetric and nondegenerated. Thus, according to [2]

$$ds^2 = 2Re\left[g_{\alpha\beta}(u)du^{\alpha} \otimes du^{\beta}\right]$$

defines an anti-Kählerian metric on G. Consequently, by (3.18), we have

Theorem 3.3. *Every semi-simple complex Lie groups is locally an anti-Kählerian Einstein space with respect to the anti-Kählerian metric defined by holomorphic metric* (3.10).

Proposition 3.4. *If the complex Lie group is semi-simple then its holomorphic curvature scalar is constant and it is given by*

$$g^{\alpha\beta}R_{\alpha\beta} = -\frac{1}{4}\left(\dim_{\mathbb{C}} G\right). \tag{3.19}$$

In the following, as in the case of holomorphic curvature of holomorphic Riemannian manifolds, it is natural to consider the type $(0,4)$ holomorphic curvature tensor associated with (3.16) as

$$R_{\alpha\beta\gamma\delta} = g_{\beta\varepsilon} R^{\varepsilon}_{\alpha\gamma\delta}. \tag{3.20}$$

According to (3.10) and (3.16) the explicit form of this holomorphic tensor is

$$R_{\alpha\beta\gamma\delta} = -\frac{1}{4} C_{\mu\rho} C^{\rho}_{\nu\varepsilon} C^{\varepsilon}_{\sigma\tau} \lambda^{\nu}_{\alpha} \lambda^{\mu}_{\beta} \lambda^{\sigma}_{\gamma} \lambda^{\tau}_{\delta}, \tag{3.21}$$

from which it result the skew-symmetry relations

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta}\,,\; R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma}. \tag{3.22}$$

Remark 3.3. In contrast to the standard holomorphic Riemann-Christoffel tensor of a holomorphic Riemannian (or an anti-Kählerian) manifold, the holomorphic tensor from (3.21) may not be expressible in terms of Christoffel symbols and their first derivatives.

Taking into account (3.22) it is reasonable to define the *holomorphic sectional curvature* $k(Z,W)$ of G with respect a pair of holomorphic vector fields $Z, W \in \mathcal{X}^{1,0}(G)$ in accordance with the standard formula

$$k(Z,W)(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma})Z^{\alpha}Z^{\gamma}W^{\beta}W^{\delta} = R_{\alpha\beta\gamma\delta}Z^{\alpha}Z^{\gamma}W^{\beta}W^{\delta}. \tag{3.23}$$

Similarly to the real case [17] we get

Theorem 3.5. *The holomorphic sectional curvature of a complex Lie group G with respect to every pair of right-invariant holomorphic vector fields is constant.*

Theorem 3.6. *The covariant derivatives of the components of $R^{\alpha}_{\beta\gamma\delta}$ with respect to the torsion-free holomorphic connection from the Theorem 3.1 vanish identically.*

3.3 Explicit expressions for metric holomorphic connection and curvature forms on $G = GL(n;\mathbb{C})$

In this subsection we shall briefly describe the explicit expressions of the notions discussed above for the case when our complex Lie group G is identified with the group $GL(n;\mathbb{C})$ of all nonsingular $n \times n$ matrices with complex entries.

Let $u, v \in GL(n;\mathbb{C})$ be complex matrices with entries $\{A^j_h\}$, $\{B^j_h\}$, $j, h = 1, \ldots, n$. The parametrization is choosen such that the parameters of u and v are respectively given by

$$u^j_h = A^j_h - \delta^j_h \; v^j_h = B^j_h - \delta^j_h, \tag{3.24}$$

which ensures that the identity matrix is represented by $\{0, \ldots, 0\}$. Evidently, we have $\dim_{\mathbb{C}} GL(n;\mathbb{C}) = r = n^2$, and then the indices $\alpha, \beta, \ldots$ from the above general notations must be replaced by pairs $\binom{j}{h}$, $\binom{l}{k}$, ..., in a manner that should be clear in the sequel. If $w = u \cdot v$ is the matrix multiplication, then the entries in this matrix are given by $C^j_h = A^j_l B^l_h = \delta^j_h + w^j_h$, and consequently the composition functions (2.1) assume the form

$$w^j_h = \phi^j_h(u,v) = u^j_h + v^j_h + u^j_l v^l_h. \tag{3.25}$$

We therefore have for $n^2 \times n^2$ matrices (2.4):

$$\Phi^{jk}_{hl}(u,v) = \frac{\partial \phi^j_h(u,v)}{\partial u^l_k} = \delta^j_l(\delta^k_h + v^k_h)\,,\ \Psi^{jp}_{hm}(u,v) = \frac{\partial \phi^j_h(u,v)}{\partial v^m_p} = \delta^p_h(\delta^j_m + u^j_m) \tag{3.26}$$

so that

$$\frac{\partial \Phi^{jk}_{hl}}{\partial v^m_p} = \delta^j_l \delta^k_m \delta^p_h.$$

The structure constants (2.2) are therefore expressible as

$$C^{\binom{j}{h}}_{\binom{l}{k}\binom{m}{p}} == \delta^j_l \delta^k_m \delta^p_h - \delta^j_m \delta^k_h \delta^p_l = C^{jkp}_{lmh}. \tag{3.27}$$

The resulting complex Cartan-Killing elements are given by

$$C_{\binom{j}{h}\binom{l}{k}} = 2(n\delta^h_l \delta^k_j - \delta^h_j \delta^k_l) = C^{hk}_{jl}. \tag{3.28}$$

Similarly to the real case it follows that the complex matrix from (3.28) has rank n^2-1, and $\det(C_{\alpha\beta}) = \det(C^{hk}_{jl}) = 0$, that is, the complex group $GL(n;\mathbb{C})$ is *not* semisimple.

Now, for the functions from (2.6) we have the following local expressions: for the second of these functions we have directly from (3.26)

$$\chi^{jk}_{hl}(u) = \Psi^{jk}_{hl}(u,0) = \delta^k_h(\delta^j_l + u^j_l) = \delta^k_h A^j_l. \tag{3.29}$$

The third functions in (2.6) depends on the parameters $\widehat{u}^h_j$ of the inverse $u^{-1} = (\widehat{A}^h_j)$ of $u = (A^j_h)$. In accordance with (3.24) we write $\widehat{A}^h_j = \delta^h_j + \widehat{u}^h_j$, so that

$$\lambda^{jk}_{hl} = \Phi^{jk}_{hl}(u,u^{-1}) = \delta^j_l(\delta^k_h + \widehat{u}^k_h) = \delta^j_l \widehat{A}^k_h. \tag{3.30}$$

The inverses of (3.29) and (3.30) are respectively given by

$$\widehat{\chi}^{mk}_{pl} = \delta^k_p \widehat{A}^m_l\,,\ \widehat{\lambda}^{mk}_{pl} = \delta^m_l A^k_p. \tag{3.31}$$

Thus, the corresponding holomorphic connection 1–forms of our torsion-free holomorphic connection from Theorem 3.1 take the form

$$\omega^{\binom{j}{h}}_{\binom{l}{k}} = \omega^{kj}_{lh} = -\frac{1}{2}(\delta^j_l \widehat{A}^k_m du^m_h + \delta^k_h \widehat{A}^m_l du^j_m), \tag{3.32}$$

from which it follows that the corresponding connection coefficients (3.7) are given by

$$\Gamma^{\binom{j}{h}}_{\binom{l}{k}\binom{p}{q}} = \Gamma^{kjq}_{lhp} = -\frac{1}{2}(\delta^j_l \delta^q_h \widehat{A}^k_p + \delta^j_p \delta^k_h \widehat{A}^q_l). \tag{3.33}$$

The holomorphic curvature 2–forms on $GL(n;\mathbb{C})$ take the form

$$\Omega^{\binom{j}{h}}_{\binom{l}{k}} = \Omega^{kj}_{lh} = \partial\omega^{kj}_{lh} + \omega^{sj}_{rh} \wedge \omega^{kr}_{ls} \tag{3.34}$$

which also can be expressed as

$$8\Omega^{kj}_{lh} = [\delta^s_h\delta^j_l\widehat{A}^k_p\widehat{A}^q_r - \delta^q_h\delta^j_l\widehat{A}^s_p\widehat{A}^k_r + \delta^j_r\delta^k_h\widehat{A}^s_p\widehat{A}^q_l - \delta^j_p\delta^k_h\widehat{A}^q_r\widehat{A}^s_l]du^p_q \wedge du^r_s. \tag{3.35}$$

Thus, if we write in accordance with (3.15)

$$\Omega^{kj}_{lh} = -\frac{1}{2}R^{kjqr}_{lhps}du^p_q \wedge du^s_r, \tag{3.36}$$

it follows that the local components of the holomorphic curvature tensor are given by

$$4R^{kjqs}_{lhpr} = \delta^j_l(\delta^q_h\widehat{A}^k_r\widehat{A}^s_p - \delta^s_h\widehat{A}^q_r\widehat{A}^k_p) + \delta^k_h(\delta^j_p\widehat{A}^s_l\widehat{A}^q_r - \delta^j_r\widehat{A}^s_p\widehat{A}^q_l). \tag{3.37}$$

Also, the same arguments as in [17], implies that a class of holomorphic metrics $g_{\alpha\beta}$ with respect to which the torsion-free holomorphic connection from Theorem 3.1 is metric for the complex Lie group $GL(n;\mathbb{C})$ is given by

$$g_{\binom{h}{j}\binom{l}{k}} = g^{jk}_{hl} = \alpha\widehat{A}^j_h\widehat{A}^k_l + \beta\widehat{A}^j_l\widehat{A}^k_h, \tag{3.38}$$

where $\alpha, \beta \in \mathbb{C}$, and according to (3.28) and (3.30) the metric holomorphic tensor from (3.10) is obtained from the choice $\alpha = -2$ and $\beta = 2n$ in (3.38). Thus the metric holomorphic tensor $g_{\alpha\beta}$ from (3.10) is not in general unique such that the torsion-free holomorphic connection from Theorem 3.1 is metric with respect to it, which justify the Remark 3.2.

Finally, for the metric holomorphic tensor from (3.38) the corresponding $(0,4)$ holomorphic tensor from (3.20) is written here in the form

$$R_{\binom{j}{h}\binom{l}{k}\binom{p}{q}\binom{r}{s}} = g_{\binom{a}{b}\binom{l}{k}}R^{\binom{a}{b}}_{\binom{j}{h}\binom{p}{q}\binom{r}{s}}, \tag{3.39}$$

and, also notice that the terms whose coefficient is α in (3.38) do not influence the final result in (3.39). It is obtained that

$$R_{\binom{j}{h}\binom{l}{k}\binom{p}{q}\binom{r}{s}} = \beta[\widehat{A}^k_j\widehat{A}^q_l\widehat{A}^h_r\widehat{A}^s_p - \widehat{A}^k_j\widehat{A}^h_p\widehat{A}^q_r\widehat{A}^s_l + \widehat{A}^h_l\widehat{A}^k_p\widehat{A}^s_j\widehat{A}^q_r - \widehat{A}^h_l\widehat{A}^k_r\widehat{A}^q_j\widehat{A}^s_p]. \tag{3.40}$$

Acknowledgements. This paper is supported by the Sectoral Operational Programme Human Resources Development (SOP HRD), ID134378 financed from the European Social Fund and by the Romanian Government.

References

[1] M. F. Atiyah, *Complex analytic connections in fibre bundles*, Trans. Amer. Math. Soc. 85 (1957), 181-207.

[2] A. Borowiec, M. Francaviglia, I. Volovich, *Anti-Kählerian manifolds*, Differential Geom. and Its Appl. 12 (2000), 281-289.

[3] A. Borowiec, M. Ferraris, M. Francaviglia, I. Volovich, *Almost complex and almost product Einstein manifolds from a variational principle*. J. Math. Phys. 40 (1999), 3446-3464.

[4] G. Ganchev, A. Borisov, *Note on the almost complex manifolds with Norden metric*, Compt. rend. l'Acad. Bulg. Sci., 39, 5 (1986), 31-34.

[5] G. Ganchev, S. Ivanov, *Connections and curvatures on complex Riemannian manifolds*, Internal Rep., Int. Centre Theor. Phys., Miramare-Trieste, 41, 1991.

[6] G. Ganchev, S. Ivanov, *Characteristic curvatures on complex Riemannian manifolds*, Rivista di Mat. della Univ. di Parma, 1, 5 (1992), 155-162.

[7] G. Ganchev, S. Ivanov, V. Mihova, *Curvatures on Anti-Kaehler manifolds*, Rivista di Mat. della Univ. di Parma, 2, 5 (1993), 249-256.

[8] S. I. Goldberg, *Curvature and Homology*, Revised Edition. ISBN 0-486-40207-X, Dover Publication, Inc. Mineola, New-York 1998.

[9] P. R. Law, *De Rham-Wu decomposition of holomorphic Riemannian manifolds*, J. Math. Phys., 43, (12), (2002), 6339-6342.

[10] C. LeBrun, *Spaces of complex null geodesics in complex–Riemannian geometry*, Trans. AMS 278, 1 (1983), 209-231.

[11] D. H. Lee, *The structure of complex Lie groups*, Research Notes in Mathematics Series, Chapman and Hall/CRC , ISBN: 1584882611 , 2001-08-31 , 232 pag.

[12] Y. Manin, *Gauge Field Theory and complex Geometry*, Springer – Verlag, Berlin, Heidelberg, New York, 1988 (translated–from the Russian).

[13] C. Meneghini, *A Holomorphic Point of View about Geodesic Completeness*, Journal of Nonlinear Mathematical Physics 11, (3) (2004), 297-324.

[14] R. Penrose, *The nonlinear graviton and curved twistor theory*, Gen. Relativity Gravitation 7 (1976), 31-52.

[15] J. F. Plebański, *Some solutions of complex Einstein equations*, J. Math. Phys. 16 (1975), 2395-2402.

[16] D. C. Robinson, *The real geometry of holomorphic four-metrics*, J. Math. Phys., 43, (2002), 2015-2028.

[17] H. Rund, *Local differential-geometric structures on Lie groups*, Tensor, N. S. 48 (1988), 64-87.

[18] H.-C. Wang, *Complex Parallisable Manifolds*, Proc. of the American Math. Soc. 5, 5 (1954), 771-776.

[19] N. Woodhouse, *The real geometry of complex space-times*, Int. J. Theor. Phys., 16 (1977), 663-670.

Author's address:

Cristian Ida and Alexandru Ionescu
Department of Mathematics and Computer Science,
University Transilvania of Braşov.
Braşov 500091, Str. Iuliu Maniu 50, România.
E-mail: cristian.ida@unitbv.ro , alexandru.codrin.ionescu@gmail.com