Optimality conditions and duality results of the nonlinear programming problems under (p, r)-invexity on differentiable manifolds

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Abstract. The main purpose of this paper is to study a pair of optimization problems on differentiable manifolds under (p, r)-invexity assumptions. By using the (p, r)-invexity assumptions on the functions involved, optimality conditions and duality results (Mond-Weir, Wolfe and mixed type) are established on differentiable manifolds. We construct counterexample to justify that our investigations are more general than the existing work available in the literature.

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1 Introduction

The field of optimization is concerned with the study of maximization and minimization of mathematical functions. Convexity plays a vital role in optimization theory. In the recent years, several generalizations have been developed for the classical properties of convexity. In 1981, Hanson [8] introduced the concept of generalized convexity by generalizing the difference $(x - y)$ in the definition of convex function to any function $\eta(x, y)$. Craven [6] named them as invex functions. Hanson's initial results inspired a great deal of subsequent work, which had greatly expanded the role and applications of invexity in nonlinear optimization and other branches of pure and applied sciences. Ben-Israel and Mond [5] introduced a new generalization of convex sets and convex functions. Jeyakumar [10] studied the properties of preinvex functions and their role in optimization and mathematical programming problems. Zalmai [21] introduced $\rho - (\eta, \theta)$-invexity and Antczak [3] introduced (p, r)-invex sets and functions. Mandal and Nahak [13] generalized the results of Zalmai [21] and Antczak [3] by introducing $(p, r) - \rho - (\eta, \theta)$-invexity.

The concept of convexity in linear topological spaces relies on the possibility of connecting any two points of the space by the line segment between them. But if we

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are given an abstract manifold then we can not join two points by a line segment. In this case, when the linear space is replaced by a manifold, the line segment is replaced by a geodesic arc. Earlier Luenberger [12] studied gradient projection methods along geodesic. Rapcsak [18] introduced optimality conditions for a minimization problem taking the constraint set as a differentiable manifold. In year 1991, Rapcsak [19] proposed geodesic convexity by taking Riemannian manifold as a definition of domain. Udriste [20] introduced the concept of duality for a convex programming problem on Riemannian manifolds. The notion of invex functions on a manifold was introduced by Pini [16]. Mititelu [15] generalized the invexity notions by introducing (ρ, η)-invex, (ρ, η)-pseudoinvex and (ρ, η)-quasiinvex functions, respectively. Mititelu [15], also had established the necessary and sufficient conditions of Karush-Kuhn-Tucker type for a vector programming problem defined on a differentiable manifold. Ferrara and Mititelu [7] developed the Mond-Weir type duality for vector programming problems on a differentiable manifold. Barani and Pouryayevali [4] introduced the geodesic invex sets, geodesic invex and geodesic preinvex functions on Riemannian manifolds. Ahmad et al. [2] extended these results by introducing geodesic η-pre-pseudo invex functions and geodesic η-pre-quasi invex functions. In year 2012, Iqbal et al. [9] defined geodesic E-convex sets, geodesic E-convex functions. Agarwal et al. [1] introduced geodesic α-invex sets, geodesic α-invex and α-preinvex functions.

In this paper we have generalized the work of Antczak [3] from functions defined on $\mathbb{R}^n$ to functions defined on differentiable manifolds. We have defined (p, r)-invex functions and studied optimality conditions and duality results (weak, strong, converse duality) on differentiable manifolds.

2 Preliminaries

In this section, we recall some definitions and known results from differentiable manifolds which can be found in ([11], [17]).

Definition 2.1. A differentiable manifold M of dimension n is a space which locally looks like $\mathbb{R}^n$, i.e., every point of it has a neighborhood homeomorphic to an open set in $\mathbb{R}^n$ and on which C^k functions for $1 \leq k \leq \infty$ are defined.

Definition 2.2. A curve on a differentiable manifold M is a smooth (i.e., C^∞) map γ from some interval $(-\delta, \delta)$ of the real line into M.

Definition 2.3. A tangent vector on a curve γ at a point p of M is defined as the map

$$\dot{\gamma}_p : C^\infty(M) \to \mathbb{R}, \qquad f \mapsto \dot{\gamma}_p(f) \equiv \frac{\mathrm{d}}{\mathrm{d}t}(fo\gamma) \mid_p .$$

Definition 2.4. The set of all tangent vectors at a point p of M is called the tangent space at p and is denoted by T_pM.

Definition 2.5. A manifold whose tangent spaces are endowed with a smoothly varying inner product with respect to a point $x \in M$ is called a Riemannian manifold. The smoothly varying inner product, denoted by $< \xi_x, \zeta_x >$ for every two elements ξ_x and ζ_x of T_xM, is called a Riemannian metric. If M is a differentiable manifold, then there always exist Riemannian metrices on M.

Definition 2.6. A geodesic is a curve whose tangents are parallel to it. A geodesic is a generalization of the notion of a straight line to curved spaces.

Definition 2.7. [19] Let M be an n-dimensional Riemannian manifold. A set $A \subset M$ is called g-convex set (geodesic convex set) if any two points of A are joined by a geodesic belonging to A, i.e., for all $x, y \in A$, there exists a geodesic $\gamma : [0,1] \to A$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Definition 2.8. [19] Let M be an n-dimensional Riemannian manifold and A be a g-convex set in M. A function $f : A \to \mathbb{R}$ is said to be geodesic convex if for any two points $x, y \in A$ and geodesic $\gamma : [0,l] \to A$ with $\gamma(0) = x$ and $\gamma(l) = y$ and all $t \in [0,1]$, we have,

$$f(\gamma(tl)) \leq tf(\gamma(0)) + (1-t)f(\gamma(l)).$$

If f is differentiable, we have

$$f(y) - f(x) \geq \nabla f(x)\dot{\gamma}(0)l,$$

where $\nabla f(x)$ and $\dot{\gamma}(0)$ mean, respectively, the gradient of f at the point x and the derivative of $\gamma(tl)$ with respect to t at the point 0.

Let M be an n-dimensional differentiable manifold and T_pM be the tangent space to M at p. Also assume that $TM = \bigcup_{p \in M} T_pM$ is the tangent bundle of M. Let α be a differentiable curve on M with $\alpha(0) = p \in M$. Then the tangent vector to the curve α at p is $v = \alpha'(0) \in T_{\alpha(0)}M = T_pM$. Assume that N is another differentiable manifold and $\phi : M \to N$ is a differentiable map.

Definition 2.9. ([7]) The linear map $\mathrm{d}\phi_p : T_pM \to T_{\phi(p)}N$ defined by $\mathrm{d}\phi_p(v) = \phi'(p)v$ is called the differential of ϕ at the point p.

Let $F : M \to \mathbb{R}$ be a differentiable function. The differential of F at p, namely $\mathrm{d}F_p : T_pM \to T_{F(p)}\mathbb{R} \equiv \mathbb{R}$, is given by $\mathrm{d}F_p(v) = \mathrm{d}F(p)v$, $v \in T_pM$.
The length of a curve $\gamma : [a,b] \to M$ is defined by

$$L(\gamma) = \int_a^b \|\gamma'(t)\|_{\gamma(t)} \mathrm{d}t.$$

For any two points $p, q \in M$, we define $\mathrm{d}(p,q) = \inf\{L(\gamma) : \gamma \text{ is curve joining } p \text{ to } q\}$. Then d is a distance which induces the original topology on M. We consider now a map $\eta : M \times M \to TM$ such that $\eta(p,q) \in T_qM$ for every $p, q \in M$. For a differentiable function $f : M \to \mathbb{R}$, Pini [16] defined invexity in the following manner.

Definition 2.10. The differentiable function f is said to be η-invex or invex on a differentiable manifold M if for any $x, y \in M$,

$$f(x) - f(y) \geq \mathrm{d}f_y(\eta(x,y)).$$

Later on Mititelu [15] generalized the above definition as follows.

Definition 2.11. The differentiable function f is said to be (ρ, η)-invex at y if there exist an $\eta : M \times M \to TM$ and $\rho \in \mathbb{R}$ such that

$$\forall x \in M : f(x) - f(y) \geq \mathrm{d}f_y(\eta(x,y)) + \rho \mathrm{d}^2(x,y).$$

Definition 2.12. ([4]) A closed η-path joining the points y and $u = \alpha_{x,y}(1)$ is a set of the form $P_{yu} = \{v : v = \alpha(t) : t \in [0,1]\}$.

3 (p,r)-Invexity

In the year 2001, Antczak [3] introduced (p, r)-invex function over $\mathbb{R}^n$ which generalizes the notion of invexity.

Definition 3.1 (Antczak (2001)). Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function and p, r be arbitrary real numbers. The function f is said to be (p, r)-invex with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ at u, if any one of the following conditions holds

$$\frac{1}{r}e^{rf(x)} \geq \frac{1}{r}e^{rf(u)}[1 + \frac{r}{p}\nabla f(u)(e^{p\eta(x,u)} - \mathbf{1})],\ for\ p \neq 0,\ r \neq 0, \tag{3.1}$$

$$\frac{1}{r}e^{rf(x)} \geq \frac{1}{r}e^{rf(u)}[1 + r\nabla f(u)\eta(x, u)], for\ p = 0,\ r \neq 0, \tag{3.2}$$

$$f(x) - f(u) \geq \frac{1}{p}\nabla f(u)(e^{p\eta(x,u)} - \mathbf{1}), for\ p \neq 0,\ r = 0, \tag{3.3}$$

$$f(x) - f(u) \geq \nabla f(u)\eta(x, u), for\ p = 0,\ r = 0. \tag{3.4}$$

We introduce the (p, r)-invex function on a differentiable manifold M. Using (p, r)-invexity assumptions, we derive optimality conditions and duality results (weak, strong and converse duality theorems) for a pair of optimization problems.

Definition 3.2. Let M be an n-dimensional differentiable manifold and $f : M \to \mathbb{R}$ be a differentiable function. Let η be a map $\eta : M \times M \to TM$ such that $\eta(x, u) \in T_uM$ for all $x, u \in M$. The exponential map on M is a map $exp_u : T_uM \to M$ and the differential of the exponential map $(\mathrm{d}exp_u)_a : T_a(T_uM) \cong T_uM \to T_cM$, where $a = t_0\eta(x, u), t_0 \in [0, 1]$, and $c \in P_{xu}$ where P_{xu} is a closed path joining the point x and u. Let p, r be arbitrary real numbers. If for all $x \in M$, the relations

$$\frac{1}{r}(e^{r(f(x)-f(u))} - 1) \geq \frac{1}{p}\mathrm{d}f_c([(\mathrm{d}exp_u)_a(p\eta(x, u))] - \mathbf{I}),\ for\ p \neq 0,\ r \neq 0, \tag{3.5}$$

$$\frac{1}{r}(e^{r(f(x)-f(u))} - 1)\ \geq\ \mathrm{d}f_u(\eta(x, u)), for\ p = 0,\ r \neq 0, \tag{3.6}$$

$$f(x) - f(u) \geq \frac{1}{p}\mathrm{d}f_c([(\mathrm{d}exp_u)_a(p\eta(x, u))] - \mathbf{I}), for\ p \neq 0,\ r = 0, \tag{3.7}$$

$$f(x) - f(u) \geq \mathrm{d}f_u(\eta(x, u)), for\ p = 0,\ r = 0, \tag{3.8}$$

hold, then f is said to be (p, r)-invex function at u on M. Here $\mathbf{I} \in T_cM$ such that for a co-ordinate chart ϕ, $\phi(\mathbf{I}) = \mathbf{1}$, where $\mathbf{1} = (1, 1, ..., 1) \in \mathbb{R}^n$.

Remark 3.3. We denote the exponential map on the manifold by $\exp(x)$ for $x \in M$ and e^x for $x \in \mathbb{R}$.

Example 3.4. The circle S can be thought of as the set $\{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\}$ of the Euclidean space $\mathbb{R}^2$. In the case of the circle S the possible co-ordinate charts are

$U_1 = \{(x,y) : x > 0\}$ $\quad\phi_1(x,y) = y$
$U_2 = \{(x,y) : x < 0\}$ $\quad\phi_2(x,y) = y$
$U_3 = \{(x,y) : y > 0\}$ $\quad\phi_3(x,y) = x$
$U_4 = \{(x,y) : y < 0\}$ $\quad\phi_4(x,y) = x.$

We define a differentiable function on S. Let $x = (x_1, x_2) \in S$ and $f : S \to \mathbb{R}$ be defined by $f(x) = x_1 + \sin x_2$. Let $u = (u_1, u_2) \in S$.
The tangent space of S at u is the set $T_uS = \{v \in \mathbb{R}^2 : u \cdot v = 0\}$.
We choose $\eta : S \times S \to T_uS$ as $\eta(x,u) = (-u_2, u_1) \in T_uS$.
Let $a = \eta(x,u) = (-u_2, u_1)$.
We now find $\mathrm{d}f_u(a)$. We take a chart $\phi_3(-u_2,u_1) = \phi(-u_2,u_1) = u_2$ at a and the identity mapping as a chart ψ at $f(a)$. Here both S and $\mathbb{R}$ are of dimension 1. We now find the Jacobian matrix $\psi o f o \phi^{-1}$ at $\phi(a)$.

$$\mathrm{d}f_a(\frac{\partial}{\partial\phi})(\psi) = \frac{\partial}{\partial\phi}(\psi o f) = \frac{\partial}{\partial\phi}(f(-u_2,u_1)) = \frac{\partial}{\partial u_2}(-u_2 + \sin u_1) = -1.$$

i.e., $\mathrm{d}f_u(\eta(x,u)) = -1$.
Now $e^{f(x)-f(u)} - 1 - \mathrm{d}f_u(\eta(x,u)) = e^{f(x)-f(u)} - 1 + 1 = e^{f(x)-f(u)} \geq 0$, $\forall x, u \in S$.
Hence f is $(0,1)$-invex on S.
But if we take $x = (1/\sqrt{2}, -1\sqrt{2}) \in S$, $u = (1/\sqrt{2}, 1\sqrt{2}) \in S$.
Then $f(x) - f(u) = 1/\sqrt{2} - \sin 1/\sqrt{2} - 1/\sqrt{2} - \sin 1/\sqrt{2} = -1.299$ and $df_u(\eta(x,u)) = -1$. Hence $f(x) - f(u) < \mathrm{d}f_u(\eta(x,u))$, i.e., f is not invex on S.

3.1 Sufficient Optimality Conditions

In recent years, many traditional optimization methods have been successfully generalized to minimize objective functions on manifolds. Mititelu [15] established necessary and sufficient conditions of Karush-Kuhn-Tucker (KKT) [14] type for a vector programming problem on differentiable manifolds. Consider the following primal optimization problem on a differentiable manifold M

(P) $\quad$ $Minimize\ f(x)$
$\quad$ subject to $g_i(x) \leq 0, \quad i = 1, ..., m,$

where $f : M \to \mathbb{R},\ g_i : M \to \mathbb{R},\ i = 1, ..., m$, are differentiable functions. Let D denote the set of all feasible solutions of **(P)**.
Let $\bar{x} \in$ D be an optimal solution of **(P)** and we define the set $J^\circ = \{j \in 1, ..., m : g_j(\bar{x}) = 0\}$. Suppose that the domain D satisfies the following constraint qualification at $\bar{x}$:

$$R(\bar{x}) : \exists v \in TM : \mathrm{d}(g_{J^\circ})_{\bar{x}}(v) \leq 0.$$

Here $\mathrm{d}(g_{J^\circ})_{\bar{x}}(v)$ is the vector components of $\mathrm{d}(g_j)_{\bar{x}}(v)$, $\forall j \in J^\circ$, taken in increasing order of j.

Lemma 3.1. *(Necessary Karush-Kuhn-Tucker (KKT) condition)([15]) If a feasible point $\bar{x} \in M$ is an optimal solution of the problem* (**P**) *and satisfies the constraint qualification $R(\bar{x})$, then there exists multiplier $\xi = (\xi_1, ..., \xi_m)^T \in \mathbb{R}^m$, such that the* *following conditions hold*

$$\mathrm{d}f_{\bar{x}} + \xi^T \mathrm{d}g_{\bar{x}} = 0, \tag{3.9}$$

$$\xi^T g(\bar{x}) = 0, \tag{3.10}$$

$$\xi \geq 0, \quad i = 1, 2, ..., m, \tag{3.11}$$

here $g = (g_1, g_2, ..., g_m)^T$.

Theorem 3.2. *Assume that a point* $\bar{x} \in M$ *is feasible for problem* **(P)**, *and let the KKT conditions* (3.9)-(3.11) *be satisfied at* $(\bar{x}, \xi)$. *If the objective function* f *and the function* $\xi^T g$ *are* (p, r)*-invex with respect to the same function* η *at* $\bar{x}$ *on D, then* $\bar{x}$ *is a global minimum point of the problem* **(P)**.

Proof. Let x be a feasible point for the problem **(P)**. Since f and $\xi^T g$ are (p, r)-invex with respect to the same function η at $\bar{x}$ on D, $\forall x$, $\bar{x} \in$ D, we have,

$$\frac{1}{r}(e^{r(f(x)-f(\bar{x}))} - 1) \geq \frac{1}{p}\mathrm{d}f_{\bar{x}}(\mathrm{d}(exp_{\bar{x}}(p\eta(x, \bar{x}))) - \mathbf{I}), \tag{3.12}$$

$$\frac{1}{r}(e^{r(\xi^T g(x)-\xi^T g(\bar{x}))} - 1) \geq \frac{\xi^T}{p}\mathrm{d}g_{\bar{x}}(\mathrm{d}(exp_{\bar{x}}(p\eta(x, \bar{x}))) - \mathbf{I}). \tag{3.13}$$

Adding (3.12) and (3.13), we have

$$\frac{1}{r}[(e^{r(f(x)-f(\bar{x}))} - 1 + e^{r(\xi^T g(x)-\xi^T g(\bar{x}))} - 1] \geq \frac{1}{p}(\mathrm{d}f_{\bar{x}} + \xi^T \mathrm{d}g_{\bar{x}})(\mathrm{d}(exp_{\bar{x}}(p\eta(x, \bar{x}))) - \mathbf{I}),$$

and by KKT condition (3.9), we have

$$\frac{1}{r}(e^{r(f(x)-f(\bar{x}))} - 1) \geq \frac{1}{r}(1 - e^{r(\xi^T g(x)-\xi^T g(\bar{x}))}),$$

or, by KKT condition (3.10)

$$\frac{1}{r}(e^{r(f(x)-f(\bar{x}))} - 1) \geq \frac{1}{r}(1 - e^{r(\xi^T g(x))}).$$

Without loss of generality, let $r > 0$ (in the case when $r < 0$ the proof is analogous; one should change only the direction of some inequalities below to the opposite one). Since x is a feasible point of **(P)**, then $g(x) \leq 0$ and $\xi \geq 0$ imply that $1 - e^{r\xi^T g(x)} \geq 0$ and $e^{r(f(x)-f(\bar{x}))} \geq 1$.
Hence $f(x) \geq f(\bar{x})$. Therefore, $\bar{x}$ is an optimal solution of the problem **(P)**. ☐

3.2 Mond-Weir Type Duality

The central part of the optimization problem is the duality theory. In several optimization problems evaluating the dual maximum is comparatively easier than solving a primal minimization problem. The concept of duality for a convex programming problem on a Riemannian manifold was first introduced by Udriste [20]. Ferrera and Mititelu [7] developed a duality of Mond-Weir type for a vector mathematical programming problem involving invex functions on a differentiable manifold. In our work, we establish the duality results for the problem **(P)** involving (p, r)-invex functions over a differentiable manifold.
For the optimization problem **(P)**, the Mond-Weir dual problem [7] **(MWD)** is defined in the following form

$$(\mathbf{MWD}) \qquad Maximize \quad f(u)$$
$$subject\ to \quad \mathrm{d}f_u + y^T \mathrm{d}g_u = 0,$$
$$y^T g(u) \geq 0, \quad y \in \mathbb{R}^m_+,$$

where $f, g_i : M \to \mathbb{R}, \quad i = 1, 2, ..., m$ are differentiable functions. Let W_1 denote the set of all feasible solutions of **(MWD)**.

Remark 3.5. Throughout the remaining sections of this paper, without loss of generality, we assume $r > 0$ (in the case when $r < 0$ the proof is analogous; one should change only the direction of some inequalities to the opposite one but finally will get same results). The theorems will be proved only in the case when $p \neq 0, r \neq 0$ (other cases can be dealt with likewise).

We have established the following duality results between **(P)** and **(MWD)**.

Theorem 3.3. *(Weak Duality) Let x and (u, y) be the feasible solutions of* **(P)** *and* **(MWD)**, *respectively. Moreover, assume that f and $y^T g$ are (p, r)-invex at u on M with respect to the same η, then* $\inf(\mathbf{P}) \geq \sup(\mathbf{MWD})$.

Proof. Since f and $y^T g$ are (p, r)-invex with respect to the same η,

$$\frac{1}{r}(e^{r(f(x)-f(u))} - 1) \geq \frac{1}{p}\mathrm{d}f_u(\mathrm{d}(exp_u(p\eta(x, u))) - \mathbf{I}), \tag{3.14}$$

$$\frac{1}{r}(e^{r(y^T g(x) - y^T g(u))} - 1) \geq \frac{y^T}{p}\mathrm{d}g_u(\mathrm{d}(exp_u(p\eta(x, u))) - \mathbf{I})\forall x \in D. \tag{3.15}$$

Adding (3.14) and (3.15), we get

$$\frac{1}{r}[(e^{r(f(x)-f(u))} - 1 + e^{r(y^T g(x) - y^T g(u))} - 1)] \geq \frac{1}{p}(\mathrm{d}f_u + y^T \mathrm{d}g_u)(\mathrm{d}(exp_u(p\eta(x, u))) - \mathbf{I}),$$

or,

$$\frac{1}{r}(e^{r(f(x)-f(u))} - 1) \geq \frac{1}{r}(1 - e^{ry^T g(x)}).$$

Since x is a feasible solution of **(P)** and $y^T \geq 0$, then we have $1 - e^{ry^T g(x)} \geq 0$, i.e., $e^{r(f(x)-f(u))} \geq 1$, or, $f(x) \geq f(u)$ holds for $\forall x \in$ D and $u \in W_1$. Therefore $\inf(\mathbf{P}) \geq \sup(\mathbf{MWD})$. □

Theorem 3.4. *(Strong Duality) Let $\bar{x}$ be an optimal solution of the problem* **(P)** *at which a constraint qualification $R(\bar{x})$ be satisfied. Then there exists $\xi \in \mathbb{R}_+^m$ such that $(\bar{x}, \xi)$ is a feasible solution of* **(MWD)**. *Suppose that the hypotheses of the Weak Duality Theorem 3.3 hold, then $(\bar{x}, \xi)$ is an optimal solution of the dual programming problem* **(MWD)**, *and the objective values of* **(P)** *and* **(MWD)** *are equal.*

Proof. Since a constraint qualification $R(\bar{x})$ is satisfied at $\bar{x}$, then from the KKT necessary conditions (3.9)-(3.11), there exists ξ such that $(\bar{x}, \xi)$ is a feasible solution of **(MWD)**. Since the conditions of the Weak Duality Theorem 3.3 hold, then $(\bar{x}, \xi)$ is an optimal solution of the dual problem **(MWD)**, and the objective values of **(P)** and **(MWD)** are equal. □

Theorem 3.5. *(Converse Duality) Let $(\bar{u}, y)$ be an optimal solution of the dual problem* **(MWD)** *such that $\bar{u} \in D$. If f and $y^T g$ are (p, r)-invex with respect to the same η at $\bar{u}$ on M. Then $\bar{u}$ is an optimal solution of* **(P)**.

Proof. Since f and $y^T g$ are (p, r)-invex with respect to the same η at $\bar{u}$, we have

$$\frac{1}{r}(e^{r(f(x)-f(\bar{u}))} - 1) \geq \frac{1}{p}\mathrm{d}f_{\bar{u}}(\mathrm{d}(exp_{\bar{u}}(p\eta(x, \bar{u}))) - \mathbf{I}), \tag{3.16}$$

$$\frac{1}{r}(e^{r(y^T g(x)-y^T g(\bar{u}))} - 1) \geq \frac{y^T}{p}\mathrm{d}g_{\bar{u}}(\mathrm{d}(exp_{\bar{u}}(p\eta(x, \bar{u}))) - \mathbf{I}), \ \forall x \in D. \tag{3.17}$$

Adding (3.16) and (3.17), we get

$$\frac{1}{r}[(e^{r(f(x)-f(\bar{u}))} - 1 + e^{r(y^T g(x)-y^T g(\bar{u}))} - 1)] \geq \frac{1}{p}(\mathrm{d}f_{\bar{u}} + y^T \mathrm{d}g_{\bar{u}})(\mathrm{d}(exp_{\bar{u}}(p\eta(x, \bar{u}))) - \mathbf{I}).$$

Since $(\bar{u}, y)$ is a feasible solution of **(MWD)**, we have $\mathrm{d}f_{\bar{u}} + y^T \mathrm{d}g_{\bar{u}} = 0$, hence

$$\frac{1}{r}(e^{r(f(x)-f(\bar{u})} - 1) \geq \frac{1}{r}(1 - e^{ry^T g(x)}).$$

Since $x \in$ D and $y \geq 0$, we have, $1 - e^{ry^T g(x)} \geq 0 \Rightarrow e^{r(f(x)-f(\bar{u}))} \geq 1 \Rightarrow f(x) \geq f(\bar{u})$. So $\bar{u}$ is an optimal solution of **(P)**. □

3.3 Wolfe Type Duality

Motivated by the classical Wolfe type duality [14], for the optimization problem **(P)**, we define the Wolfe type dual **(WD)** in the following form

$$\begin{aligned} (\mathbf{WD}) \qquad & Maximize \quad f(u) + \sum_{i=1}^{m} \xi_i g_i(u) \\ & subject\ to \quad \mathrm{d}f_u + \sum_{i=1}^{m} \xi_i \mathrm{d}g_{iu} = 0, \\ & \xi_i \geq 0, \quad i = 1, 2, ..., m, \end{aligned}$$

where $f, g_i : M \to \mathbb{R}, \quad i = 1, 2, ..., m$ are differentiable functions. Let W_2 be the set of all feasible solutions of **(WD)**.

We have proved the following duality results between **(P)** and **(WD)**.

Theorem 3.6. *(Weak Duality) Let x and (u, ξ) be feasible solutions for* **(P)** *and* **(WD)***, respectively. Moreover, assume that f and $\sum_{i=1}^{m} \xi_i g_i$ are (p, r)-invex and $(p, -r)$-invex respectively, at u on M with respect to the same η, then* $\inf(\mathbf{P}) \geq \sup(\mathbf{WD})$.

Proof. Since f and $\sum_{i=1}^{m} \xi_i g_i$ are (p, r)-invex and $(p, -r)$-invex respectively, with respect to the same η, then $\forall x \in \mathrm{D}$, we have

$$\frac{1}{r}(e^{r(f(x)-f(u))} - 1) \geq \frac{1}{p}\mathrm{d}f_u(\mathrm{d}(exp_u(p\eta(x,u))) - \mathbf{I}), \tag{3.18}$$

$$\frac{-1}{r}(e^{-r(\sum_{i=1}^{m} \xi_i g_i(x) - \sum_{i=1}^{m} \xi_i g_i(u))} - 1) \geq \frac{1}{p}\sum_{i=1}^{m} \xi_i \mathrm{d}g_{iu}(\mathrm{d}(exp_u(p\eta(x,u))) - \mathbf{I}). \tag{3.19}$$

Adding (3.18) and (3.19), we get

$$\frac{1}{r}[(e^{r(f(x)-f(u))} - 1 - e^{-r(\sum_{i=1}^{m} \xi_i g_i(x) - \sum_{i=1}^{m} \xi_i g_i(u))} + 1)] \geq \frac{1}{p}(\mathrm{d}f_u + \sum_{i=1}^{m} \xi_i \mathrm{d}g_{iu})(\mathrm{d}(exp_u(p\eta(x,u))) - \mathbf{I}).$$

Since (u, ξ) is a feasible solution of **(WD)**, we have

$$\mathrm{d}f_u + \sum_{i=1}^{m} \xi_i \mathrm{d}g_{iu} = 0.$$

Hence

$$e^{r(f(x)-f(u))} \geq e^{r(-\sum_{i=1}^{m} \xi_i g_i(x) + \sum_{i=1}^{m} \xi_i g_i(u))}. \tag{3.20}$$

Since x is a feasible solution of **(P)** and $\xi_i \geq 0$, we have $\sum_{i=1}^{m} \xi_i g_i(x) \leq 0$. Hence

$$\begin{aligned} e^{r(f(x)-f(u))} &\geq e^{r(\sum_{i=1}^{m} \xi_i g_i(u))} \\ \Rightarrow f(x) - f(u) &\geq \sum_{i=1}^{m} \xi_i g_i(u), \\ \Rightarrow f(x) &\geq f(u) + \sum_{i=1}^{m} \xi_i g_i(u), \end{aligned}$$

holds for $\forall x \in \mathrm{D}$ and $u \in W_2$. Therefore $\inf(\mathbf{P}) \geq \sup(\mathbf{WD})$. □

Theorem 3.7. *(Strong Duality) Let $\bar{x}$ be an optimal solution of the problem* **(P)** *at which a constraint qualification $R(\bar{x})$ be satisfied. Then there exists $\xi \in \mathbb{R}^m_+$ such that $(\bar{x}, \xi)$ is a feasible solution of* **(WD)***. Suppose that the hypotheses of the Weak Duality Theorem 3.6 hold, then $(\bar{x}, \xi)$ is an optimal solution of the dual programming problem* **(WD)***, and the objective values of* **(P)** *and* **(WD)** *are equal.*

Proof. Since a constraint qualification $R(\bar{x})$ is satisfied at $\bar{x}$, then from the KKT necessary conditions (3.9)-(3.11), there exists ξ such that $(\bar{x}, \xi)$ is a feasible solution of **(WD)**. Since the conditions of the Weak Duality Theorem 3.6 hold, then $(\bar{x}, \xi)$ is an optimal solution of the dual problem **(WD)** and the objective values of **(P)** and **(WD)** are equal. □

Theorem 3.8. *(Converse Duality) Let (u, ξ) be an optimal solution of the dual problem* **(WD)** *such that $u \in D$. If f is (p, r)-invex and $\sum_{i=1}^{m} \xi_i g_i$ is $(p, -r)$-invex with respect to the same η at u on M. Then u is an optimal solution of* **(P)**.

Proof. We prove it by contradiction. Let u is not an optimal solution of **(P)**. Hence $\exists x \in D \ni f(x) < f(u)$. Since (u, ξ) is an optimal solution of **(WD)**, we have

$$f(u) + \sum_{i=1}^{m} \xi_i g_i(u) > f(x) + \sum_{i=1}^{m} \xi_i g_i(x), \tag{3.21}$$

or,

$$f(x) - f(u) < -\sum_{i=1}^{m} \xi_i g_i(x) + \sum_{i=1}^{m} \xi_i g_i(u). \tag{3.22}$$

Since f is (p, r)-invex and $\sum_{i=1}^{m} \xi_i g_i$ is $(p, -r)$- invex we have from (3.20)

$$e^{r(f(x)-f(u))} \geq e^{r(-\sum_{i=1}^{m} \xi_i g_i(x) + \sum_{i=1}^{m} \xi_i g_i(u))},$$

$$\Rightarrow f(x) - f(u) \geq -\sum_{i=1}^{m} \xi_i g_i(x) + \sum_{i=1}^{m} \xi_i g_i(u), \tag{3.23}$$

which is a contradiction to (3.22).
Hence u is an optimal solution of **(P)**. $\square$

3.4 Mixed Type Duality

For the problem **(P)**, we consider the Mixed type dual problem **(MDP)** in the following form

$$(\mathbf{MDP}) \qquad Maximize \quad f(u) + \sum_{i=1}^{m} \xi_i g_i(u)$$

$$subject\ to \quad \mathrm{d}f_u + \sum_{i=1}^{m} \xi_i \mathrm{d}g_{iu} = 0,$$

$$\sum_{i=1}^{m} \xi_i g_i(u) \geq 0, \quad \xi_i \geq 0, \qquad i = 1, 2, ..., m.$$

Let W_3 be the set of all feasible solutions of **(MDP)**.
We have established the following duality results between **(P)** and **(MDP)**, whose proofs are omitted as they are very similar to Theorem 3.6 to Theorem 3.8.

Theorem 3.9. *(Weak Duality). Let x and (u, ξ) be feasible solutions for* **(P)** *and* **(MDP)***, respectively. Moreover, we assume that f and $\sum_{i=1}^{m} \xi_i g_i$ are (p, r)-invex and $(p, -r)$-invex, respectively at u on M with respect to the same η, then* $\inf(\mathbf{P}) \geq \sup(\mathbf{MDP})$.

Theorem 3.10. *(Strong Duality) Let $\bar{x}$ be an optimal solution of the problem* **(P)** *at which a constraint qualification $R(\bar{x})$ be satisfied. Then there exists $\bar{\xi} \in \mathbb{R}^m_+$, such that $(\bar{x}, \xi)$ is a feasible solution of* **(MDP)***. Suppose that the hypotheses of the Weak Duality Theorem 3.9 hold, then $(\bar{x}, \xi)$ is an optimal solution of the dual programming problem* **(MDP)***, and the objective values of* **(P)** *and* **(MDP)** *are equal.*

Theorem 3.11. *(Converse Duality) Let (u, ξ) be an optimal solution of the dual problem* **(MDP)** *such that $u \in D$. If f is (p, r)-invex and $\sum_{i=1}^m \xi_i g_i$ is $(p, -r)$-invex with respect to the same η at u on M. Then u is an optimal solution of* **(P)***.*

4 Conclusions

In this paper, we introduce the notion of (p, r)-invex functions on differentiable manifolds. We establish optimality conditions and duality results under (p, r)-invexity assumptions for a general nonlinear programming problem that is built upon on differentiable manifolds. Variational problems and control problems on differentiable manifolds under generalized invexity will orient future research of the authors.

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