Jet Berwald-Riemann-Lagrange geometrization for affine maps between Finsler manifolds

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Abstract. In this paper we introduce a natural definition for the affine maps between two Finsler manifolds (M, F) and $(N, \widetilde{F})$ and we give some geometrical properties of these affine maps. Starting from the equations of the affine maps, we construct a natural Berwald-Riemann-Lagrange geometry on the 1-jet space $J^1(TM, N)$, in the sense of a Berwald nonlinear connection $\Gamma^{\text{b}}_{\text{jet}}$, a Berwald $\Gamma^{\text{b}}_{\text{jet}}$-linear d-connection $B\Gamma^{\text{b}}_{\text{jet}}$, together with its d-torsions and d-curvatures, which geometrically characterizes the initial affine maps between Finsler manifolds.

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1 Introduction

It is well known that the harmonic maps between Riemannian manifolds are defined as extremals of the energy functional. Because these harmonic maps are very important in differential geometry and mathematical physics, they were intensively studied by Eells and Lemaire [4].

By using the volume element induced on the projective sphere bundle SM of a Finsler manifold (M, F), the harmonic maps between a Finsler manifold and a Riemannian manifold were considered by Mo [7].

Recent studies on Finsler geometry led to the investigation of the nondegenerate harmonic maps between two Finsler manifolds (M, F) and $(N, \widetilde{F})$, as critical points of a natural energy functional on the sphere bundle SM. Thus, Shen and Zhang studied the variation formulas [11] for harmonic maps between Finsler manifolds, and He and Shen established a corresponding generalized Weitzenböck formula [5].

A different general geometrical approach for harmonic maps between two generalized Lagrange spaces $(M, g_{\alpha\beta}(t^\gamma, s^\gamma))$ and $(N, \widetilde{g}_{ij}(x^k, y^k))$ is given by the author of this paper in [8].

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In this work we investigate affine maps between Finsler manifolds, as particular cases of nondegenerate harmonic maps. From a geometrical point of view, we believe that our particular case of nondegenerate harmonic maps (we refer to the affine maps) is not too restrictive one, because we consider that there exist enough interesting geometrical results which characterize the nondegenerate affine maps between Finsler manifolds. In this direction, using the *Riemann-Lagrange geometry on 1-jet spaces* recently developed by the author of this paper in [9] and [10], we will show that the equations of the nondegenerate affine maps between two Finsler manifolds (M, F) and $(N, \widetilde{F})$ produce some natural d-torsions and d-curvatures on the 1-jet space $J^1(TM, N)$, where TM is the tangent bundle of the smooth manifold M.

We would like to point out that the jet Riemann-Lagrange geometrical ideas presented in detail in the works [9] and [10] were initially stated by Asanov in the paper [2].

2 Basic formulas on Finsler manifolds

Let us denote by M a p-dimensional smooth manifold, which induces on its tangent bundle TM the local coordinates (t^α, s^α). Throughout this paper the greek indices $\alpha, \beta, \gamma, \ldots$ run from 1 to p. Let us consider that the manifold M is endowed with a Finsler structure $F : TM \to [0, \infty)$, such that (M, F) is a Finsler manifold.

The *fundamental metrical d-tensor* of the Finsler manifold (M, F) is defined on $TM\backslash\{0\}$ by

$$g_{\alpha\beta}(t^\varepsilon, s^\varepsilon) = \frac{1}{2}\frac{\partial^2 F^2}{\partial s^\alpha \partial s^\beta}.$$

Remark 2.1. Taking into account that F^2 is 2-positive homogenous, we immediately deduce, via the Euler theorem, that we have $F^2 = g_{\alpha\beta}s^\alpha s^\beta$.

The fundamental metrical d-tensor $(g_{\alpha\beta})$ produces the *Cartan d-tensor* of the Finsler manifold (M, F), putting

$$C_{\alpha\beta\gamma} = \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial s^\gamma} = \frac{1}{4}\frac{\partial^3 F^2}{\partial s^\alpha \partial s^\beta \partial s^\gamma}.$$

It is obvious that the Cartan d-tensor is totally symmetric in the indices α, β and γ. Moreover, because $g_{\alpha\beta}$'s are positive homogenous of degree zero, the Euler theorem implies the equalities

$$C_{0\beta\gamma} = C_{\alpha 0\gamma} = C_{\alpha\beta 0} = 0,$$

where by the index 0 we understand the contraction with s^μ. For instance, we have $C_{\alpha\beta 0} = C_{\alpha\beta\mu}s^\mu$.

Let us consider the *formal Christoffel symbols* of the second kind

$$\gamma^\mu_{\alpha\beta} = \frac{g^{\mu\varepsilon}}{2}\left(\frac{\partial g_{\varepsilon\alpha}}{\partial t^\beta} + \frac{\partial g_{\varepsilon\beta}}{\partial t^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial t^\varepsilon}\right), \tag{2.1}$$

where $(g^{\mu\varepsilon})$ denotes the inverse matrix of $(g_{\mu\varepsilon})$. The formal Christoffel symbols produce on $TM\backslash\{0\}$ the *nonlinear Cartan connection* of the Finsler manifold (M, F), taking (see [3, p. 33])

$$N^\beta_\alpha = \gamma^\beta_{\alpha\varepsilon}s^\varepsilon - C^\beta_{\alpha\varepsilon}\gamma^\varepsilon_{\mu\nu}s^\mu s^\nu, \tag{2.2}$$

where $C^{\beta}_{\alpha\varepsilon} = g^{\beta\lambda}C_{\lambda\alpha\varepsilon}$. An important geometrical concept in Finsler geometry is given by the following notion:

Definition 2.2. A curve $c : [a,b] \to M$, locally expressed by $t^\alpha = t^\alpha(t)$, where $t \in [a,b]$, is called an *autoparallel curve of the nonlinear Cartan connection of the Finsler manifold* (M,F) or, briefly, an *autoparallel curve* on (M,F), if and only if

$$\frac{d^2t^\alpha}{dt^2} + N^\alpha_\beta\left(t^\mu(t), \frac{dt^\mu}{dt}\right)\frac{dt^\beta}{dt} = 0, \ \forall\ \alpha = \overline{1,p}. \tag{2.3}$$

Using the nonlinear Cartan connection (N^α_β) we can construct the *generalized Christoffel symbols*

$$\Gamma^\gamma_{\alpha\beta} = \frac{g^{\gamma\mu}}{2}\left(\frac{\delta g_{\mu\alpha}}{\delta t^\beta} + \frac{\delta g_{\mu\beta}}{\delta t^\alpha} - \frac{\delta g_{\alpha\beta}}{\delta t^\mu}\right),$$

where

$$\frac{\delta}{\delta t^\alpha} = \frac{\partial}{\partial t^\alpha} - N^\varepsilon_\alpha \frac{\partial}{\partial s^\varepsilon}.$$

Remark 2.3. The set of d-vector fields

$$\left\{\frac{\delta}{\delta t^\alpha}, \frac{\partial}{\partial s^\alpha}\right\} \subset \mathcal{X}(TM\backslash\{0\})$$

represents a basis in the set of vector fields on $TM\backslash\{0\}$, which is called the *adapted basis* produced by the Finsler structure F. The transformation rules of the elements of the adapted basis are tensorial ones.

From a geometrical point of view, the generalized Christoffel symbols can be regarded as, in Miron and Anastasiei's terminology [6, p. 149], the *adapted components* on $TM\backslash\{0\}$ of the *distinguished linear Rund connection*

$$R\Gamma = \left(N^\beta_\alpha, \Gamma^\gamma_{\alpha\beta}, 0\right),$$

or, in Bao, Chern and Shen's terminology [3, p. 39], as the coefficients of the *Chern connection* on the pulled-back tangent bundle $\pi^*TM \to TM\backslash\{0\}$, where $\pi : TM\backslash\{0\} \to M$ is the canonical projection.

Remark 2.4. (i) For practitioners of Finsler geometry, Anastasiei pointed out in [1] that the *Rund connection* and the *Chern connection* coincide. In a such geometrical context, we underline that in this paper we follow the terminology and the notations used by Miron and Anastasiei in [6].

(ii) It is important to note that, on the Finsler manifold (M,F), the formula

$$N^\beta_\alpha = \Gamma^\beta_{\alpha\gamma}s^\gamma$$

is always true and very useful. Consequently, the *autoparallel curves* of the Finsler manifold (M,F) are the solutions of the system of differential equations

$$\frac{d^2t^\mu}{dt^2} + \Gamma^\mu_{\alpha\beta}\left(t^\varepsilon(t), \frac{dt^\varepsilon}{dt}\right)\frac{dt^\alpha}{dt}\frac{dt^\beta}{dt} = 0, \ \forall\ \mu = \overline{1,p}, \tag{2.4}$$

which is equivalent to the ODEs system of second order

$$\frac{d^2t^\mu}{dt^2} + \gamma^\mu_{\alpha\beta}\left(t^\varepsilon(t), \frac{dt^\varepsilon}{dt}\right)\frac{dt^\alpha}{dt}\frac{dt^\beta}{dt} = 0, \ \forall\ \mu = \overline{1,p}. \tag{2.5}$$

It follows that an *autoparallel curve* $c(t) = (t^\alpha(t))$ having a *natural parameter*, that is $F(c(t), dc/dt) =$ constant, is equivalent with a *constant speed geodesic* on the Finsler manifold (M, F). For more details, see [3, p. 125-128] and [6, p. 132-138].

In the Finsler geometry literature, a famous linear d-connection on the Finsler manifold (M, F) is the *Berwald connection*

$$B\Gamma = \left(N^\beta_\alpha, B^\gamma_{\alpha\beta}, 0\right),$$

whose *adapted components* are defined by

$$B^\gamma_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta} + C^\gamma_{\alpha\beta|\mu}s^\mu = \Gamma^\gamma_{\alpha\beta} + C^\gamma_{\alpha\beta|0}, \tag{2.6}$$

where

$$C^\gamma_{\alpha\beta|\mu} = \frac{\delta C^\gamma_{\alpha\beta}}{\delta t^\mu} + C^\varepsilon_{\alpha\beta}\Gamma^\gamma_{\varepsilon\mu} - C^\gamma_{\varepsilon\beta}\Gamma^\varepsilon_{\alpha\mu} - C^\gamma_{\alpha\varepsilon}\Gamma^\varepsilon_{\beta\mu}$$

represent the *local horizontal covariant derivatives* produced by the Rund connection $R\Gamma$.

Remark 2.5. (i) The local horizontal covariant derivatives produced by the Rund connection $R\Gamma$ behave as differentiations, in the sense that they work by the Leibniz rule upon the tensorial product of d-tensors and they commute with the contractions of d-tensors.

(ii) The following special properties of the local horizontal covariant derivatives associated to the Rund connection $R\Gamma$ hold good (see [6, p. 140]):

$$g_{\alpha\beta|\gamma} = 0, \ s^\alpha{}_{|\gamma} = 0, \ F_{|\gamma} = 0.$$

(iii) If we define the *spray* of the Finsler manifold (M, F) by

$$G^\mu = \frac{1}{2}\gamma^\mu_{\alpha\beta}s^\alpha s^\beta = \frac{1}{2}\Gamma^\mu_{\alpha\beta}s^\alpha s^\beta, \tag{2.7}$$

then, by a direct calculation, we find the following relations:

$$2G^\gamma = N^\gamma_\alpha s^\alpha, \ N^\gamma_\alpha = \frac{\partial G^\gamma}{\partial s^\alpha} \text{ and } B^\gamma_{\alpha\beta} = \frac{\partial N^\gamma_\alpha}{\partial s^\beta} = \frac{\partial^2 G^\gamma}{\partial s^\alpha \partial s^\beta}. \tag{2.8}$$

(iv) The equations of the autoparallel curves of the Finsler manifold (M, F) have the following simple form:

$$\frac{d^2t^\alpha}{dt^2} + 2G^\alpha\left(t^\mu(t), \frac{dt^\mu}{dt}\right) = 0, \ \forall\ \alpha = \overline{1,p}.$$

Moreover, following the geometrical ideas from [6, p. 132-136], it is important to note that, in fact, the autoparallel curves of the Finsler manifold (M, F) coincide exactly with the extremal curves of the integral action of the absolute energy

$$\mathbf{E}(c) = \int_a^b F^2\left(t^\alpha(t), \frac{dt^\alpha}{dt}\right)dt.$$

In this context, taking into account that $F^2 = g_{\alpha\beta}s^\alpha s^\beta$, a direct calculation of the Euler-Lagrange equations of the preceding energy functional leads us to the formula

$$G^\gamma(t^\varepsilon, s^\varepsilon) = \frac{g^{\gamma\mu}}{4}\left[\frac{\partial^2 F^2}{\partial s^\mu \partial t^\nu}s^\nu - \frac{\partial F^2}{\partial t^\mu}\right]. \tag{2.9}$$

From a Finsler geometrical point of view, we point out that the Berwald connection $B\Gamma$ is characterized in the adapted basis $\{\delta/\delta t^\alpha, \partial/\partial s^\alpha\}$ by *one local torsion* d-tensor

$$^{\mathbf{b}}\mathcal{R}^\alpha_{\beta\gamma} = \frac{\delta N^\alpha_\beta}{\delta t^\gamma} - \frac{\delta N^\alpha_\gamma}{\delta t^\beta} \tag{2.10}$$

and *two essential local curvature* d-tensors

$$\begin{aligned} ^{\mathbf{b}}\mathcal{R}^\alpha_{\beta\gamma\varepsilon} &= \frac{\delta B^\alpha_{\beta\gamma}}{\delta t^\varepsilon} - \frac{\delta B^\alpha_{\beta\varepsilon}}{\delta t^\gamma} + B^\mu_{\beta\gamma}B^\alpha_{\mu\varepsilon} - B^\mu_{\beta\varepsilon}B^\alpha_{\mu\gamma} \\ ^{\mathbf{b}}\mathcal{P}^{\ \alpha}_{\beta\gamma\varepsilon} &= \frac{\partial B^\alpha_{\beta\gamma}}{\partial s^\varepsilon}. \end{aligned} \tag{2.11}$$

For more details, the reader is invited to compare the book [6, p. 48, 122, 149] with the book [3, p. 52, 67].

Remark 2.6. Taking into account the relations (2.8), note that the Berwald curvature d-tensor $^{\mathbf{b}}\mathcal{P}^{\ \alpha}_{\beta\gamma\varepsilon}$ is totally symmetric in the indices β, γ and ε.

3 Affine maps between Finsler manifolds

Let (M, F) and $(N, \widetilde{F})$ be two Finsler manifolds, where the dimension of N is n, and let $\varphi : (M, F) \to (N, \widetilde{F})$ be a smooth map which is *nondegenerate*, that is its induced tangent map verifies the condition $Ker(d\varphi) = \{0\}$ (i.e., it is an immersion).

Remark 3.1. (i) We suppose that the tangent bundle TN has the local coordinates (x^i, y^i), where $i = \overline{1,n}$. Moreover, throughout this paper, we suppose that the latin indices $i, j, k, ...$ run from 1 to n.

(ii) On the source Finsler manifold (M, F) we will use the notations and indices from Section 2, and on the target Finsler manifold $(N, \widetilde{F})$ we will denote the same geometrical entities by the same letters, but with tilde and corresponding indices.

Let us suppose that the nondegenerate smooth map φ is locally expressed by $\varphi^i = \varphi^i(t^\mu)$ and let us introduce the notations

$$\varphi^i_\alpha = \frac{\partial \varphi^i}{\partial t^\alpha} \text{ and } \varphi^i_{\alpha\beta} = \frac{\partial^2 \varphi^i}{\partial t^\alpha \partial t^\beta}.$$

In this geometrical context, we introduce the following concept:

Definition 3.2. The nondegenerate smooth map $\varphi : (M, F) \to (N, \widetilde{F})$ is called an *affine map between the Finsler manifolds* (M, F) *and* $(N, \widetilde{F})$ if and only if

$$\varphi^i_{\alpha\beta} - B^\gamma_{\alpha\beta}\varphi^i_\gamma + \widetilde{B}^i_{jk}\varphi^j_\alpha\varphi^k_\beta = 0, \ \forall\ \alpha, \beta = \overline{1,p}, \ \forall\ i = \overline{1,n}, \tag{3.1}$$

where

$$B^{\gamma}_{\alpha\beta} = B^{\gamma}_{\alpha\beta}(t^{\mu}, s^{\mu}) \text{ and } \widetilde{B}^{i}_{jk} = \widetilde{B}^{i}_{jk}(\varphi^{l}(t^{\mu}), \varphi^{l}_{\varepsilon}(t^{\mu})s^{\varepsilon})$$

are the adapted components of the Berwald connection on (M, F) and $(N, \widetilde{F})$, respectively.

Remark 3.3. (i) If the target Finsler manifold $(N, \widetilde{F})$ is a Riemannian one, then we have $\widetilde{C}^{i}_{jk} = 0$ and $\widetilde{B}^{i}_{jk} = \widetilde{\gamma}^{i}_{jk}(\varphi^{l}(t^{\mu}))$. In this case, the assumption on the nondegeneration of φ is not necessary in our definition.

(ii) If (M, F) and $(N, \widetilde{F})$ are both Riemannian manifolds, then we recover the classical definition of the affine maps between two Riemannian manifolds.

In order to give some geometrical examples of affine maps, let us introduce the following definition:

Definition 3.4. A smooth map $\varphi : (M, F) \to (N, \widetilde{F})$ is called an *isometry between the Finsler manifolds* (M, F) *and* $(N, \widetilde{F})$ or, briefly, *Finsler isometry*, if it satisfies the conditions:

(i) φ is a difeomorphism;

(ii) $F(t, s) = \widetilde{F}(\varphi(t), d\varphi(s))$, $\forall\ (t, s) \in TM\backslash\{0\}$.

Remark 3.5. If the smooth map $\varphi : M \to N$ is a difeomorphism, then its induced tangent map $d\varphi_t : T_tM \to T_{\varphi(t)}N$, $\forall\ t \in M$, is an isomorphism of vector spaces. It follows that we have $p = n$ and $\det(\varphi^{i}_{\alpha}) \neq 0$, that is φ is a nondegenerate map.

Theorem 3.1. *Any Finsler isometry* $\varphi : (M, F) \to (N, \widetilde{F})$ *is an affine map between the Finsler manifolds* (M, F) *and* $(N, \widetilde{F})$.

Proof. Let $\varphi : (M, F) \to (N, \widetilde{F})$ be a Finsler isometry and let (ψ^{α}_{i}) be the inverse of the matrix (φ^{i}_{α}). It follows that we have $\psi^{\alpha}_{i}\varphi^{j}_{\alpha} = \delta^{j}_{i}$ and $\psi^{\alpha}_{i}\varphi^{i}_{\beta} = \delta^{\alpha}_{\beta}$.

Because φ is a Finsler isometry, a direct calculation leads us to the relations

$$g_{\alpha\beta} = \widetilde{g}_{ij}\varphi^{i}_{\alpha}\varphi^{j}_{\beta} \text{ and } g^{\alpha\beta} = \widetilde{g}^{ij}\psi^{\alpha}_{i}\psi^{\beta}_{j},\ \forall\ \alpha, \beta = \overline{1, p}. \tag{3.2}$$

Using formula (2.1) and the relations (3.2), a new direct calculation gives us the relations

$$\begin{aligned} \gamma^{\mu}_{\alpha\beta} \quad = \quad & \widetilde{\gamma}^{m}_{ij}\varphi^{i}_{\alpha}\varphi^{j}_{\beta}\psi^{\mu}_{m} + \widetilde{C}^{m}_{ij}\left(\varphi^{i}_{\alpha}\varphi^{j}_{\beta\nu} + \varphi^{i}_{\beta}\varphi^{j}_{\alpha\nu}\right)\psi^{\mu}_{m}s^{\nu} - \\ & -\widetilde{C}_{ijm}g^{\mu\varepsilon}\varphi^{i}_{\alpha}\varphi^{j}_{\beta}\varphi^{m}_{\varepsilon\nu}s^{\nu} + \varphi^{m}_{\alpha\beta}\psi^{\mu}_{m},\ \forall\ \alpha, \beta, \mu = \overline{1, p}. \end{aligned} \tag{3.3}$$

Contracting the relations (3.3) with s^{α} and s^{β}, the formula (2.7) implies the equalities

$$2G^{\gamma} = 2\widetilde{G}^{m}\psi^{\gamma}_{m} + \varphi^{m}_{\alpha\beta}\psi^{\gamma}_{m}s^{\alpha}s^{\beta},\ \forall\ \gamma = \overline{1, p}, \tag{3.4}$$

where, taking into account that we have $\widetilde{C}_{ijk} = \widetilde{C}_{ijk}(\varphi^{l}, \varphi^{l}_{\varepsilon}s^{\varepsilon})$, we used the relation

$$\widetilde{C}_{ijk}\varphi^{k}_{\varepsilon}s^{\varepsilon} = 0.$$

In the sequel, a double differentiation in (3.4), together with the relation (2.8), imply the equalities

$$B^{\gamma}_{\alpha\beta} = \widetilde{B}^{m}_{jk}\varphi^{j}_{\alpha}\varphi^{k}_{\beta}\psi^{\gamma}_{m} + \varphi^{m}_{\alpha\beta}\psi^{\gamma}_{m},\ \forall\ \alpha,\beta,\gamma = \overline{1,p}. \tag{3.5}$$

It is obvious now that the equalities (3.5) imply the equalities (3.1), which represent the equations of the affine maps. In conclusion, φ is an affine map between the Finsler manifolds (M, F) and $(N, \widetilde{F})$. □

Corollary 3.2. *The identity map $\mathcal{I} : (N, \widetilde{F}) \to (N, \widetilde{F})$ is an affine map.*

Now, let us study the affinity of the identity map $\mathcal{I}$ when it works with two different Finsler structures F and $\widetilde{F}$ on the manifold N.

Proposition 3.3. *The identity map $\mathcal{I} : (N, F) \to (N, \widetilde{F})$ is an affine map if and only if*

$$G^{i} = \widetilde{G}^{i},\ \forall\ i = \overline{1,n}, \tag{3.6}$$

where G^i and $\widetilde{G}^i$ are the spray coefficients of the Finsler manifolds (N, F) and $(N, \widetilde{F})$, respectively.

Proof. Let us suppose that we locally have

$$\mathcal{I}^{i} = \mathcal{I}^{i}(x^{k}) = x^{i},\ \forall\ i = \overline{1,n}.$$

Then, it immediately follows that we have $\mathcal{I}^{i}_{j} = \delta^{i}_{j}$ and $\mathcal{I}^{i}_{jk} = 0$. Consequently, $\mathcal{I}$ is an affine map between the Finsler manifolds (N, F) and $(N, \widetilde{F})$ if and only if we have

$$B^{i}_{jk}(x^{l}, y^{l}) = \widetilde{B}^{i}_{jk}(x^{l}, y^{l}) \Leftrightarrow \frac{\partial^{2} G^{i}}{\partial y^{j}\partial y^{k}} = \frac{\partial^{2} \widetilde{G}^{i}}{\partial y^{j}\partial y^{k}},\ \forall\ i,j,k = \overline{1,n}, \tag{3.7}$$

where

$$G^{i} = \frac{1}{2}\gamma^{i}_{pq}y^{p}y^{q} \text{ and } \widetilde{G}^{i} = \frac{1}{2}\widetilde{\gamma}^{i}_{pq}y^{p}y^{q}.$$

Taking into account that the spray coefficients are 2-positive homogenous, by contractions with y^j and y^k, the equalities (3.7) imply the equalities (3.6). Conversely, it is obvious that the equalities (3.6) imply the equalities (3.7). In conclusion, we obtain what we were looking for. □

Corollary 3.4. *The identity map $\mathcal{I} : (N, F) \to (N, \widetilde{F})$ is an affine map if and only if*

$$g^{ip}\left[\frac{\partial^{2} F^{2}}{\partial y^{p}\partial x^{q}}y^{q} - \frac{\partial F^{2}}{\partial x^{p}}\right] = \widetilde{g}^{ip}\left[\frac{\partial^{2} \widetilde{F}^{2}}{\partial y^{p}\partial x^{q}}y^{q} - \frac{\partial \widetilde{F}^{2}}{\partial x^{p}}\right],\ \forall\ i = \overline{1,n}.$$

Proof. The Corollary is an immediate consequence of Proposition 3.3 and formula (2.9). □

Corollary 3.5. *If (N, h) is a flat Riemannian manifold and $(N, \widetilde{F})$ is a locally Minkowski manifold, then the identity maps*

$$\mathcal{I} : (N, h) \to (N, \widetilde{F})$$

and

$$\mathcal{I} : (N, \widetilde{F}) \to (N, h)$$

are affine maps.

Proof. If $\widetilde{F}$ is a local Minkowski structure, then there exists a system of local coordinates (x^i) such that $\widetilde{F}(x, y) = \widetilde{F}(y)$, that is $\widetilde{g}_{ij}(x, y) = \widetilde{g}_{ij}(y)$. It follows that we have $\widetilde{\gamma}^k_{ij} = 0$, that is $\widetilde{G}^k = 0$, $\forall\ k = \overline{1, n}$.

On the other hand, if the Riemannian metric h is flat, then we also have $\gamma^k_{ij} = 0$, that is $G^k = 0$, $\forall\ k = \overline{1, n}$. □

4 Geometrical properties of the affine maps between Finsler manifolds

In our geometrical context, let us consider the particular case when our source Finsler manifold is the Euclidian manifold

$$(M, F) = (\mathbb{R}, F(t, s) = |s|).$$

Then, we can prove the following result:

Proposition 4.1. *Any affine map $c : (\mathbb{R}, F) \to (N, \widetilde{F})$ is an autoparallel curve of the Finsler manifold $(N, \widetilde{F})$.*

Proof. If $M = \mathbb{R}$ is regarded as the Euclidian manifold $(\mathbb{R}, 1)$, then the equations (3.1) of the affine maps become

$$\frac{d^2c^i}{dt^2} + \widetilde{B}^i_{jk}\left(c^l(t), \frac{dc^l}{dt}s\right)\frac{dc^j}{dt}\frac{dc^k}{dt} = 0,\ \forall\ i = \overline{1, n},\ \forall\ s \in \mathbb{R}^*, \tag{4.1}$$

where $c(t) = (c^i(t))$ is an affine map. Taking into account that $\widetilde{B}^i_{jk}$'s are 0-positive homogenous, we deduce from equations (4.1) that an affine map $c(t)$ must verify the equations

$$\frac{d^2c^i}{dt^2} + \widetilde{B}^i_{jk}\left(c^l(t), \frac{dc^l}{dt}\right)\frac{dc^j}{dt}\frac{dc^k}{dt} = 0,\ \forall\ i = \overline{1, n}. \tag{4.2}$$

Now, using the formula (2.6) and the fact that $\widetilde{C}^i_{jk|0}(dc^j/dt) = 0$, it follows that the equations (4.2) become exactly the equations (2.4) of the autoparallel curves of the Finsler manifold $(N, \widetilde{F})$. □

In order to obtain a geometrical result which characterizes the affine maps between Finsler manifolds, let us prove the following helpful statement:

Lemma 4.2. *Let $u_{\alpha\beta} : TM\backslash\{0\} \to \mathbb{R}$, where $\alpha, \beta = \overline{1,p}$, be a family of smooth maps which have the following four properties:*

(i) $u_{\alpha\beta} = u_{\beta\alpha}$;

(ii) $u_{\alpha\beta}$*'s are* 0*-positive homogenous;*

(iii) $\dfrac{\partial u_{\alpha\beta}}{\partial s^\gamma}$ *is totally symmetric in* α, β, γ;

(iv) $u_{\alpha\beta} s^\alpha s^\beta = 0$.

Then, we have

$$u_{\alpha\beta} = 0, \ \forall\ \alpha, \beta = \overline{1,p}.$$

Proof. Differentiating (iv) with respect to s^γ and using (i), we obtain the equalities

$$\frac{\partial u_{\alpha\beta}}{\partial s^\gamma} s^\alpha s^\beta + 2u_{\alpha\gamma}s^\alpha = 0, \ \forall\ \gamma = \overline{1,p}. \tag{4.3}$$

Using now (iii) and (ii), we immediately deduce that

$$\frac{\partial u_{\alpha\beta}}{\partial s^\gamma} s^\alpha s^\beta = \frac{\partial u_{\beta\gamma}}{\partial s^\alpha} s^\alpha s^\beta = 0, \ \forall\ \gamma = \overline{1,p}.$$

Consequently, the equalities (4.3) become

$$u_{\alpha\gamma}s^\alpha = 0, \ \forall\ \gamma = \overline{1,p}. \tag{4.4}$$

Applying the same procedure to the equalities (4.4), we find that

$$u_{\gamma\varepsilon} = 0, \ \forall\ \gamma, \varepsilon = \overline{1,p}.$$

□

Theorem 4.3 (characterization of affine maps). *A nondegenerate mapping $\varphi : (M, F) \to (N, \widetilde{F})$ is an affine map between the Finsler manifolds (M, F) and $(N, \widetilde{F})$ if and only if the map φ carries autoparallel curves from (M, F) into autoparallel curves on $(N, \widetilde{F})$.*

Proof. Let $c(t) = (t^\alpha(t))$ be an autoparallel curve on the Finsler manifold (M, F), that is it verifies the equations (2.4).

Let us consider that the curve $\widetilde{c}(t) = (\varphi \circ c)(t)$ is locally expressed by the components $x^i(t) = \varphi^i(t^\alpha(t))$. Then, differentiating by t, we immediately find for each $i \in \{1, ..., n\}$ the equalities

$$\frac{dx^i}{dt} = \varphi^i_\alpha \frac{dt^\alpha}{dt} \text{ and } \frac{d^2x^i}{dt^2} = \varphi^i_{\alpha\beta}\frac{dt^\alpha}{dt}\frac{dt^\beta}{dt} + \varphi^i_\mu \frac{d^2t^\mu}{dt^2}. \tag{4.5}$$

Using the equalities (2.4), it follows that we have

$$\begin{aligned} \frac{d^2x^i}{dt^2} + \widetilde{\Gamma}^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} &= \varphi^i_\mu \frac{d^2t^\mu}{dt^2} + \left(\varphi^i_{\alpha\beta} + \widetilde{\Gamma}^i_{jk}\varphi^j_\alpha\varphi^k_\beta\right)\frac{dt^\alpha}{dt}\frac{dt^\beta}{dt} = \\ &= \left(\varphi^i_{\alpha\beta} - \Gamma^\gamma_{\alpha\beta}\varphi^i_\gamma + \widetilde{\Gamma}^i_{jk}\varphi^j_\alpha\varphi^k_\beta\right)\frac{dt^\alpha}{dt}\frac{dt^\beta}{dt}. \end{aligned} \tag{4.6}$$

" $\Longrightarrow$ " If φ is an affine map, then it verifies the equations

$$\varphi^i_{\alpha\beta} - \left(\Gamma^\gamma_{\alpha\beta} + C^\gamma_{\alpha\beta|0}\right)\varphi^i_\gamma + \left(\widetilde{\Gamma}^i_{jk} + \widetilde{C}^i_{jk|0}\right)\varphi^j_\alpha\varphi^k_\beta = 0, \tag{4.7}$$

for any $\alpha, \beta = \overline{1,p}$ and $i = \overline{1,n}$. Contracting the equations (4.7) with dt^α/dt and dt^β/dt and taking into account the relations

$$C^\gamma_{\alpha\beta|0}\frac{dt^\alpha}{dt}\frac{dt^\beta}{dt} = 0,\ \forall\ \gamma = \overline{1,p}, \tag{4.8}$$

and

$$\widetilde{C}^i_{jk|0}\varphi^j_\alpha\varphi^k_\beta\frac{dt^\alpha}{dt}\frac{dt^\beta}{dt} = 0,\ \forall\ i = \overline{1,n}, \tag{4.9}$$

we deduce from equalities (4.6) that we have

$$\frac{d^2x^i}{dt^2} + \widetilde{\Gamma}^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0,\ \forall\ i = \overline{1,n}.$$

This is exactly what we were looking for.

" $\Longleftarrow$ " Conversely, if φ carries autoparallel curves from (M, F) into autoparallel curves on $(N, \widetilde{F})$, then the equalities (4.6) imply the relations

$$\left(\varphi^i_{\alpha\beta} - \Gamma^\gamma_{\alpha\beta}\varphi^i_\gamma + \widetilde{\Gamma}^i_{jk}\varphi^j_\alpha\varphi^k_\beta\right)\frac{dt^\alpha}{dt}\frac{dt^\beta}{dt} = 0,\ \forall\ i = \overline{1,n},$$

for an arbitrary d-tensor field $s^\alpha = dt^\alpha/dt$. Obviously, the relations (4.8) and (4.9) lead us to the equalities

$$\left[\varphi^i_{\alpha\beta} - \left(\Gamma^\gamma_{\alpha\beta} + C^\gamma_{\alpha\beta|0}\right)\varphi^i_\gamma + \left(\widetilde{\Gamma}^i_{jk} + \widetilde{C}^i_{jk|0}\right)\varphi^j_\alpha\varphi^k_\beta\right]s^\alpha s^\beta = 0, \tag{4.10}$$

for any $i = \overline{1,n}$. Denoting now the square parentheses from the left side of the equalities (4.10) with $u^i_{\alpha\beta}$, via the relations (2.6) and (2.8), we remark that we can apply the Lemma 4.2 to $u^i_{\alpha\beta}$, for any $i = \overline{1,n}$. In conclusion, we obtain

$$u^i_{\alpha\beta} = 0,\ \forall\ \alpha, \beta = \overline{1,p},\ \forall\ i = \overline{1,n},$$

that is the nondegenerate map φ is an affine map between the Finsler manifolds (M, F) and $(N, \widetilde{F})$. □

In the sequel, we will show that the affine maps are only particular cases of *harmonic maps between Finsler manifolds*, studied by Mo [7] or Shen and Zhang [11].

Proposition 4.4. *Supposing that M is a compact oriented smooth manifold without boundary, then any affine map $\varphi : (M, F) \to (N, \widetilde{F})$ is a harmonic map between the Finsler manifolds (M, F) and $(N, \widetilde{F})$, with vanishing tension field.*

Proof. Following the geometrical ideas developed in [11], a particular case of harmonic maps between the Finsler manifolds (M, F) and $(N, \widetilde{F})$ is when the tension field of the nondegenerate smooth map φ vanishes identically. But, the *tension field* of φ is given by the components (see [11, p. 45, (2.25)])

$$\begin{aligned}(4.11)\qquad \tau^i(\varphi) &= g^{\alpha\beta}\left\{\varphi^i_{\alpha\beta} - B^\gamma_{\alpha\beta}\varphi^i_\gamma + \widetilde{B}^i_{jk}\varphi^j_\alpha\varphi^k_\beta\right\} + \\ &+4g^{\alpha\beta}\widetilde{C}^i_{jk}\varphi^k_\alpha\left\{\varphi^j_{\beta\gamma} - \Gamma^\mu_{\beta\gamma}\varphi^j_\mu + \widetilde{\Gamma}^j_{pq}\varphi^p_\beta\varphi^q_\gamma\right\}s^\gamma + \\ &+g^{\alpha\beta}\widetilde{C}^i_{jkl}\varphi^k_\alpha\varphi^l_\beta\left\{\varphi^j_{\gamma\varepsilon} - \Gamma^\mu_{\gamma\varepsilon}\varphi^j_\mu + \widetilde{\Gamma}^j_{pq}\varphi^p_\gamma\varphi^q_\varepsilon\right\}s^\gamma s^\varepsilon,\end{aligned}$$

where

$$\widetilde{C}^i_{jkl}(x,y) = \widetilde{g}^{im}\frac{\partial\widetilde{C}_{jkl}}{\partial y^m}$$

and, we underline that, in the expression (4.11), the Finsler geometrical entities on $(N,\widetilde{F})$ are computed in

$$(x^l, y^l) = (\varphi^l(t^\mu), \varphi^l_\mu s^\mu) \in TN\backslash\{0\}.$$

Because we have

$$C^\alpha_{\beta\gamma|0}s^\gamma = \widetilde{C}^i_{jk|0}y^k = 0,$$

the formula (2.6) implies that the components (4.11) of the tension field of the nondegenerate smooth map φ take the simpler form

$$(4.12)\qquad \tau^i(\varphi) = g^{\alpha\beta}\tau^i_{\alpha\beta} + 4g^{\alpha\beta}\widetilde{C}^i_{jk}\varphi^k_\alpha\tau^j_{\beta\gamma}s^\gamma + g^{\alpha\beta}\widetilde{C}^i_{jkl}\varphi^k_\alpha\varphi^l_\beta\tau^j_{\gamma\varepsilon}s^\gamma s^\varepsilon,$$

where

$$\tau^i_{\alpha\beta} = \varphi^i_{\alpha\beta} - B^\gamma_{\alpha\beta}\varphi^i_\gamma + \widetilde{B}^i_{jk}\varphi^j_\alpha\varphi^k_\beta,\ \forall\ \alpha,\beta = \overline{1,p},\ \forall\ i = \overline{1,n}.$$

It is obvious now that if the nondegenerate smooth map φ is an affine map, then its tension field vanishes identically. In conclusion, we obtained what we were looking for. □

Remark 4.1. In Proposition 4.4 the assumptions upon the source manifold M were imposed by the good definition of the harmonic maps between two Finsler manifolds (see [11]). We point out yet that our definition of the affine maps between Finsler manifolds did not require these assumptions.

Corollary 4.5. *Under the assumptions of Proposition 4.4, any nondegenerate smooth map* $\varphi : (M,F) \to (N,\widetilde{F})$*, which carries autoparallel curves from* (M,F) *into autoparallel curves on* $(N,\widetilde{F})$*, is a harmonic map between the Finsler manifolds* (M,F) *and* $(N,\widetilde{F})$*, with vanishing tension field.*

5 Berwald-Riemann-Lagrange geometry on the 1-jet space $J^1(TM,N)$, produced by the equations of the affine maps between two Finsler manifolds

The aim of this Section is to associate to the affine maps equations (3.1) some geometrical objects which could characterize the affine maps between two Finsler manifolds.

Taking into account that starting from the Asanov's geometrical ideas [2] one has recently elaborated a *Riemann-Lagrange geometry on 1-jet spaces*, in the sense of d-connections, d-torsions and d-curvatures (see [9]), our geometrical construction is made on the 1-jet space $J^1(TM, N)$. For a more clear exposition of our jet geometrical ideas, it is important as the tangent bundle TM to be regarded as having the local coordinates (t^α, s^a), where the indices $\alpha, \beta, \gamma, ...$ and $a, b, c, ...$ have the same range: $1, 2, ..., p$.

Remark 5.1. (i) In our geometrical context, it is more convenient to have two kinds of indices, in order to mark the distinct elements of the *adapted basis*

$$\left\{\frac{\delta}{\delta t^\alpha} = \frac{\partial}{\partial t^\alpha} - N^b_\alpha \frac{\partial}{\partial s^b}, \frac{\partial}{\partial s^a}\right\} \subset \mathcal{X}(TM\backslash\{0\}) \tag{5.1}$$

provided by the canonical nonlinear Cartan connection (2.2) of the Finsler structure $F : TM \to \mathbb{R}_+$ on the smooth manifold M.

(ii) Note that we will use the formal notation $A = (\alpha, a)$ for an index which run two times from 1 to p, namely, firstly by α, corresponding to the coordinates t^α or, equivalent, to the *horizontal d-vector fields* $\delta/\delta t^\alpha$, and after that by a, corresponding to the coordinates s^a or, equivalent, to the *vertical d-vector fields* $\partial/\partial s^a$. Moreover, throughout this paper, the capital latin letters $A, B, C, ...$ will denote indices like the previous one.

Using the notation $\left(T^A\right) = (t^\alpha, s^a)$ for the local coordinates on the tangent bundle TM, we recall that the coordinates on the 1-jet space $J^1(TM, N)$ are $\left(T^A, x^i, X^i_A\right)$, where the coordinates $\left(X^i_A\right) = \left(x^i_\alpha, y^i_a\right)$ have the meaning of the *partial derivatives* of the functions x^i with respect to t^α (these are denoted by x^i_α) and with respect to s^a (these are denoted by y^i_a), respectively. The coordinate transformation rules on $J^1(TM, N)$ are given by the general transformation laws [9]

$$\begin{cases} \overline{T}^A = \overline{T}^A\left(T^B\right) \\ \overline{x}^i = \overline{x}^i\left(x^j\right) \\ \overline{X}^i_A = \dfrac{\partial \overline{x}^i}{\partial x^j}\dfrac{\partial T^B}{\partial \overline{T}^A} X^j_B. \end{cases}$$

Consequently, direct local computations say us that the local coordinates on $J^1(TM, N)$ are $\left(t^\alpha, s^a, x^i, x^i_\alpha, y^i_a\right)$ and that they transform by the rules

$$\begin{cases} \overline{t}^\alpha = \overline{t}^\alpha\left(t^\beta\right) \\ \overline{s}^a = \dfrac{\partial \overline{t}^a}{\partial t^b} s^b \\ \overline{x}^i = \overline{x}^i\left(x^j\right) \\ \overline{x}^i_\alpha = \dfrac{\partial \overline{x}^i}{\partial x^j}\dfrac{\partial t^\beta}{\partial \overline{t}^\alpha} x^j_\beta + \dfrac{\partial \overline{x}^i}{\partial x^j}\dfrac{\partial^2 t^b}{\partial \overline{t}^\alpha \partial \overline{t}^c}\dfrac{\partial \overline{t}^c}{\partial t^d} s^d y^j_b \\ \overline{y}^i_a = \dfrac{\partial \overline{x}^i}{\partial x^j}\dfrac{\partial t^b}{\partial \overline{t}^a} y^j_b. \end{cases} \tag{5.2}$$

Firstly, let us suppose that

$$\left(\Gamma^C_{AB}\right) = \left({}^1\Gamma^\gamma_{\alpha\beta},\ {}^2\Gamma^c_{\alpha\beta},\ {}^3\Gamma^\gamma_{a\beta},\ {}^4\Gamma^c_{a\beta},\ {}^5\Gamma^\gamma_{\alpha b},\ {}^6\Gamma^c_{\alpha b},\ {}^7\Gamma^\gamma_{ab},\ {}^8\Gamma^c_{ab}\right)$$

are the *normal components* of a linear connection ∇ on $TM\backslash\{0\}$, in the sense that we have

$$\nabla_{\frac{\partial}{\partial T^B}} \frac{\partial}{\partial T^A} = \Gamma^C_{AB}\frac{\partial}{\partial T^C},$$

that is

$$\nabla_{\frac{\partial}{\partial t^\beta}} \frac{\partial}{\partial t^\alpha} = {}^1\Gamma^\gamma_{\alpha\beta}\frac{\partial}{\partial t^\gamma} + {}^2\Gamma^c_{\alpha\beta}\frac{\partial}{\partial s^c}, \quad \nabla_{\frac{\partial}{\partial t^\beta}} \frac{\partial}{\partial s^a} = {}^3\Gamma^\gamma_{a\beta}\frac{\partial}{\partial t^\gamma} + {}^4\Gamma^c_{a\beta}\frac{\partial}{\partial s^c},$$
$$\nabla_{\frac{\partial}{\partial s^b}} \frac{\partial}{\partial t^\alpha} = {}^5\Gamma^\gamma_{\alpha b}\frac{\partial}{\partial t^\gamma} + {}^6\Gamma^c_{\alpha b}\frac{\partial}{\partial s^c}, \quad \nabla_{\frac{\partial}{\partial s^b}} \frac{\partial}{\partial s^a} = {}^7\Gamma^\gamma_{ab}\frac{\partial}{\partial t^\gamma} + {}^8\Gamma^c_{ab}\frac{\partial}{\partial s^c}.$$

Then, following the geometrical ideas from [9, p. 25], we underline that the components

$$M^{(j)}_{(B)A} = -\Gamma^C_{AB}X^j_C, \tag{5.3}$$

where

$$X^j_C = \left(x^j_\gamma, y^j_c\right),$$

represent a *temporal nonlinear connection* on the 1-jet space $J^1(TM, N)$, *naturally attached to the linear connection* ∇ *from* $TM\backslash\{0\}$.

In this geometrical context, we point out that the Berwald adapted components $B^\gamma_{\alpha\beta} = B^\gamma_{\alpha\beta}(t^\mu, s^a)$ define a linear d-connection ${}^{\mathbf{b}}\nabla$ on $TM\backslash\{0\}$, given in the adapted basis (5.1) by the relations

$$\begin{aligned} &{}^{\mathbf{b}}\nabla_{\frac{\delta}{\delta t^\beta}} \frac{\delta}{\delta t^\alpha} = B^\gamma_{\alpha\beta}\frac{\delta}{\delta t^\gamma}, \quad &&{}^{\mathbf{b}}\nabla_{\frac{\delta}{\delta t^\beta}} \frac{\partial}{\partial s^a} = B^c_{a\beta}\frac{\partial}{\partial s^c},\\ &{}^{\mathbf{b}}\nabla_{\frac{\partial}{\partial s^b}} \frac{\delta}{\delta t^\alpha} = 0, &&{}^{\mathbf{b}}\nabla_{\frac{\partial}{\partial s^b}} \frac{\partial}{\partial s^a} = 0. \end{aligned} \tag{5.4}$$

Remark 5.2. The adapted torsion and curvature d-tensors of the Berwald linear d-connection ${}^{\mathbf{b}}\nabla$ on $TM\backslash\{0\}$ are given by the formulas (2.10) and (2.11).

Taking into account the relations (5.4), together with the equality

$$\frac{\partial}{\partial t^\alpha} = \frac{\delta}{\delta t^\alpha} + N^b_\alpha\frac{\partial}{\partial s^b},$$

by direct computations, we deduce that the Berwald d-connection ${}^{\mathbf{b}}\nabla$ on $TM\backslash\{0\}$ has, in the *normal basis*

$$\left\{\frac{\partial}{\partial t^\alpha}, \frac{\partial}{\partial s^a}\right\} \subset \mathcal{X}(TM\backslash\{0\}),$$

the *normal components* $\left({}^{\mathbf{b}}\Gamma^{C}_{AB}\right) =$

$$
\begin{aligned}
&{}^{1\mathbf{b}}\Gamma^{\gamma}_{\alpha\beta} = B^{\gamma}_{\alpha\beta},\ {}^{2\mathbf{b}}\Gamma^{c}_{\alpha\beta} = N^{c}_{\alpha:\beta},\ {}^{3\mathbf{b}}\Gamma^{\gamma}_{a\beta} = 0,\ {}^{4\mathbf{b}}\Gamma^{c}_{a\beta} = B^{c}_{a\beta},\\
&{}^{5\mathbf{b}}\Gamma^{\gamma}_{\alpha b} = 0,\ {}^{6\mathbf{b}}\Gamma^{c}_{\alpha b} = B^{c}_{\alpha b},\ {}^{7\mathbf{b}}\Gamma^{\gamma}_{ab} = 0,\ {}^{8\mathbf{b}}\Gamma^{c}_{ab} = 0,
\end{aligned}
\tag{5.5}
$$

where

$$N^{c}_{\alpha:\beta} = \frac{\partial N^{c}_{\alpha}}{\partial t^{\beta}} + N^{d}_{\alpha}B^{c}_{d\beta} - N^{c}_{\gamma}B^{\gamma}_{\alpha\beta}.$$

As a consequence, via the formula (5.3) applied to the Berwald linear d-connection ${}^{\mathbf{b}}\nabla$ from $TM\backslash\{0\}$, we find the following geometrical result and concept:

Definition 5.3. The set of local functions

$$\left({}^{\mathbf{b}}M^{(j)}_{(B)A}\right) = \left({}^{1\mathbf{b}}M^{(j)}_{(\beta)\alpha},\ {}^{2\mathbf{b}}M^{(j)}_{(b)\alpha},\ {}^{3\mathbf{b}}M^{(j)}_{(\beta)a},\ {}^{4\mathbf{b}}M^{(j)}_{(b)a}\right),$$

where

$$
\begin{aligned}
&{}^{1\mathbf{b}}M^{(j)}_{(\beta)\alpha} = -B^{\gamma}_{\alpha\beta}x^{j}_{\gamma} - N^{c}_{\alpha:\beta}y^{j}_{c}, \quad &&{}^{2\mathbf{b}}M^{(j)}_{(b)\alpha} = -B^{c}_{\alpha b}y^{j}_{c}\\
&{}^{3\mathbf{b}}M^{(j)}_{(\beta)a} = -B^{c}_{a\beta}y^{j}_{c}, &&{}^{4\mathbf{b}}M^{(j)}_{(b)a} = 0,
\end{aligned}
\tag{5.6}
$$

represents a temporal nonlinear connection on $J^1(TM,N)$, which may be called the *Berwald temporal nonlinear connection* on $J^1(TM,N)$.

Secondly, let us consider that $\varphi : (M,F) \to (N,\widetilde{F})$ is an affine map between the Finsler manifolds (M,F) and $(N,\widetilde{F})$. Then, it is important to note that, under a change of coordinates on M and N, the behaviour on $J^1(TM,N)$ of the coordinates $\left(y^i_a\right)$ (see the relations (5.2)) is the same with that of the components $\left(\varphi^i_\alpha\right)$. Consequently, by a convenient jet extension, the Berwald adapted components

$$\widetilde{B}^{i}_{jk} = \widetilde{B}^{i}_{jk}(\varphi^l, \varphi^l_\varepsilon s^\varepsilon),$$

which appear in the equations of the affine maps (3.1), can be well represented on the 1-jet space $J^1(TM,N)$ by the geometrical objects

$$\widetilde{B}^{i}_{jk} = \widetilde{B}^{i}_{jk}(x^l, y^l_a s^a),$$

whose transformation rules on $J^1(TM,N)$ are (for more details, please consult [6, p. 122] or [3, p. 43])

$$\overline{\widetilde{B}}^{q}_{rs} = \widetilde{B}^{i}_{jk}\frac{\partial x^j}{\partial \overline{x}^r}\frac{\partial x^k}{\partial \overline{x}^s}\frac{\partial \overline{x}^q}{\partial x^i} + \frac{\partial^2 x^i}{\partial \overline{x}^r \partial \overline{x}^s}\frac{\partial \overline{x}^q}{\partial x^i}. \tag{5.7}$$

Following the jet geometrical ideas from [9] or [10], it immediately follows that the components [9, p. 25]

$${}^{\mathbf{b}}N^{(j)}_{(B)i} = \widetilde{B}^{j}_{ik}X^{k}_{B}, \tag{5.8}$$

where $X^k_B = \left(x^k_\beta, y^k_b\right)$, represent a *spatial nonlinear connection* on the 1-jet space $J^1(TM,N)$. In conclusion, putting $B=\beta$ and $B=b$, respectively, into the formula (5.8), we can enunciate the following geometrical result and concept:

Definition 5.4. The set of local functions

$$\left({}^{\mathbf{b}}N^{(j)}_{(B)i} \right) = \left({}^{\mathbf{1b}}N^{(j)}_{(\beta)i},\ {}^{\mathbf{2b}}N^{(j)}_{(b)i} \right),$$

where

$$ {}^{\mathbf{1b}}N^{(j)}_{(\beta)i} = \widetilde{B}^{j}_{ik} x^{k}_{\beta}, \quad {}^{\mathbf{2b}}N^{(j)}_{(b)i} = \widetilde{B}^{j}_{ik} y^{k}_{b}, \tag{5.9}$$

represents a spatial nonlinear connection on $J^1(TM, N)$, which may be called the *Berwald spatial nonlinear connection* on $J^1(TM, N)$.

Remark 5.5. The set of local functions

$$\Gamma^{\mathbf{b}}_{\text{jet}} = \left({}^{\mathbf{b}}M^{(j)}_{(B)A},\ {}^{\mathbf{b}}N^{(j)}_{(B)i} \right)$$

is a nonlinear connection on $J^1(TM, N)$, which may be called the *jet Berwald nonlinear connection* on $J^1(TM, N)$.

The Berwald nonlinear connection $\Gamma^{\mathbf{b}}_{jet}$, whose local components are given by (5.6) and (5.9), produces the *jet adapted basis* [9, p. 24]

$$\left\{ \frac{\delta^{\mathbf{J}}}{\delta T^A}, \frac{\delta^{\mathbf{J}}}{\delta x^i}, \frac{\partial}{\partial X^i_A} \right\} \subset \mathcal{X}(J^1(TM, N)),$$

where

$$\frac{\delta^{\mathbf{J}}}{\delta T^A} = \frac{\partial}{\partial T^A} - {}^{\mathbf{b}}M^{(j)}_{(B)A} \frac{\partial}{\partial X^j_B}, \quad \frac{\delta^{\mathbf{J}}}{\delta x^i} = \frac{\partial}{\partial x^i} - {}^{\mathbf{b}}N^{(j)}_{(B)i} \frac{\partial}{\partial X^j_B}.$$

Taking into account that $A = (\alpha, a)$ and using the formulas (5.6) and (5.9), it is easy to deduce

Proposition 5.1. *The elements of the* jet adapted basis *are*

$$\left\{ \frac{\delta^{\mathbf{J}}}{\delta t^\alpha}, \frac{\delta^{\mathbf{J}}}{\delta s^a}, \frac{\delta^{\mathbf{J}}}{\delta x^i}, \frac{\partial}{\partial x^i_\alpha}, \frac{\partial}{\partial y^i_a} \right\} \subset \mathcal{X}(J^1(TM, N)),$$

where

$$\begin{aligned}
\frac{\delta^{\mathbf{J}}}{\delta t^\alpha} &= \frac{\partial}{\partial t^\alpha} + \left(B^{\gamma}_{\alpha\beta} x^j_\gamma + N^{c}_{\alpha:\beta} y^j_c \right) \frac{\partial}{\partial x^j_\beta} + B^{c}_{\alpha b} y^j_c \frac{\partial}{\partial y^j_b}, \\
\frac{\delta^{\mathbf{J}}}{\delta s^a} &= \frac{\partial}{\partial s^a} + B^{c}_{a\beta} y^j_c \frac{\partial}{\partial x^j_\beta}, \\
\frac{\delta^{\mathbf{J}}}{\delta x^i} &= \frac{\partial}{\partial x^i} - \widetilde{B}^{j}_{ik} x^k_\beta \frac{\partial}{\partial x^j_\beta} - \widetilde{B}^{j}_{ik} y^k_b \frac{\partial}{\partial y^j_b}.
\end{aligned} \tag{5.10}$$

Following again the jet geometrical ideas exposed in [9] and [10], the Berwald linear d-connection ${}^{\mathbf{b}}\nabla$ from $TM \backslash \{0\}$, together with the Berwald linear d-connection ${}^{\mathbf{b}}\widetilde{\nabla}$ from $TN \backslash \{0\}$, produce a *jet Berwald linear d-connection* $B\Gamma^{\mathbf{b}}_{jet}$ on $J^1(TM, N)$, taking as its *jet adapted components* the following coefficients [9, p. 30]:

$$B\Gamma^{\mathbf{b}}_{jet} = \Big(\overline{G}^{A}_{BC} = {}^{\mathbf{b}}\Gamma^{A}_{BC},\ G^{k}_{iC} = 0,\ G^{(i)(B)}_{(A)(j)C} = -\delta^{i}_{j}\ {}^{\mathbf{b}}\Gamma^{B}_{CA},$$
$$\overline{L}^{A}_{Bj} = 0,\ L^{k}_{ij} = \widetilde{B}^{k}_{ij},\ L^{(i)(B)}_{(A)(j)k} = \delta^{B}_{A}\widetilde{B}^{i}_{jk},$$
$$\overline{C}^{A(C)}_{B(k)} = 0,\ C^{j(C)}_{i(k)} = 0,\ C^{(i)(B)(C)}_{(A)(j)(k)} = 0 \Big),$$

where

$$\delta^{B}_{A} = \begin{cases} \delta^{\beta}_{\alpha}, & \text{if } A = \alpha,\ B = \beta \\ \delta^{b}_{a}, & \text{if } A = a,\ B = b \\ 0, & \text{otherwise.} \end{cases}$$

Remark 5.6. The jet *Berwald linear d-connection* $B\Gamma^{\mathbf{b}}_{jet}$ is a $\Gamma^{\mathbf{b}}_{jet}$-linear connection on $J^1(TM,N)$. For more details, please consult [9, p. 28] or [10].

Consequently, using the relations (5.5) and taking into account that the indices $A, B, C, ...$ have the form $(\alpha, a), (\beta, b), (\gamma, c), ...$, we obtain

Proposition 5.2. *The essential adapted components of the* jet Berwald linear d-connection $B\Gamma^{\mathbf{b}}_{\text{jet}}$ *are only the following* eleven *components:*

$$(5.11)\quad \begin{aligned} B\Gamma^{\mathbf{b}}_{\text{jet}} = \Big(& \overline{G}^{\alpha}_{\beta\gamma} = B^{\alpha}_{\beta\gamma},\ \overline{G}^{a}_{\beta\gamma} = N^{a}_{\beta:\gamma},\ \overline{G}^{a}_{b\gamma} = B^{a}_{b\gamma},\ \overline{G}^{a}_{\beta c} = B^{a}_{\beta c}, \\ & G^{(i)(\beta)}_{(\alpha)(j)\gamma} = -\delta^{i}_{j}B^{\beta}_{\gamma\alpha},\ G^{(i)(b)}_{(\alpha)(j)\gamma} = -\delta^{i}_{j}N^{b}_{\gamma:\alpha}, \\ & G^{(i)(b)}_{(\alpha)(j)c} = -\delta^{i}_{j}B^{b}_{c\alpha},\ G^{(i)(b)}_{(a)(j)\gamma} = -\delta^{i}_{j}B^{b}_{\gamma a},\ L^{k}_{ij} = \widetilde{B}^{k}_{ij}, \\ & L^{(i)(\beta)}_{(\alpha)(j)k} = \delta^{\beta}_{\alpha}\widetilde{B}^{i}_{jk},\ L^{(i)(b)}_{(a)(j)k} = \delta^{b}_{a}\widetilde{B}^{i}_{jk} \Big). \end{aligned}$$

Because the Riemann-Lagrange geometry of the general Γ-linear connections on 1-jet spaces, in the sense of their d-torsions and d-curvatures, is now completely done in [9] and [10], it follows that we can compute on $J^1(TM,N)$ the adapted components of the torsion and curvature d-tensors produced by the Berwald linear d-connection $B\Gamma^{\mathbf{b}}_{jet}$. In a such jet Riemann-Lagrange geometrical context, using the formulas (2.10) and (2.11), we can give the following geometrical results:

Theorem 5.3. *The jet Berwald linear d-connection* $B\Gamma^{\mathbf{b}}_{\text{jet}}$ *on* $J^1(TM,N)$ *is characterized by* fifteen *essential local adapted* d-torsions*:*

(T1) $T^{a}_{\beta\gamma} = N^{a}_{\beta:\gamma} - N^{a}_{\gamma:\beta} = {}^{\mathbf{b}}\mathcal{R}^{\alpha}_{\beta\gamma}$,

(T2) $P^{(m)\ (b)}_{(\mu)i(j)} = {}^{\mathbf{b}}\widetilde{\mathcal{P}}^{m}_{ikj}x^{k}_{\mu}s^{b}$,

(T3) $P^{(m)\ (b)}_{(c)i(j)} = {}^{\mathbf{b}}\widetilde{\mathcal{P}}^{m}_{ikj}y^{k}_{c}s^{b}$,

(T4) $$R^{(m)}_{(\mu)\alpha\beta} = -\left[{}^{\mathbf{b}}\mathcal{R}^{\varepsilon}_{\mu\alpha\beta} + {}^{\mathbf{b}}\mathcal{P}^{\varepsilon}_{\mu\alpha c}N^{c}_{\beta} - {}^{\mathbf{b}}\mathcal{P}^{\varepsilon}_{\mu\beta c}N^{c}_{\alpha}\right]x^{m}_{\varepsilon} +$$
$$+\mathcal{A}_{\{\alpha,\beta\}}\left[\frac{\partial N^{c}_{\beta:\mu}}{\partial t^{\alpha}} + N^{d}_{\beta:\mu}B^{c}_{d\alpha} - N^{c}_{\beta:\gamma}B^{\gamma}_{\alpha\mu}\right]y^{m}_{c},$$

(T5) $R^{(m)}_{(\mu)\alpha b} = -\ {}^{\mathbf{b}}\mathcal{P}^{\varepsilon}_{\mu\alpha b}x^m_{\varepsilon} + \left[\dfrac{\partial B^c_{b\mu}}{\partial t^\alpha} - \dfrac{\partial N^c_{\alpha:\mu}}{\partial s^b} + B^d_{b\mu}B^c_{d\alpha} - B^\gamma_{\alpha\mu}B^c_{\gamma b}\right] y^m_c,$

(T6) $R^{(m)}_{(\mu)a\beta} = {}^{\mathbf{b}}\mathcal{P}^{\varepsilon}_{\mu a\beta}x^m_{\varepsilon} - \left[\dfrac{\partial B^c_{a\mu}}{\partial t^\beta} - \dfrac{\partial N^c_{\beta:\mu}}{\partial s^a} + B^d_{a\mu}B^c_{d\beta} - B^\gamma_{\beta\mu}B^c_{\gamma a}\right] y^m_c,$

(T7) $R^{(m)}_{(c)\alpha\beta} = -\left[{}^{\mathbf{b}}\mathcal{R}^d_{c\alpha\beta} + {}^{\mathbf{b}}\mathcal{P}^d_{c\alpha f}N^f_\beta - {}^{\mathbf{b}}\mathcal{P}^d_{c\beta f}N^f_\alpha\right] y^m_d,$

(T8) $R^{(m)}_{(c)\alpha b} = -\ {}^{\mathbf{b}}\mathcal{P}^d_{c\alpha b}y^m_d,$

(T9) $R^{(m)}_{(c)a\beta} = {}^{\mathbf{b}}\mathcal{P}^d_{ca\beta}y^m_d,$

(T10) $R^{(m)}_{(\mu)\alpha j} = -\ {}^{\mathbf{b}}\widetilde{\mathcal{P}}^m_{jkl}B^c_{\alpha b}s^b x^k_\mu y^l_c,$

(T11) $R^{(m)}_{(\mu)aj} = -\ {}^{\mathbf{b}}\widetilde{\mathcal{P}}^m_{jkl}x^k_\mu y^l_a,$

(T12) $R^{(m)}_{(c)\alpha j} = -\ {}^{\mathbf{b}}\widetilde{\mathcal{P}}^m_{jkl}B^d_{\alpha b}s^b y^k_c y^l_d,$

(T13) $R^{(m)}_{(c)aj} = -\ {}^{\mathbf{b}}\widetilde{\mathcal{P}}^m_{jkl}y^k_c y^l_a,$

(T14) $R^{(m)}_{(\mu)ij} = \left[{}^{\mathbf{b}}\widetilde{\mathcal{R}}^m_{kij} + {}^{\mathbf{b}}\widetilde{\mathcal{P}}^m_{kil}\widetilde{N}^l_j - {}^{\mathbf{b}}\widetilde{\mathcal{P}}^m_{kjl}\widetilde{N}^l_i\right] x^k_\mu -$

$-\left[{}^{\mathbf{b}}\widetilde{\mathcal{P}}^m_{kil}\widetilde{B}^l_{jp} - {}^{\mathbf{b}}\widetilde{\mathcal{P}}^m_{kjl}\widetilde{B}^l_{ip}\right] x^k_\mu y^p_a s^a$

(T15) $R^{(m)}_{(c)ij} = \left[{}^{\mathbf{b}}\widetilde{\mathcal{R}}^m_{kij} + {}^{\mathbf{b}}\widetilde{\mathcal{P}}^m_{kil}\widetilde{N}^l_j - {}^{\mathbf{b}}\widetilde{\mathcal{P}}^m_{kjl}\widetilde{N}^l_i\right] y^k_c -$

$-\left[{}^{\mathbf{b}}\widetilde{\mathcal{P}}^m_{kil}\widetilde{B}^l_{jp} - {}^{\mathbf{b}}\widetilde{\mathcal{P}}^m_{kjl}\widetilde{B}^l_{ip}\right] y^k_c y^p_a s^a$

where $\mathcal{A}_{\{\alpha,\beta\}}$ *means an alternate sum;*

Proof. The essential local adapted d-torsions of the Berwald linear distinguished connection $B\Gamma^{\mathbf{b}}_{jet}$ on $J^1(TM,N)$ are given by the general formulas [9, p. 34]:

(t1) $T^M_{AB} = \overline{G}^M_{AB} - \overline{G}^M_{BA},$

(t2) $P^{(m)\ (B)}_{(M)A(j)} = \dfrac{\partial\left[{}^{\mathbf{b}}M^{(m)}_{(M)A}\right]}{\partial X^j_B} - G^{(m)(B)}_{(M)(j)A},$

(t3) $P^{(m)\ (B)}_{(M)i(j)} = \dfrac{\partial\left[{}^{\mathbf{b}}N^{(m)}_{(M)i}\right]}{\partial X^j_B} - L^{(m)(B)}_{(M)(j)i},$

(t4) $R^{(m)}_{(M)AB} = \dfrac{\delta^{\mathbf{J}}\left[{}^{\mathbf{b}}M^{(m)}_{(M)A}\right]}{\delta T^B} - \dfrac{\delta^{\mathbf{J}}\left[{}^{\mathbf{b}}M^{(m)}_{(M)B}\right]}{\delta T^A},$

(t5) $R^{(m)}_{(M)Aj} = \dfrac{\delta^{\mathbf{J}}\left[{}^{\mathbf{b}}M^{(m)}_{(M)A}\right]}{\delta x^j} - \dfrac{\delta^{\mathbf{J}}\left[{}^{\mathbf{b}}N^{(m)}_{(M)j}\right]}{\delta T^A},$

(t6) $R^{(m)}_{(M)ij} = \dfrac{\delta^{\mathbf{J}}\left[{}^{\mathbf{b}}N^{(m)}_{(M)i}\right]}{\delta x^j} - \dfrac{\delta^{\mathbf{J}}\left[{}^{\mathbf{b}}N^{(m)}_{(M)j}\right]}{\delta x^i},$

where $\left(T^A\right)=(t^\alpha,s^a)$ and $\left(X^i_A\right)=\left(x^i_\alpha,y^i_a\right)$. Taking into account that the indices A, B, ... are indices of kind (α,a), (β,b), ... and using the fomulas (5.11), (5.6), (5.9) and (5.10), by laborious local computations, we find the required result. □

Theorem 5.4. *The jet Berwald linear d-connection* $B\Gamma^{\mathbf{b}}_{\text{jet}}$ *on* $J^1(TM,N)$ *is characterized by* thirty *essential local adapted* d-curvatures*:*

- **(C1)** $\overline{R}^\delta_{\alpha\beta\gamma}={}^{\mathbf{b}}\mathcal{R}^\delta_{\alpha\beta\gamma}+{}^{\mathbf{b}}\mathcal{P}^\delta_{\alpha\beta c}N^c_\gamma-{}^{\mathbf{b}}\mathcal{P}^\delta_{\alpha\gamma c}N^c_\beta,$
- **(C2)** $\overline{R}^d_{\alpha\beta\gamma}=\mathcal{A}_{\{\beta,\gamma\}}\left[\dfrac{\partial N^d_{\alpha:\beta}}{\partial t^\gamma}+N^c_{\alpha:\beta}B^d_{c\gamma}-N^d_{\mu:\beta}B^\mu_{\alpha\gamma}\right],$
- **(C3)** $\overline{R}^\delta_{\alpha\beta c}={}^{\mathbf{b}}\mathcal{P}^\delta_{\alpha\beta c},$
- **(C4)** $\overline{R}^d_{\alpha\beta c}=\dfrac{\partial N^d_{\alpha:\beta}}{\partial s^c}-\dfrac{\partial B^d_{\alpha c}}{\partial t^\beta}+B^\mu_{\alpha\beta}B^d_{\mu c}-B^f_{\alpha c}B^d_{f\beta},$
- **(C5)** $\overline{R}^\delta_{\alpha b\gamma}=-{}^{\mathbf{b}}\mathcal{P}^\delta_{\alpha b\gamma},$
- **(C6)** $\overline{R}^d_{\alpha b\gamma}=\dfrac{\partial B^d_{\alpha b}}{\partial t^\gamma}-\dfrac{\partial N^d_{\alpha:\gamma}}{\partial s^b}+B^c_{\alpha b}B^d_{c\gamma}-B^\mu_{\alpha\gamma}B^d_{\mu b},$
- **(C7)** $\overline{R}^d_{a\beta\gamma}={}^{\mathbf{b}}\mathcal{R}^d_{a\beta\gamma}+{}^{\mathbf{b}}\mathcal{P}^d_{a\beta c}N^c_\gamma-{}^{\mathbf{b}}\mathcal{P}^d_{a\gamma c}N^c_\beta,$
- **(C8)** $\overline{R}^d_{a\beta c}={}^{\mathbf{b}}\mathcal{P}^d_{a\beta c},$
- **(C9)** $\overline{R}^d_{ab\gamma}=-{}^{\mathbf{b}}\mathcal{P}^d_{ab\gamma},$
- **(C10)** $R^l_{i\beta k}=-{}^{\mathbf{b}}\widetilde{\mathcal{P}}^l_{ikj}B^c_{\beta a}s^a y^j_c,$
- **(C11)** $R^l_{ibk}=-{}^{\mathbf{b}}\widetilde{\mathcal{P}}^l_{ikj}y^j_b,$
- **(C12)** $R^l_{ijk}={}^{\mathbf{b}}\widetilde{\mathcal{R}}^l_{ijk}+{}^{\mathbf{b}}\widetilde{\mathcal{P}}^l_{ijr}\widetilde{N}^r_k-{}^{\mathbf{b}}\widetilde{\mathcal{P}}^l_{ikr}\widetilde{N}^r_j+\left[{}^{\mathbf{b}}\widetilde{\mathcal{P}}^l_{ikr}\widetilde{B}^r_{jp}-{}^{\mathbf{b}}\widetilde{\mathcal{P}}^l_{ijr}\widetilde{B}^r_{kp}\right]y^p_b s^b,$
- **(C13)** $P^{l\ (c)}_{ij(k)}={}^{\mathbf{b}}\widetilde{\mathcal{P}}^l_{ijk}s^c,$
- **(C14)** $R^{(l)(\alpha)}_{(\varepsilon)(i)\beta\gamma}=-\delta^l_i\cdot\left[{}^{\mathbf{b}}\mathcal{R}^\alpha_{\varepsilon\beta\gamma}+{}^{\mathbf{b}}\mathcal{P}^\alpha_{\varepsilon\beta c}N^c_\gamma-{}^{\mathbf{b}}\mathcal{P}^\alpha_{\varepsilon\gamma c}N^c_\beta\right]=-\delta^l_i\cdot\overline{R}^\alpha_{\varepsilon\beta\gamma},$
- **(C15)** $R^{(l)(\alpha)}_{(\varepsilon)(i)\beta c}=-\delta^l_i\cdot{}^{\mathbf{b}}\mathcal{P}^\alpha_{\varepsilon\beta c}=-\delta^l_i\cdot\overline{R}^\alpha_{\varepsilon\beta c},$
- **(C16)** $R^{(l)(\alpha)}_{(\varepsilon)(i)b\gamma}=\delta^l_i\cdot{}^{\mathbf{b}}\mathcal{P}^\alpha_{\varepsilon b\gamma}=-\delta^l_i\cdot\overline{R}^\alpha_{\varepsilon b\gamma},$
- **(C17)** $R^{(l)(a)}_{(\varepsilon)(i)\beta\gamma}=-\delta^l_i\cdot\mathcal{A}_{\{\beta,\gamma\}}\left[\dfrac{\partial N^a_{\beta:\varepsilon}}{\partial t^\gamma}+N^c_{\beta:\varepsilon}B^a_{c\gamma}-N^a_{\beta:\mu}B^\mu_{\varepsilon\gamma}\right],$
- **(C18)** $R^{(l)(a)}_{(\varepsilon)(i)\beta c}=-\delta^l_i\cdot\left[\dfrac{\partial N^a_{\beta:\varepsilon}}{\partial s^c}-\dfrac{\partial B^a_{c\varepsilon}}{\partial t^\beta}+B^\mu_{\beta\varepsilon}B^a_{\mu c}-B^d_{c\varepsilon}B^a_{d\beta}\right],$
- **(C19)** $R^{(l)(a)}_{(\varepsilon)(i)b\gamma}=-\delta^l_i\cdot\left[\dfrac{\partial B^a_{b\varepsilon}}{\partial t^\gamma}-\dfrac{\partial N^a_{\gamma:\varepsilon}}{\partial s^b}+B^c_{b\varepsilon}B^a_{c\gamma}-B^\mu_{\gamma\varepsilon}B^a_{\mu b}\right],$
- **(C20)** $R^{(l)(a)}_{(d)(i)\beta\gamma}=-\delta^l_i\cdot\left[{}^{\mathbf{b}}\mathcal{R}^a_{d\beta\gamma}+{}^{\mathbf{b}}\mathcal{P}^a_{d\beta c}N^c_\gamma-{}^{\mathbf{b}}\mathcal{P}^a_{d\gamma c}N^c_\beta\right]=-\delta^l_i\cdot\overline{R}^a_{d\beta\gamma},$
- **(C21)** $R^{(l)(a)}_{(d)(i)\beta c}=-\delta^l_i\cdot{}^{\mathbf{b}}\mathcal{P}^a_{d\beta c}=-\delta^l_i\cdot\overline{R}^a_{d\beta c},$
- **(C22)** $R^{(l)(a)}_{(d)(i)b\gamma}=\delta^l_i\cdot{}^{\mathbf{b}}\mathcal{P}^a_{db\gamma}=-\delta^l_i\cdot\overline{R}^a_{db\gamma},$

(C23) $R^{(l)(\alpha)}_{(\varepsilon)(i)\beta k} = -\delta^{\alpha}_{\varepsilon} \cdot {}^{\mathbf{b}}\widetilde{\mathcal{P}}^{l}_{ikj} B^{c}_{\beta b} s^{b} y^{j}_{c} = \delta^{\alpha}_{\varepsilon} \cdot R^{l}_{i\beta k}$,

(C24) $R^{(l)(\alpha)}_{(\varepsilon)(i)bk} = -\delta^{\alpha}_{\varepsilon} \cdot {}^{\mathbf{b}}\widetilde{\mathcal{P}}^{l}_{ikj} y^{j}_{b} = \delta^{\alpha}_{\varepsilon} \cdot R^{l}_{ibk}$,

(C25) $R^{(l)(a)}_{(d)(i)\beta k} = -\delta^{a}_{d} \cdot {}^{\mathbf{b}}\widetilde{\mathcal{P}}^{l}_{ikj} B^{c}_{\beta b} s^{b} y^{j}_{c} = \delta^{a}_{d} \cdot R^{l}_{i\beta k}$,

(C26) $R^{(l)(a)}_{(d)(i)bk} = -\delta^{a}_{d} \cdot {}^{\mathbf{b}}\widetilde{\mathcal{P}}^{l}_{ikj} y^{j}_{b} = \delta^{a}_{d} \cdot R^{l}_{ibk}$,

(C27) $R^{(l)(\alpha)}_{(\varepsilon)(i)jk} = \delta^{\alpha}_{\varepsilon} \cdot R^{l}_{ijk}$,

(C28) $R^{(l)(a)}_{(d)(i)jk} = \delta^{a}_{d} \cdot R^{l}_{ijk}$,

(C29) $P^{(l)(\alpha)\ (c)}_{(\varepsilon)(i)j(k)} = \delta^{\alpha}_{\varepsilon} \cdot {}^{\mathbf{b}}\widetilde{\mathcal{P}}^{l}_{ijk} s^{c} = \delta^{\alpha}_{\varepsilon} \cdot P^{l\ (c)}_{ij(k)}$,

(C30) $P^{(l)(a)\ (c)}_{(d)(i)j(k)} = \delta^{a}_{d} \cdot {}^{\mathbf{b}}\widetilde{\mathcal{P}}^{l}_{ijk} s^{c} = \delta^{a}_{d} \cdot P^{l\ (c)}_{ij(k)}$.

Proof. The essential local adapted d-curvatures of the Berwald linear distinguished connection $B\Gamma^{\mathbf{b}}_{jet}$ on $J^1(TM, N)$ are given by the general formulas [9, p. 36]:

(c1) $\overline{R}^{D}_{ABC} = \dfrac{\delta^{\mathbf{J}} \overline{G}^{D}_{AB}}{\delta T^{C}} - \dfrac{\delta^{\mathbf{J}} \overline{G}^{D}_{AC}}{\delta T^{B}} + \overline{G}^{M}_{AB} \overline{G}^{D}_{MC} - \overline{G}^{M}_{AC} \overline{G}^{D}_{MB}$,

(c2) $R^{l}_{iBk} = -\dfrac{\delta^{\mathbf{J}} L^{l}_{ik}}{\delta T^{B}}$,

(c3) $R^{l}_{ijk} = \dfrac{\delta^{\mathbf{J}} L^{l}_{ij}}{\delta x^{k}} - \dfrac{\delta^{\mathbf{J}} L^{l}_{ik}}{\delta x^{j}} + L^{m}_{ij} L^{l}_{mk} - L^{m}_{ik} L^{l}_{mj}$,

(c4) $P^{l\ (G)}_{ij(k)} = \dfrac{\partial L^{l}_{ij}}{\partial X^{k}_{G}}$,

(c5) $R^{(l)(A)}_{(D)(i)BC} = \dfrac{\delta^{\mathbf{J}} G^{(l)(A)}_{(D)(i)B}}{\delta T^{C}} - \dfrac{\delta^{\mathbf{J}} G^{(l)(A)}_{(D)(i)C}}{\delta T^{B}} +$

$+ G^{(m)(A)}_{(M)(i)B} \cdot G^{(l)(M)}_{(D)(m)C} - G^{(m)(A)}_{(M)(i)C} \cdot G^{(l)(M)}_{(D)(m)B}$,

(c6) $R^{(l)(A)}_{(D)(i)Bk} = \dfrac{\delta^{\mathbf{J}} G^{(l)(A)}_{(D)(i)B}}{\delta x^{k}} - \dfrac{\delta^{\mathbf{J}} L^{(l)(A)}_{(D)(i)k}}{\delta T^{B}} +$

$+ G^{(m)(A)}_{(M)(i)B} \cdot L^{(l)(M)}_{(D)(m)k} - L^{(m)(A)}_{(M)(i)k} \cdot G^{(l)(M)}_{(D)(m)B}$,

(c7) $R^{(l)(A)}_{(D)(i)jk} = \dfrac{\delta^{\mathbf{J}} L^{(l)(A)}_{(D)(i)j}}{\delta x^{k}} - \dfrac{\delta^{\mathbf{J}} L^{(l)(A)}_{(D)(i)k}}{\delta x^{j}} +$

$+ L^{(m)(A)}_{(M)(i)j} \cdot L^{(l)(M)}_{(D)(m)k} - L^{(m)(A)}_{(M)(i)k} \cdot L^{(l)(M)}_{(D)(m)j}$,

(c8) $P^{(l)(A)\ (G)}_{(D)(i)j(k)} = \dfrac{\partial L^{(l)(A)}_{(D)(i)j}}{\partial X^{k}_{G}}$,

where $\left(T^{B}\right) = \left(t^{\beta}, s^{b}\right)$ and $\left(X^{k}_{G}\right) = \left(x^{k}_{\gamma}, y^{k}_{c}\right)$.

Taking into account that the indices A, B, C, ... are indices of kind (α, a), (β, b), (γ, c)... and using the fomulas (5.11) and (5.10), by laborious local computations, we find what we were looking for. □

Remark 5.7. In order to obtain geometrical informations on $J^1(TM, N)$ about our starting affine map $\varphi : (M, F) \to (N, \widetilde{F})$, we can replace again y^i_α with φ^i_α. In this jet geometrical context, the nondegenerate affine map φ is effectively *"characterized"* by *eight jet d-torsions (*we refer to **(T2)**, **(T3)**, **(T10)**$-$**(T15)**) and *twelve jet d-curvatures (*we refer to **(C10)**$-$**(C13)**, **(C23)**$-$**(C30)**). It is an open problem what is the real geometrical meaning of this intimate connection between the affine maps between two Finsler manifolds and their attached d-torsions and d-curvatures on the 1-jet space $J^1(TM, N)$.

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