Some dynamical aspects on the Lie group $SO(4)$

Camelia Pop, Camelia Petrişor

Abstract. An optimal control problem on $SO(4)$ is discussed and some of its dynamical and geometrical properties are pointed out.

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1 Introduction

In the last time there was a great deal of interest in the study of control problems on matrix Lie group due to their applications in spacecraft dynamics [17] and subacvatic dynamics [3]. The goal of our paper is to study an optimal control problem on the Lie group $SO(4)$ and to point out some of its dynamical and geometrical properties. Similar problems have been studied in [4], [5], [15] and [16].

2 The geometrical picture of the problem

Let $SO(4)$ be the set of all matrices $A \in \mathcal{M}_{4\times 4}(R)$ such that $A^t \cdot A = I_4$ and $det(A) = 1$.

It is a 6-dimensional Lie group and a basis of its Lie algebra $so(4)$:

$$so(4) = \left\{ \left[\begin{array}{cccc} 0 & a_1 & a_2 & a_3 \\ -a_1 & 0 & a_4 & a_5 \\ -a_2 & -a_4 & 0 & a_6 \\ -a_3 & -a_5 & -a_6 & 0 \end{array} \right] \middle| \; a_1, a_2, a_3, a_4, a_5, a_6 \in \mathbb{R} \right\}$$

is given by:

$$A_1 = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right], \; A_2 = \left[\begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right], \; A_3 = \left[\begin{array}{cccc} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array} \right],$$

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$$A_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; A_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \; A_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Proposition 2.1. *The Lie algebra structure of $so(4)$ is given by the following table:*

$[.,.]$	A_1	A_2	A_3	A_4	A_5	A_6
A_1	0	$-A_4$	$-A_5$	A_2	A_3	0
A_2	A_4	0	$-A_6$	$-A_1$	0	A_3
A_3	A_5	A_6	0	0	$-A_1$	$-A_2$
A_4	$-A_2$	A_1	0	0	$-A_6$	A_5
A_5	$-A_3$	0	A_1	A_6	0	$-A_4$
A_6	0	$-A_3$	A_2	$-A_5$	A_4	0

Now, a general left-invariant, drift free, control system on $SO(4)$ with fewer controls than state variables can be written in the following form:

$$\dot{X} = X\left(\sum_{i=1}^{m} A_i u_i\right),$$

where $X \in SO(4)$, $u_i, i = 1, 2, ..., m$ are the controls and $m < 6$.

In all that follows we shall concentrate to the following left-invariant, drift-free control system on $SO(4)$ with 3 controls:

$$\dot{X} = X\left(A_1 u_1 + A_3 u_3 + A_4 u_4\right), \tag{2.1}$$

Then, we have:

Proposition 2.2. *The system (2.1) is controllable.*

Proof. Since the span of the set of Lie brackets generated by A_1, A_3, A_4 coincides with $so(4)$ the Proposition is a consequence of a result due to Jurdjevic and Sussman ([10]). □

Let J be the cost function given by:

$$J(u_1, u_3, u_4) = \frac{1}{2}\int_0^{t_f} [c_1 u_1^2(t) + c_3 u_3^2(t) + c_4 u_4^2(t)]dt,$$

$$c_1 > 0, c_3 > 0, c_4 > 0.$$

Then we have:

Proposition 2.3. *The controls that minimize J and steer the system (2.1) from $X = X_0$ at $t = 0$ to $X = X_f$ at $t = t_f$ are given by:*

$$u_1 = \frac{1}{c_1}x_1, \; u_3 = \frac{1}{c_3}x_3, \; u_4 = \frac{1}{c_4}x_4,$$

where $x_i's$ are solutions of:

$$
(2.2) \qquad \begin{cases}
\dot{x}_1 = \dfrac{1}{c_3}x_3x_5 - \dfrac{1}{c_4}x_2x_4 \\[2ex]
\dot{x}_2 = (\dfrac{1}{c_4} - \dfrac{1}{c_1})x_1x_4 + \dfrac{1}{c_3}x_3x_6 \\[2ex]
\dot{x}_3 = -\dfrac{1}{c_1}x_1x_5 \\[2ex]
\dot{x}_4 = \dfrac{1}{c_1}x_1x_2 \\[2ex]
\dot{x}_5 = \left(\dfrac{1}{c_1} - \dfrac{1}{c_3}\right)x_1x_3 - \dfrac{1}{c_4}x_4x_6 \\[2ex]
\dot{x}_6 = -\dfrac{1}{c_3}x_2x_3 + \dfrac{1}{c_4}x_4x_5.
\end{cases}
$$

Proof. Let us apply Krishnaprasad's theorem (see [12]). It follows that the optimal Hamiltonian is given by:

$$H(x_1, x_3, x_4) = \frac{1}{2}\left(\frac{x_1^2}{c_1} + \frac{x_3^2}{c_3} + \frac{x_4^2}{c_4}\right).$$

It is in fact the controlled Hamiltonian $\overline{H}$ given by:

$$\begin{aligned}\overline{H}(x_1, x_3, x_4) &= x_1u_1 + x_3u_3 + x_4u_4 \\ &\quad - \frac{1}{2}(c_1u_1^2 + c_3u_3^2 + c_4u_4^2),\end{aligned}$$

which is reduced to $so(4)^*_-$ via Poisson reduction. Here $so(4)^*_-$ is $so(4)^* \simeq \mathbb{R}^6$ together with the minus Lie-Poisson structure generated by the matrix:

$$\Pi_- = \begin{bmatrix} 0 & x_4 & x_5 & -x_2 & -x_3 & 0 \\ -x_4 & 0 & x_6 & x_1 & 0 & -x_3 \\ -x_5 & -x_6 & 0 & 0 & x_1 & x_2 \\ x_2 & -x_1 & 0 & 0 & x_6 & -x_5 \\ x_3 & 0 & -x_1 & -x_6 & 0 & x_4 \\ 0 & x_3 & -x_2 & x_5 & -x_4 & 0 \end{bmatrix}.$$

Then the optimal controls are given by:

$$u_1 = \frac{1}{c_1}x_1, u_3 = \frac{1}{c_3}x_3, u_4 = \frac{1}{c_4}x_4,$$

where $x_i's$ are solutions of the reduced Hamilton's equations given by:

$$[\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4, \dot{x}_5, \dot{x}_6]^t = \Pi_- \cdot \nabla H$$

which are nothing else then the required equations (2.2). □

Proposition 2.4. *The functions C_1 and C_2 given by:*

$$C_1 = \frac{1}{2}\sum_{i=1}^{6} x_i^2,$$

$$C_2 = x_1x_6 - x_2x_5 + x_3x_4,$$

are Casimirs of our Poisson configuration $(\mathbb{R}^6, \Pi_-)$.

Proof. Indeed, we have successively: $(\nabla C_i)^t \Pi_- = 0,\ i = 1, 2$, as required. $\square$

The goal of our paper is to study some dynamical properties for the system (2.2).

3 Stability

Let us suppose that:

$$c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = c > 0.$$

Then our dynamics (2.2) takes the following form:

$$(3.1)\qquad \begin{cases} \dot{x}_1 = \frac{1}{c}(x_3x_5 - x_2x_4) \\ \dot{x}_2 = \frac{1}{c}x_3x_6 \\ \dot{x}_3 = -\frac{1}{c}x_1x_5 \\ \dot{x}_4 = \frac{1}{c}x_1x_2 \\ \dot{x}_5 = -\frac{1}{c}x_4x_6 \\ \dot{x}_6 = -\frac{1}{c}(x_2x_3 - x_4x_5). \end{cases}$$

Using MATHEMATICA 7 we are leading to:

Proposition 3.1. *The dynamics (3.1) has the following equilibrium states:*

$$e_1^{M,N,P} = (M, 0, N, P, 0, 0),\ M, N, P \in \mathbb{R};$$

$$e_2^{M,N,P} = (0, M, 0, 0, N, P),\ M, N, P \in \mathbb{R};$$

$$e_3^{M,N} = (M, 0, 0, 0, 0, N),\ M, N \in \mathbb{R};$$

$$e_4^{M,N} = (0, M, N, -N, -M, 0),\ M, N \in \mathbb{R};$$

$$e_5^{M,N} = (0, M, N, N, M, 0),\ M, N \in \mathbb{R}.$$

First consider the system linearized about e_1. Its eigenvalues are given by:

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0,\ \lambda_{5,6} = \pm i\frac{\sqrt{N^2 + P^2}}{c}.$$

So we can conclude:

Proposition 3.2. *The equilibrium states* $e_1^{M,N,P}$, $M, N, P \in \mathbb{R}$, *are spectrally stable for any* $M, N, P \in \mathbb{R}$.

Let us consider now the system linearized about e_2. Its eigenvalues are given by:

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0,\ \lambda_{5,6} = \pm i\frac{\sqrt{N^2 - M^2}}{c}.$$

Hence we infer the following:

Proposition 3.3. *The equilibrium states* $e_2^{M,N,P}$, $M, N, P \in \mathbb{R}$, *have the following behavior:*

- *If* $N < M$ *they are spectrally stable;*
- *If* $N > M$ *they are unstable.*

Let us consider now the system linearized about $e_3^{M,N}$ Its eigenvalues are given by:

$$\lambda_1 = \lambda_2 = 0,\ \lambda_3 = \lambda_4 = \pm\frac{\sqrt{MN}}{c},\ \lambda_{5,6} = \pm i\frac{\sqrt{MN}}{c}.$$

Because the characteristic polynomial has a root with positive real part, we can conclude that:

Proposition 3.4. *The equilibrium states* $e_3^{M,N}$, $M, N \in \mathbb{R}$, *are unstable for any* $M, N \in \mathbb{R}$.

Let us consider now the system linearized about $e_4^{M,N}$ Its eigenvalues are given by:

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0,\ \lambda_{5,6} = \pm i\frac{N\sqrt{2}}{c}.$$

About their spectral stability, we have the next result:

Proposition 3.5. *The equilibrium states* $e_4^{M,N}$, $M, N \in \mathbb{R}$, *are spectrally stable for any* $M, N \in \mathbb{R}$.

The same eigenvalues are obtained for the equilibrium states $e_5^{M,N}$, $M, N \in \mathbb{R}$, so we can conclude:

Proposition 3.6. *The equilibrium states* $e_5^{M,N}$, $M, N \in \mathbb{R}$, *are spectrally stable for any* $M, N \in \mathbb{R}$.

We can now pass to discuss the nonlinear stability of the equilibrium states $e_2^{M,N,P}$, $e_4^{M,N}$ and $e_5^{M,N}$, $M, N, P \in \mathbb{R}$.

Proposition 3.7. *The equilibrium states* $e_2^{M,N,P}$, $M,N,P\in\mathbb{R}$ *are nonlinearly stable if* $P=0, M,N\neq 0$ *or* $M=N, P\neq 0$.

Proposition 3.8. *The equilibrium states* $e_4^{M,N}$, $M,N\in\mathbb{R}$, *are nonlinearly stable if* $N=0, M\neq 0$ *or* $M=0, N\neq 0$.

Proof. Let us consider first the case $N=0$. We shall make the proof using Arnold's technique, see [2] and [6]. Let $F_{\lambda,\mu}\in C^{\infty}(\mathbb{R}^6,\mathbb{R})$ be the smooth function given by:

$$F_{\lambda,\mu}(x_1,x_2,x_3,x_4,x_5,x_6)\overset{def}{=}\frac{1}{2c}(x_1^2+x_3^2+x_4^2)+$$
$$+\frac{\lambda}{2}(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)+\mu\left(x_1x_6-x_2x_5+x_3x_4\right).$$

Then we have successively:

(i)$\nabla F_\lambda(e_4^M)=0$ iff $\lambda=-\mu$;

(ii)$W=\{(a,b,c,d,b,e),\ a,b,c,d,e\in\mathbb{R}.\}$;

(iii) $(\forall)v\in W$, i.e, $v=[a,b,c,d,b,e]^t$, $a,b,c,d,e\in\mathbb{R}$, we have:

$$v^t\nabla^2F_{\lambda,-\lambda}(e_4^M)v=\frac{1-\lambda c}{c}dx_1^2-dx_2^2+\frac{1-\lambda c}{c}dx_3^2+\frac{1-\lambda c}{c}dx_4^2-$$
$$-\lambda dx_5^2-\lambda dx_6^2+2\lambda dx_3dx_4-2\lambda dx_2dx_5+2\lambda dx_1dx_6$$

and hence

$$\nabla^2F_{\lambda,-\lambda}(e_4^M)\big|_{W\times W}$$

is positive definite for any $M<0$. Therefore, via Arnold's technique, the equilibrium states $e_4^M=(0,M,0,0,-M,0)$, $M\in\mathbb{R}^*$, are nonlinearly stable as required. Using similar arguments, we can conclude that the equilibrium states $e_4^N=(0,0,N,-N,0,0)$, $N\in\mathbb{R}^*$, are nonlinearly stable. □

Proposition 3.9. *The equilibrium states* $e_5^{M,N}$, $M,N\in\mathbb{R}$, *are nonlinearly stable if* $N=0, M\neq 0$ *or* $M=0, N\neq 0$.

Proof. Let us consider first the case $N=0$. We shall make the proof using Arnold's technique (see [2] and [6]). Let $F_{\lambda,\mu}\in C^{\infty}(\mathbb{R}^6,\mathbb{R})$ be the smooth function given by:

$$F_{\lambda,\mu}(x_1,x_2,x_3,x_4,x_5,x_6)\overset{def}{=}(x_1x_6-x_2x_5+x_3x_4)+\frac{\lambda}{2c}(x_1^2+x_3^2+x_4^2)+$$
$$+\frac{\mu}{2}\left(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2\right).$$

Then we have successively:

(i)$\nabla F_\lambda(e_5^M)=0$ iff $\lambda=1$;

(ii)$W=\{(a,b,c,d,-b,e),\ a,b,c,d,e\in\mathbb{R}.\}$;

(iii) $(\forall)v\in W$, i.e, $v=[a,b,c,d,-b,e]^t$, $a,b,c,d,e\in\mathbb{R}$, we have:

$$v^t\nabla^2F_{1,\mu}(e_5^M)v=\frac{c+\mu}{c}dx_1^2+dx_2^2+\frac{c+\mu}{c}dx_3^2+\frac{c+\mu}{c}dx_4^2+$$

$$+dx_5^2 + dx_6^2 + 2dx_3dx_4 - 2dx_2dx_5 + 2dx_1dx_6$$

and hence

$$\nabla^2 F_{1,\mu}(e_5^M)\big|_{W\times W}$$

is positive definite for any $M > 0$. Therefore, via Arnold's technique, the equilibrium states $e_5^M = (0, M, 0,$
$0, M, 0)$, $M \in \mathbb{R}^*$, are nonlinear stable as required. Using similar arguments, we can conclude that the equilibrium states $e_5^N = (0, 0, N, N, 0, 0)$, $N \in \mathbb{R}^*$, are nonlinear stable. □

Remark 3.1. The nonlinear stability of the equilibrium states $e_1^{M,N,P}$, $M, N,$
$P \in \mathbb{R}$, is an open problem. In this case, the energy methods are inconclusive.

4 Numerical integration of the dynamics (2.2)

It is easy to see that for the equations (2.2), Kahan's integrator (see [11]) can be written in the following form:

$$(4.1)\quad \begin{cases} x_1^{n+1} - x_1^n = \dfrac{h}{2c_3}(x_3^{n+1}x_5^n + x_5^{n+1}x_3^n) - \dfrac{h}{2c_4}(x_2^{n+1}x_4^n + x_4^{n+1}x_2^n) \\ x_2^{n+1} - x_2^n = \dfrac{h}{2}(\dfrac{1}{c_4} - \dfrac{1}{c_1})(x_1^{n+1}x_4^n + x_4^{n+1}x_1^n) + \dfrac{h}{2c_3}(x_3^{n+1}x_6^n + x_6^{n+1}x_3^n) \\ x_3^{n+1} - x_3^n = -\dfrac{h}{2c_1}(x_1^{n+1}x_5^n + x_5^{n+1}x_1^n) \\ x_4^{n+1} - x_4^n = \dfrac{h}{2c_1}(x_1^{n+1}x_2^n + x_2^{n+1}x_1^n) \\ x_5^{n+1} - x_5^n = \dfrac{h}{2}\left(\dfrac{1}{c_1} - \dfrac{1}{c_3}\right)(x_1^{n+1}x_3^n + x_3^{n+1}x_1^n) - \dfrac{h}{2c_4}(x_4^{n+1}x_6^n + x_6^{n+1}x_4^n) \\ x_6^{n+1} - x_6^n = -\dfrac{h}{2c_3}(x_3^{n+1}x_2^n + x_2^{n+1}x_3^n) + \dfrac{h}{2c_4}(x_5^{n+1}x_4^n + x_4^{n+1}x_5^n) \end{cases}$$

Using MATHEMATICA 7 we obtain the following result:

Proposition 4.1. *Kahan's integrator (4.1) has the following properties:*

(i) It is not Poisson preserving.

(ii) It does not preserve the Casimirs C_1 and C_2 of our Poisson configuration $(\mathbb{R}^6, \Pi)$.

(iii) It does not preserve the Hamiltonian H of our system (2.2).

We shall discuss now the numerical integration of the dynamics (2.2) via the Lie-Trotter integrator [18]. For the beginning, let us observe that the Hamiltonian vector field X_H splits as follows:

$$X_H = X_{H_1} + X_{H_2} + X_{H_3},$$

where

$$H_1 = \frac{1}{2c_1}x_1^2,\ H_2 = \frac{1}{2c_3}x_3^2,\ H_3 = \frac{1}{2c_4}x_4^2.$$

Their corresponding integral curves are respectively given by:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \\ x_6(t) \end{bmatrix} = A_i \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \\ x_5(0) \\ x_6(0) \end{bmatrix},\ i = 1, 2, 3,$$

where

$$\begin{cases} A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos at & 0 & -\sin at & 0 & 0 \\ 0 & 0 & \cos at & 0 & -\sin at & 0 \\ 0 & \sin at & 0 & \cos at & 0 & 0 \\ 0 & 0 & \sin at & 0 & \cos at & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \\ \\ a = \dfrac{1}{c_1}x_1(0) \end{cases}$$

$$\begin{cases} A_2 = \begin{bmatrix} \cos bt & 0 & 0 & 0 & \sin bt & 0 \\ 0 & \cos bt & 0 & 0 & 0 & \sin bt \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\sin bt & 0 & 0 & 0 & \cos bt & 0 \\ 0 & -\sin bt & 0 & 0 & 0 & \cos bt \end{bmatrix}, \\ \\ b = \dfrac{1}{c_3}x_3(0) \end{cases}$$

and

$$\begin{cases} A_3 = \begin{bmatrix} \cos ct & -\sin ct & 0 & 0 & 0 & 0 \\ \sin ct & \cos ct & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos ct & -\sin ct \\ 0 & 0 & 0 & 0 & \sin ct & \cos ct \end{bmatrix}. \\ \\ c = \dfrac{1}{c_4}x_4(0) \end{cases}$$

Then the Lie-Trotter integrator is given by:

$$\begin{bmatrix} x_1^{n+1} \\ x_2^{n+1} \\ x_3^{n+1} \\ x_4^{n+1} \\ x_5^{n+1} \\ x_6^{n+1} \end{bmatrix} = A_1 A_2 A_3 \begin{bmatrix} x_1^{n} \\ x_2^{n} \\ x_3^{n} \\ x_4^{n} \\ x_5^{n} \\ x_6^{n} \end{bmatrix}, \tag{4.2}$$

i.e.

$$\begin{cases} x_1^{n+1} = \cos bt\ \cos ct\ x_1^n - \cos bt\ \sin ct\ x_2^n + \sin bt\ \cos ct\ x_5^n \\ \qquad -\sin bt\ \sin ct\ x_6^n \\ x_2^{n+1} = \cos at\ \cos bt \cos ct\ x_1^n + \cos at\ \cos bt \cos ct\ x_2^n - \sin at\ \ x_4^n + \\ \qquad \cos at\ \sin bt\ \sin ct\ x_5^n + \cos at\ \sin bt\ \cos ct\ x_6^n \\ x_3^{n+1} = \sin at\ \sin bt\ \cos ct\ x_1^n + \sin at\ \sin bt\ \sin\ ct\ x_2^n + \\ \qquad + \cos at\ x_3^n - \sin at\ \cos bt\ \cos ct\ x_5^n + \sin at\ \cos bt\ \sin ct\ x_6^n \\ x_4^{n+1} = \sin at\ \cos bt\ \sin ct\ x_1^n + \sin at\ \cos bt\ \cos ct\ x_2^n + \\ \qquad + \cos at\ x_4^n + \sin at\ \sin bt\ \sin ct\ x_5^n + \sin at\ \sin bt\ \cos ct\ x_6^n \\ x_5^{n+1} = -\cos at\ \sin bt\ \cos ct\ x_1^n + \cos at\ \sin bt\ \sin ct\ x_2^n + \\ \qquad + \sin at\ x_3^n + \cos at\ \cos bt\ \cos ct\ x_5^n - \cos at\ \cos bt\ \sin ct\ x_6^n \\ x_6^{n+1} = -sin\ bt\ sin\ ct\ x_1^n - sin\ bt\ cos\ ct\ x_2^n + cos\ bt\ sin\ ct\ x_5^n + \\ \qquad + \cos bt\ \cos ct\ x_6^n \end{cases} \tag{4.3}$$

Now, using MATHEMATICA 7 we obtain the following results:

Proposition 4.2. *The Lie-Trotter integrator (4.3) has the following properties:*

(i) It preserves the Poisson structure Π.

(ii) It preserves the Casimirs C_1 *and* C_2 *of our Poisson configuration* $(\mathbb{R}^6, \Pi)$.

(iii) It does not preserve the Hamiltonian H *of our system (2.2).*

(iv) Its restriction to the coadjoint orbit $(\mathcal{O}_k, \omega_k)$*, where*

$$\mathcal{O}_k = \{(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6 |\ x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + \\ + x_6^2 = const,\ x_1x_6 - x_2x_5 + x_3x_4 = const.\}$$

and ω_k *is the Kirilov-Kostant-Souriau symplectic structure on* $\mathcal{O}_k$ *gives rise to a symplectic integrator.*

Remark 4.1. If we make a comparison with the 4th-step Runge-Kutta method we can see that Lie-Trotter integrator gives us quite different results. But, in this case, Kahan's integrator has failed, see Figures 4.1-4.2. However, Lie-Trotter and Kahan integrators have the advantage to be easier implemented.

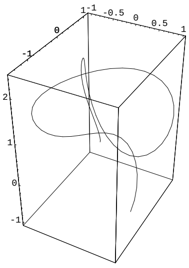

Figure 4.1. The 4th-step Runge-Kutta, projection on $Ox_1x_2x_3$

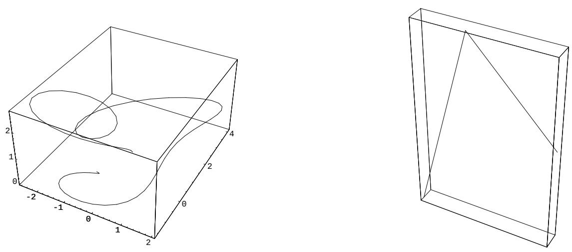

Figure 4.2. a) Lie-Trotter integrator; b) Kahan integrator (projections on $Ox_1x_2x_3$)

5 Conclusions

The purpose of our paper is to study a class of left-invariant, drift-free optimal control problem on the special orthogonal group $SO(4)$. The class of all control-affine left-invariant, drift-free optimal control problems on $SO(4)$ can be reduced to a class of 37 typical controllable left-invariant control systems on $SO(4)$. The left-invariant, drift-free optimal control problems involves finding a trajectory-control pair on $SO(4)$, which minimize a cost function and satisfies the given dynamical constrains and boundary conditions in a fixed time. The problem is lifted to the cotangent bundle $T^*SO(4)$ using the optimal Hamiltonian on $so(4)^*$, where the maximum principle yields the optimal control. The Arnold's method is used to give sufficient conditions for nonlinear stability of the equilibrium states. Unfortunately, for this time, we were not able to give this conditions for the general case (like in [15]), but for some specific values of the real parameters. In the last paragraph we have study the numerical integration via three methods: Lie-Trotter algorithm, Kahan's algorithm and Runge-Kutta 4th steps method. Unlike other studied systems (see [14], [15]) the three results are quite different. This is happened because our system is 6-dimensional, unlike the two mentioned which are 3-dimensional.

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Authors' address:

Politehnica University of Timisoara
Math Department, Pta. Victoriei No. 2, 300006, Timisoara, Romania
E-mail: cariesanu@yahoo.com , galeacamelia@yahoo.com