On the inverse problem of Lagrangian dynamics on Lie algebroids

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Abstract. In the present paper we start the study of the inverse problem of Lagrangian dynamics on Lie algebroids. Using the notion of $\mathcal{J}$-regular section we prove the equivalence between the Helmholtz conditions and a Hamiltonian section on the prolongations of a Lie algebroid.

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1 Introduction

The inverse problem of Lagrangian mechanics is a subject which has been studied intensively in the several decades and can be formulated as follows. Under what conditions a system of second order differential equations on a n-dimensional manifold M can be derived from a variational principle? One solution of this problem is known as the Helmholtz conditions. There are various approaches to derive the Helmholtz conditions and we refer to the survey [6] and the monographs [1], [5] for comments on the history of the problem. Generally, the framework of these studies is the tangent bundle TM of the manifold M. A generalization of the Helmholtz conditions can be found in [7] and the case when M is a Lie group is given in [2]. A Lie algebroid is a generalization of the tangent bundle. Using the geometry of Lie algebroids, Weinstain [14] developed a generalized theory of Lagrangian mechanics and obtained the equations of motions, using the Poisson structure on the dual of a Lie algebroid and Legendre transformation associated with a regular Lagrangian. Thus it is natural to extend the study of the inverse problem to the more general framework of Lie algebroids. In [12] the expressions of the Helmholtz conditions on Lie algebroids are given. In this paper we prove the equivalence between the Helmholtz conditions and a Hamiltonian section on the prolongations of a Lie algebroids over the vector bundle projections.

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2 Preliminaries on Lie algebroids

Let M be a differentiable, n-dimensional manifold and (TM,π_M,M) its tangent bundle. A Lie algebroid [9] over the manifold M is the triple $(E,[\cdot,\cdot]_E,\sigma)$ where $\pi:E\to M$ is a vector bundle of rank m over M, whose $C^\infty(M)$-module of sections $\Gamma(E)$ is equipped with a Lie algebra structure $[\cdot,\cdot]_E$ and $\sigma:E\to TM$ is a vector bundle homomorphism (called *the anchor*) which induces a Lie algebra homomorphism (also denoted σ) from $\Gamma(E)$ to $\chi(M)$, satisfying the Leibniz rule

$$[s_1,fs_2]_E=f[s_1,s_2]_E+(\sigma(s_1)f)s_2,$$

for every $f\in C^\infty(M)$ and $s_1,\ s_2\in\Gamma(E)$. Therefore, we get

$$\sigma[s_1,s_2]_E=[\sigma(s_1),\sigma(s_2)],\quad [s_1,[s_2,s_3]_E]_E+[s_2,[s_3,s_1]_E]_E+[s_3,[s_1,s_2]_E]_E=0.$$

If $\omega\in\bigwedge^k(E^*)$ then the *exterior derivative* $d^E\omega\in\bigwedge^{k+1}(E^*)$ is given by the formula

$$\begin{aligned} d^E\omega(s_1,...,s_{k+1}) &= \sum_{i=1}^{k+1}(-1)^{i+1}\sigma(s_i)\omega(s_1,...,\hat{s}_i,...,s_{k+1})+\\ &\quad+\sum_{1\leq i<j\leq k+1}(-1)^{i+j}\omega([s_i,s_j]_E,s_1,...,\hat{s}_i,...,\hat{s}_j,...s_{k+1}), \end{aligned}$$

where $s_i\in\Gamma(E)$, $i=\overline{1,k+1}$, and it follows that $(d^E)^2=0$. Also, for $\xi\in\Gamma(E)$ on can define the *Lie derivative* with respect to ξ by $\mathcal{L}_\xi=i_\xi\circ d^E+d^E\circ i_\xi$, where i_ξ is the contraction with ξ. If we take the local coordinates (x^i) on an open subset $U\subset M$, a local basis $\{s_\alpha\}$ of sections of the bundle $\pi^{-1}(U)\to U$ generates local coordinates (x^i,y^α) on E. The local functions $\sigma^i_\alpha(x)$, $L^\gamma_{\alpha\beta}(x)$ on M given by

$$\sigma(s_\alpha)=\sigma^i_\alpha\frac{\partial}{\partial x^i},\quad [s_\alpha,s_\beta]_E=L^\gamma_{\alpha\beta}s_\gamma,\quad i=\overline{1,n},\quad \alpha,\beta,\gamma=\overline{1,m},$$

are called the *structure functions* of Lie algebroids.

2.1 The prolongation of a Lie algebroid over the vector bundle projection

Let (E,π,M) be a vector bundle. For the projection $\pi:E\to M$ we can construct the prolongation of E (see [3], [10], [8]). The associated vector bundle is $(\mathcal{T}E,\pi_2,E)$ where $\mathcal{T}E=\cup_{w\in E}\mathcal{T}_wE$ with

$$\mathcal{T}_wE=\{(u_x,v_w)\in E_x\times T_wE\mid \sigma(u_x)=T_w\pi(v_w),\quad \pi(w)=x\in M\},$$

and the projection $\pi_2(u_x,v_w)=\pi_E(v_w)=w$, where $\pi_E:TE\to E$ is the tangent projection. We have also the canonical projection $\pi_1:\mathcal{T}E\to E$ given by $\pi_1(u,v)=u$. The projection onto the second factor $\sigma^1:\mathcal{T}E\to TE$, $\sigma^1(u,v)=v$ will be the anchor of a new Lie algebroid over manifold E. An element of $\mathcal{T}E$ is said to be vertical if it is in the kernel of the projection π_1. We will denote $(V\mathcal{T}E,\pi_{2|_{V\mathcal{T}E}},E)$ the vertical bundle of $(\mathcal{T}E,\pi_2,E)$. The local basis of $\Gamma(\mathcal{T}E)$ is given by $\{\mathcal{X}_\alpha,\mathcal{V}_\alpha\}$, where [10]

$$\mathcal{X}_\alpha(u)=\left(s_\alpha(\pi(u)),\sigma^i_\alpha\left.\frac{\partial}{\partial x^i}\right|_u\right),\quad \mathcal{V}_\alpha(u)=\left(0,\left.\frac{\partial}{\partial y^\alpha}\right|_u\right),$$

and $(\partial/\partial x^i, \partial/\partial y^\alpha)$ is the local basis on TE. The structure functions of $\mathcal{T}E$ are given by the following formulas

$$\sigma^1(\mathcal{X}_\alpha) = \sigma^i_\alpha \frac{\partial}{\partial x^i}, \quad \sigma^1(\mathcal{V}_\alpha) = \frac{\partial}{\partial y^\alpha},$$

$$[\mathcal{X}_\alpha, \mathcal{X}_\beta]_{\mathcal{T}E} = L^\gamma_{\alpha\beta}\mathcal{X}_\gamma, \quad [\mathcal{X}_\alpha, \mathcal{V}_\beta]_{\mathcal{T}E} = 0, \quad [\mathcal{V}_\alpha, \mathcal{V}_\beta]_{\mathcal{T}E} = 0.$$

The differential of sections of $(\mathcal{T}E)^*$ is determined by

$$d^E\mathcal{X}^\alpha = -\frac{1}{2}L^\alpha_{\beta\gamma}\mathcal{X}^\beta \wedge \mathcal{X}^\gamma, \quad d^E\mathcal{V}^\alpha = 0.$$

Other canonical geometric objects (see [8]) are *Euler section* $C = y^\alpha\mathcal{V}_\alpha$ and the *vertical endomorphism* or *tangent structure* $J = \mathcal{X}^\alpha \otimes \mathcal{V}_\alpha$. A section $\mathcal{S}$ of $\mathcal{T}E$ is called *semispray* (or *second order differential equation -SODE*) if $J(\mathcal{S}) = C$. In local coordinates a semispray has the expression

$$\mathcal{S}(x,y) = y^\alpha\mathcal{X}_\alpha + \mathcal{S}^\alpha(x,y)\mathcal{V}_\alpha \tag{2.1}$$

A nonlinear connection N on $\mathcal{T}E$ is an $m-$dimensional distribution (called *horizontal distribution*) $N : u \in E \to H\mathcal{T}_uE \subset \mathcal{T}E$ that is supplementary to the vertical distribution. This means that we have the following decomposition $\mathcal{T}_uE = H\mathcal{T}_uE \oplus V\mathcal{T}_uE$, for $u \in E$. A connection N on $\mathcal{T}E$ induces two projectors $\mathrm{h}, \mathrm{v} : \mathcal{T}E \to \mathcal{T}E$ such that $\mathrm{h}(\rho) = \rho^{\mathrm{h}}$ and $\mathrm{v}(\rho) = \rho^{\mathrm{v}}$ for every $\rho \in \Gamma(\mathcal{T}E)$, where $\mathrm{h} = \frac{1}{2}(id + N)$, $\mathrm{v} = \frac{1}{2}(id - N)$. Locally, a connection can be expressed as $N(\mathcal{X}_\alpha) = \mathcal{X}_\alpha - 2N^\beta_\alpha\mathcal{V}_\beta$, $N(\mathcal{V}_\beta) = -\mathcal{V}_\beta$, where $N^\beta_\alpha = N^\beta_\alpha(x,y)$ are the local coefficients of N. The sections

$$\delta_\alpha = (\mathcal{X}_\alpha)^{\mathrm{h}} = \mathcal{X}_\alpha - N^\beta_\alpha\mathcal{V}_\beta,$$

generate a basis of $H\mathcal{T}E$. The frame $\{\delta_\alpha, \mathcal{V}_\alpha\}$ is a local basis of $\mathcal{T}E$ called the Berwald basis. The dual basis is $\{\mathcal{X}^\alpha, \delta\mathcal{V}^\alpha\}$ where $\delta\mathcal{V}^\alpha = \mathcal{V}^\alpha - N^\alpha_\beta\mathcal{X}^\beta$.
A semispray $\mathcal{S}$ with local coefficients $\mathcal{S}^\alpha$ determines an associated nonlinear connection $N = -\mathcal{L}_S J$ with local coefficients

$$N^\beta_\alpha = \frac{1}{2}\left(-\frac{\partial\mathcal{S}^\beta}{\partial y^\alpha} + y^\varepsilon L^\beta_{\alpha\varepsilon}\right). \tag{2.2}$$

The inverse problem of Lagrangian dynamics on Lie algebroids is to give necessary and sufficient conditions for a system of second order differential equation to be the Euler-Lagrange equations of some regular Lagrangian function. One solution is known as the Helmholtz conditions. If $\mathcal{S}$ is a semispray on $\mathcal{T}E$ given by (2.1) and N^α_β are the coefficients of the associated nonlinear connection (2.2) then the Helmholtz conditions are concerned with the local existence of the functions $g_{\alpha\beta}$ such that [12]

$$\begin{aligned} &i)\ \det g_{\alpha\beta} \neq 0,\ g_{\alpha\beta} = g_{\beta\alpha},\ \frac{\partial g_{\alpha\beta}}{\partial y^\varepsilon} = \frac{\partial g_{\alpha\varepsilon}}{\partial y^\beta}, \\ &ii)\ \mathcal{S}(g_{\alpha\beta}) - g_{\gamma\beta}N^\gamma_\alpha - g_{\gamma\alpha}N^\gamma_\beta = y^\varepsilon\left(g_{\gamma\beta}L^\gamma_{\varepsilon\alpha} + g_{\gamma\alpha}L^\gamma_{\varepsilon\beta}\right), \\ &iii)\ g_{\alpha\gamma}\left(\sigma^i_\beta\frac{\partial\mathcal{S}^\gamma}{\partial x^i} + \mathcal{S}N^\gamma_\beta + N^\varepsilon_\beta N^\gamma_\varepsilon - (L^\delta_{\varepsilon\beta}N^\gamma_\delta + L^\gamma_{\delta\varepsilon}N^\delta_\beta)y^\varepsilon\right) = \\ &\qquad g_{\beta\gamma}\left(\sigma^i_\alpha\frac{\partial\mathcal{S}^\gamma}{\partial x^i} + \mathcal{S}N^\gamma_\alpha + N^\varepsilon_\alpha N^\gamma_\varepsilon - (L^\delta_{\varepsilon\alpha}N^\gamma_\delta + L^\gamma_{\delta\varepsilon}N^\delta_\alpha)y^\varepsilon\right). \end{aligned} \tag{2.3}$$

2.2 The prolongation of a Lie algebroid to its dual bundle

Let $\tau : E^* \to M$ be the dual bundle of $\pi : E \to M$ and $(E, [\cdot,\cdot]_E, \sigma)$ a Lie algebroid structure over M. One can construct a Lie algebroid structure over E^*, by taking the prolongation over $\tau : E^* \to M$ (see [3], [8], [4]). The associated vector bundle is $(\mathcal{T}E^*, \tau_1, E^*)$ where $\mathcal{T}E^* = \underset{u^* \in E^*}{\cup} \mathcal{T}_{u^*}E^*$ and

$$\mathcal{T}_{u^*}E^* = \{(u_x, v_{u^*}) \in E_x \times T_{u^*}E^* | \sigma(u_x) = T_{u^*}\tau(v_{u^*}), \tau(u^*) = x \in M\},$$

and the projection $\tau_1 : \mathcal{T}E^* \to E^*$, $\tau_1(u_x, v_{u^*}) = u^*$. The anchor is the projection $\sigma^1 : \mathcal{T}E^* \to TE^*$, $\sigma^1(u, v) = v$. Notice that if $\mathcal{T}\tau : \mathcal{T}E^* \to E$, $\mathcal{T}\tau(u, v) = u$ then $(V\mathcal{T}E^*, \tau_{1|V\mathcal{T}E^*}, E^*)$ with $V\mathcal{T}E^* = Ker\mathcal{T}\tau$ is a subbundle of $(\mathcal{T}E^*, \tau_1, E^*)$, called the *vertical subbundle*. If (q^i, μ_α) are local coordinates on E^* at u^* and $\{s_\alpha\}$ is a local basis of sections of $\pi : E \to M$ then a local basis of $\Gamma(\mathcal{T}E^*)$ is $\{\mathcal{Q}_\alpha, \mathcal{P}^\alpha\}$ where [8]

$$\mathcal{Q}_\alpha(u^*) = \left(s_\alpha(\tau(u^*)), \sigma^i_\alpha \frac{\partial}{\partial q^i}|_{u^*}\right), \quad \mathcal{P}^\alpha(u^*) = \left(0, \frac{\partial}{\partial \mu_\alpha}|_{u^*}\right).$$

The structure functions on $\mathcal{T}E^*$ are given by the following formulas

$$\sigma^1(\mathcal{Q}_\alpha) = \sigma^i_\alpha \frac{\partial}{\partial q^i}, \quad \sigma^1(\mathcal{P}^\alpha) = \frac{\partial}{\partial \mu_\alpha},$$

$$[\mathcal{Q}_\alpha, \mathcal{Q}_\beta]_{\mathcal{T}E^*} = L^\gamma_{\alpha\beta}\mathcal{Q}_\gamma, \quad [\mathcal{Q}_\alpha, \mathcal{P}^\alpha]_{\mathcal{T}E^*} = 0, \quad [\mathcal{P}^\alpha, \mathcal{P}^\beta]_{\mathcal{T}E^*} = 0,$$

and therefore

$$d^E\mathcal{Q}^\gamma = -\frac{1}{2}L^\gamma_{\alpha\beta}\mathcal{Q}^\alpha \wedge \mathcal{Q}^\beta, \quad d^E\mathcal{P}_\alpha = 0, \quad d^E q^i = \sigma^i_\alpha \mathcal{Q}^\alpha, \quad d^E\mu_\alpha = \mathcal{P}_\alpha,$$

where $\{\mathcal{Q}^\alpha, \mathcal{P}_\alpha\}$ is the dual basis of $\{\mathcal{Q}_\alpha, \mathcal{P}^\alpha\}$. In local coordinates the *Liouville section* is given by $\theta_E = \mu_\alpha \mathcal{Q}^\alpha$. The *canonical symplectic structure* ω_E is defined by $\omega_E = -d^E\theta_E$. It follows that ω_E is a non degenerate 2-section, $d^E\omega_E = 0$ and

$$\omega_E = \mathcal{Q}^\alpha \wedge \mathcal{P}_\alpha + \frac{1}{2}\mu_\alpha L^\alpha_{\beta\gamma}\mathcal{Q}^\beta \wedge \mathcal{Q}^\gamma. \tag{2.4}$$

3 Regular sections and the inverse problem of Lagrangian dynamics on Lie algebroids

An almost tangent structure $\mathcal{J}$ on $\mathcal{T}E^*$ is a bundle morphism $\mathcal{J} : \mathcal{T}E^* \to \mathcal{T}E^*$ of $\tau_1 : \mathcal{T}E^* \to E^*$ of rank m, such that $\mathcal{J}^2 = 0$. An almost tangent structure $\mathcal{J}$ on $\mathcal{T}E^*$ is called *adapted* if $im\mathcal{J} = \ker \mathcal{J} = V\mathcal{T}E^*$. Locally, an adapted almost tangent structure is given by

$$\mathcal{J} = t_{\alpha\beta}\mathcal{Q}^\alpha \otimes \mathcal{P}^\beta,$$

where the matrix $(t_{\alpha\beta}(x, \mu))$ is nondegenerate. It follows that $\mathcal{J}$ is an integrable structure if and only if [4]

$$\frac{\partial t^{\alpha\gamma}}{\partial \mu_\beta} = \frac{\partial t^{\beta\gamma}}{\partial \mu_\alpha}, \tag{3.1}$$

where $t^{\alpha\gamma}t_{\gamma\beta} = \delta^\alpha_\beta$.

Definition 3.1. An adapted almost tangent structure $\mathcal{J}$ on $\mathcal{T}E^*$ is called symmetric if

$$\omega_E(\mathcal{J}\rho_1, \rho_2) = \omega_E(\mathcal{J}\rho_2, \rho_1), \quad \forall \rho_1, \rho_2 \in \Gamma(\mathcal{T}E^*). \tag{3.2}$$

Considering $\rho_1 = \xi_1^\alpha \mathcal{Q}_\alpha + \rho_{1\beta}\mathcal{P}^\beta$ and $\rho_2 = \xi_2^\alpha \mathcal{Q}_\alpha + \rho_{2\beta}\mathcal{P}^\beta$ we obtain

$$\omega_E\left(\xi_1^\alpha t_{\alpha\beta}\mathcal{P}^\beta, \xi_2^\alpha \mathcal{Q}_\alpha + \rho_{2\beta}\mathcal{P}^\beta\right) = \omega_E\left(\xi_2^\alpha t_{\alpha\beta}\mathcal{P}^\beta, \xi_1^\alpha \mathcal{Q}_\alpha + \rho_{1\beta}\mathcal{P}^\beta\right),$$

which lead to the symmetry of $t_{\alpha\beta}$.
If g is a pseudo-Riemannian metric on the vertical subbundle $V\mathcal{T}E^*$ (i.e. a $(0,2)$-type symmetric E-tensor $g = g^{\alpha\beta}(q,\mu)\mathcal{P}_\alpha \otimes \mathcal{P}_\beta$ of rank m on $\mathcal{T}E^*$) then there exists a unique symmetric adapted almost tangent structure on $\mathcal{T}E^*$ such that

$$g(\mathcal{J}\rho, \mathcal{J}\upsilon) = -\omega_E(\mathcal{J}\rho, \upsilon), \quad \forall \rho_1, \rho_2 \in \Gamma(\mathcal{T}E^*), \tag{3.3}$$

and we say that $\mathcal{J}$ is induced by the metric g. Locally, the relation (3.3) implies $t^{\alpha\beta} = g^{\alpha\beta}$.

Definition 3.2. Let $\mathcal{J}$ be an adapted almost tangent structure on $\mathcal{T}E^*$. A section ρ of $\mathcal{T}E^*$ is called $\mathcal{J}-$ *regular* if

$$\mathcal{J}[\rho, \mathcal{J}\nu]_{\mathcal{T}E^*} = -\mathcal{J}\nu, \quad \forall \nu \in \Gamma(\mathcal{T}E^*). \tag{3.4}$$

Locally, the section $\rho = \xi^\alpha \mathcal{Q}_\alpha + \rho_\beta \mathcal{P}^\beta$ is $\mathcal{J}-$*regular* if and only if [4]

$$t^{\alpha\beta} = \frac{\partial \xi^\beta}{\partial \mu_\alpha},$$

where $t^{\alpha\beta}t_{\alpha\gamma} = \delta_\gamma^\beta$. We have to remark that if the equation (3.4) is satisfied for any section $\nu \in \Gamma(\mathcal{T}E^*)$ with $rank[t^{\alpha\beta}] = m$, then $\mathcal{J}$ is an integrable structure.

Definition 3.3. A section ψ on $\mathcal{T}E^*$ is called the *Hamilton section* if it is $\mathcal{J}-$regular and

$$\mathcal{L}_\psi \omega_E = 0,$$

where ω_E is the canonical symplectic section.

If $\psi = \xi^\alpha \mathcal{Q}_\alpha + \rho_\alpha \mathcal{P}^\alpha$ then by direct computation we obtain

$$\mathcal{L}_\psi \omega_E = \frac{\partial \xi^\beta}{\partial \mu_\alpha}\mathcal{P}_\alpha \wedge \mathcal{P}_\beta + \left(\sigma_\alpha^i \frac{\partial \xi^\beta}{\partial q^i} + \frac{\partial \rho_\alpha}{\partial \mu_\beta} - \xi^\gamma L_{\gamma\alpha}^\beta - \mu_\varepsilon L_{\gamma\alpha}^\varepsilon \frac{\partial \xi^\gamma}{\mu_\beta}\right)\mathcal{Q}^\alpha \wedge \mathcal{P}_\beta +$$

$$\left(-\sigma_\alpha^i \frac{\partial \rho_\beta}{\partial q^i} + \frac{1}{2}\rho_\gamma L_{\alpha\beta}^\gamma + \mu_\varepsilon \xi^\gamma \sigma_\alpha^i \frac{\partial L_{\gamma\beta}^\varepsilon}{\partial q^i} + \mu_\varepsilon L_{\gamma\beta}^\varepsilon \sigma_\alpha^i \frac{\partial \xi^\gamma}{\partial q^i} - \frac{1}{2}\mu_\varepsilon \xi^\theta L_{\theta\gamma}^\varepsilon L_{\alpha\beta}^\gamma\right)\mathcal{Q}^\alpha \wedge \mathcal{Q}^\beta,$$

and $\mathcal{L}_\psi \omega_E = 0$ leads to the equations [13]

$$\frac{\partial \xi^\beta}{\partial \mu_\alpha} = \frac{\partial \xi^\alpha}{\partial \mu_\beta}, \tag{3.5}$$

$$\sigma_\alpha^i \frac{\partial \xi^\beta}{\partial q^i} + \frac{\partial \rho_\alpha}{\partial \mu_\beta} = \xi^\gamma L_{\gamma\alpha}^\beta + \mu_\varepsilon L_{\gamma\alpha}^\varepsilon \frac{\partial \xi^\gamma}{\mu_\beta}, \tag{3.6}$$

$$\sigma^i_\beta \frac{\partial \rho_\alpha}{\partial q^i} - \sigma^i_\alpha \frac{\partial \rho_\beta}{\partial q^i} = \mu_\varepsilon \xi^\gamma \left(\sigma^i_\beta \frac{\partial L^\varepsilon_{\gamma\alpha}}{\partial q^i} - \sigma^i_\alpha \frac{\partial L^\varepsilon_{\gamma\beta}}{\partial q^i} + L^\varepsilon_{\nu\gamma} L^\nu_{\alpha\beta} \right) + \mu_\varepsilon \frac{\partial \xi^\gamma}{\partial q^i} \left(\sigma^i_\beta L^\varepsilon_{\gamma\alpha} - \sigma^i_\alpha L^\varepsilon_{\gamma\beta} \right) - \rho_\gamma L^\gamma_{\alpha\beta}. \tag{3.7}$$

Next, we consider a local diffeomorphism Φ to E^* to E given locally by

$$x^i = q^i, \quad y^\alpha = \xi^\alpha(q, \mu), \tag{3.8}$$

and its inverse Φ^{-1} has the following local coordinates expression

$$q^i = x^i, \quad \mu_\alpha = \zeta_\alpha(x, y), \tag{3.9}$$

There always exists a local diffeomorphism Φ to E^* to E given, for instance, by Legendre transformation associated with a regular Hamiltonian on E^* and the Lagrangian is $\mathcal{L}(x,y) = \zeta_\alpha y^\alpha - \mathcal{H}(x,\mu)$ where the components $\zeta_\alpha(x,y)$ define a 1-section on E. From the condition for Φ^{-1} to be the inverse of Φ we get the following formulas:

$$\mathcal{V}_\beta(\zeta_\alpha) \circ \Phi = g_{\alpha\beta}, \quad \mathcal{X}_\beta(\zeta_\alpha) \circ \Phi = -g_{\alpha\gamma} \mathcal{Q}_\beta(\xi^\gamma), \tag{3.10}$$

$$\Phi_* \mathcal{P}^\alpha = (g^{\alpha\beta} \circ \Phi^{-1}) \mathcal{V}_\beta, \quad \Phi_*(\mathcal{Q}_\alpha) = \mathcal{X}_\alpha + (\mathcal{Q}_\alpha(\xi^\beta) \circ \Phi^{-1}) \mathcal{V}_\beta, \tag{3.11}$$

$$\Phi_*^{-1}(\mathcal{V}_\alpha) = g_{\alpha\beta} \mathcal{P}^\beta, \quad \Phi_*^{-1}(\mathcal{X}_\alpha) = \mathcal{Q}_\alpha - g_{\gamma\varepsilon} \mathcal{Q}_\alpha(\xi^\varepsilon) \mathcal{P}^\gamma, \tag{3.12}$$

where Φ_* is the tangent map of Φ and $g^{\alpha\beta} = \partial \xi^\alpha / \partial \mu_\beta$, $g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha_\gamma$. We denote $g_{\alpha\beta} = g_{\alpha\beta} \circ \Phi^{-1}$ by abuse.
Let us consider $\mathcal{S} = y^\alpha \mathcal{X}_\alpha + \mathcal{S}^\alpha(x,y) \mathcal{V}_\alpha$ a semispray on $\mathcal{T}E$ and $\Phi^{-1} : E \to E^*$ the diffeomorphism given by (3.9). Then we set:

Theorem 3.1. *The section $\rho = \Phi_*^{-1} \mathcal{S}$ is a Hamiltonian section on $\mathcal{T}E^*$ if and only if $\mathcal{S}$ and the function $g_{\alpha\beta} = \partial \zeta_\alpha / \partial y^\beta$ satisfy the Helmholtz conditions and the equation*

$$\mathcal{X}_\alpha(\zeta_\theta) - \mathcal{X}_\theta(\zeta_\alpha) = N^\varepsilon_\theta g_{\varepsilon\alpha} - N^\varepsilon_\alpha g_{\varepsilon\theta} + \zeta_\varepsilon L^\varepsilon_{\alpha\theta}.$$

Proof. Since Φ is a local diffeomorphism it results $\det(g_{\alpha\beta}) \neq 0$. Using (3.12) we obtain

$$\begin{aligned} \Phi_*^{-1} \mathcal{S} &= \Phi_*^{-1} \left(y^\alpha \mathcal{X}_\alpha \right) + \Phi_*^{-1} \left(\mathcal{S}^\alpha \mathcal{V}_\alpha \right) = \xi^\alpha \left(Q_\alpha - g_{\gamma\varepsilon} \mathcal{Q}_\alpha(\xi^\varepsilon) \mathcal{P}^\gamma \right) + \mathcal{S}^\alpha g_{\alpha\beta} \mathcal{P}^\beta \\ &= \xi^\alpha Q_\alpha + \left(-g_{\gamma\varepsilon} \xi^\alpha \mathcal{Q}_\alpha(\xi^\varepsilon) + \mathcal{S}^\alpha g_{\alpha\gamma} \right) \mathcal{P}^\gamma. \end{aligned}$$

From the condition (3.5) it results $\frac{\partial \xi^\beta}{\partial \mu_\alpha} = \frac{\partial \xi^\alpha}{\partial \mu_\beta}$ which leads to the first Helmholtz condition. The condition (3.6) lead to

$$\sigma^i_\gamma \frac{\partial \xi^\alpha}{\partial q^i} + \frac{\partial}{\partial \mu_\alpha} \left(-\xi^\beta g_{\gamma\varepsilon} \mathcal{Q}_\beta(\xi^\varepsilon) + \mathcal{S}^\beta g_{\beta\gamma} \right) = \xi^\beta L^\alpha_{\beta\gamma} + \mu_\varepsilon L^\varepsilon_{\beta\gamma} \frac{\partial \xi^\beta}{\partial \mu_\alpha},$$

and using (3.10) it results

$$\mathcal{Q}_\gamma(\xi^\alpha) + g^{\beta\alpha} \mathcal{X}_\beta(\zeta_\gamma) + \xi^\beta \mathcal{X}_\beta(g_{\varepsilon\gamma}) g^{\alpha\varepsilon} + \mathcal{P}^\alpha(\mathcal{S}^\beta) g_{\beta\gamma} + \mathcal{S}^\beta \mathcal{P}^\alpha(g_{\beta\gamma}) + \xi^\beta L^\alpha_{\gamma\beta} + \mu_\varepsilon L^\varepsilon_{\gamma\beta} g^{\beta\alpha} = 0$$

which is equivalent with

$$g^{\theta\beta}\left(\mathcal{X}_\theta(\zeta_\alpha)-\mathcal{X}_\alpha(\zeta_\theta)+\mathcal{V}_\theta(\mathcal{S}^\varepsilon)g_{\varepsilon\alpha}+\xi^\varepsilon\mathcal{X}_\varepsilon(g_{\alpha\theta})+\mathcal{S}^\varepsilon\mathcal{V}_\varepsilon(g_{\alpha\theta})+\zeta_\varepsilon L^\varepsilon_{\alpha\theta}\right)+y^\varepsilon L^\beta_{\alpha\varepsilon}=0.$$

From (2.2) we obtain

$$\mathcal{X}_\alpha(\zeta_\theta)-\mathcal{X}_\theta(\zeta_\alpha)=\mathcal{S}(g_{\alpha\theta})-2N^\varepsilon_\theta g_{\varepsilon\alpha}+y^\beta L^\varepsilon_{\theta\beta}g_{\varepsilon\alpha}+y^\beta L^\varepsilon_{\alpha\beta}g_{\varepsilon\theta}+\zeta_\varepsilon L^\varepsilon_{\alpha\theta}. \tag{3.13}$$

The antisymmetric part of (3.13) leads to the equation

$$\mathcal{S}(g_{\alpha\theta})-N^\varepsilon_\theta g_{\varepsilon\alpha}-N^\varepsilon_\alpha g_{\varepsilon\theta}+y^\beta\left(L^\varepsilon_{\theta\beta}g_{\varepsilon\alpha}+L^\varepsilon_{\alpha\beta}g_{\varepsilon\theta}\right)=0$$

which is the second Helmholtz condition. The symmetric part of (3.13) leads also to the equation

$$\mathcal{X}_\alpha(\zeta_\theta)-\mathcal{X}_\theta(\zeta_\alpha)=N^\varepsilon_\theta g_{\varepsilon\alpha}-N^\varepsilon_\alpha g_{\varepsilon\theta}+\zeta_\varepsilon L^\varepsilon_{\alpha\theta}.$$

Finally, the section $\rho=\Phi_*^{-1}\mathcal{S}$ is a Hamilton section if the condition (3.7) is satisfied. In the same way, by straightforward computation the third Helmholdz condition is obtained, which ends the proof. □

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