Foliations compatible with Hamiltonians

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Abstract. The aim of the paper is to construct projectable Bott linear connections in the lifted foliation on the transverse bundle of a foliation, using linear and nonlinear transverse connections. Considering a connection adapted to a hamiltonian foliation, one lift it on the transverse bundle and one prove that the lifted foliation is a Riemannian one (as proved by one of authors, the last property is fulfilled automatically if the hamiltonian is 2–homogeneous). The results extend similar ones of Miernowski and Mozgawa.

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1 Preliminaries

Let us consider M an $(n+m)$-dimensional manifold which will be assumed to be connected and orientable.

A codimension n foliation $\mathcal{F}$ on M is defined by a foliated cocycle $\{U_i, \varphi_i, f_{i,j}\}$ such that:

(i) $\{U_i\}$, $i \in I$ is an open covering of M;

(ii) For every $i \in I$, $\varphi_i : U_i \to T$ are submersions, where T is an m-dimensional manifold, called transversal manifold;

(iii) The maps $f_{i,j} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ satisfy

$$\varphi_j = f_{i,j} \circ \varphi_i \tag{1.1}$$

for every $(i,j) \in I \times I$ such that $U_i \cap U_j \neq \varnothing$.

Every fiber of φ_i is called a *plaque* of the foliation. Condition (1.1) says that, on the intersection $U_i \cap U_j$ the plaques defined respectively by φ_i and φ_j coincides. The

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manifold M is decomposed into a family of disjoint immersed connected submanifolds of dimension m; each of these submanifolds is called a *leaf* of $\mathcal{F}$.

We say that $\mathcal{F}$ is *transversely orientable* if on T can be given an orientation which is preserved by all $f_{i,j}$. By $T\mathcal{F}$ we denote the tangent bundle to $\mathcal{F}$ and $\Gamma(\mathcal{F})$ is the space of its global sections i.e. vector fields tangent to $\mathcal{F}$.

In this paper a system of local coordinates adapted to the foliation $\mathcal{F}$ means coordinates $(x^u, x^{\bar{u}})$ $u = 1, \ldots, m, \bar{u} = 1, \ldots, n$ on an open subset U on which the foliation is trivial and defined by the equations $dx^{\bar{u}} = 0$, $\bar{u} = 1, \ldots, n$.

We notice that the total spaces of the conormal bundle $Q^*\mathcal{F}$ of $\mathcal{F}$ carries a natural foliation $\widetilde{\mathcal{F}}$ of codimension $2n$ such that the leaves of $\widetilde{\mathcal{F}}$ are covering spaces of the leaves of $\mathcal{F}$, and it is called the *natural lift* of $\mathcal{F}$ to its conormal bundle $Q^*\mathcal{F}$.

If we denote by $\{dx^{\bar{u}}\}$, $\bar{u} = 1, \ldots, n$ the corresponding local coframe on $Q^*\mathcal{F}$ then we can induce a chart $(x^{\bar{u}}, p_{\bar{u}}, x^u)$ on $Q^*\mathcal{F}$ where $p = p_{\bar{u}} dx^{\bar{u}} \in \Gamma(Q^*\mathcal{F})$, and the system of equations $x^{\bar{u}} = const.$, $p_{\bar{u}} = const.$ defines the foliation $\widetilde{\mathcal{F}}$.

Let $Q\widetilde{\mathcal{F}} = T(Q^*\mathcal{F})/T\widetilde{\mathcal{F}}$ be the normal bundle of the foliated manifold $(Q^*\mathcal{F}, \widetilde{\mathcal{F}})$. The vectors $\left\{\frac{\partial}{\partial x^{\bar{u}}}, \frac{\partial}{\partial p_{\bar{u}}}\right\}$, $\bar{u} = 1, \ldots, n$ form a natural frame of $Q\widetilde{\mathcal{F}}$ at the point $(x^{\bar{u}}, p_{\bar{u}}, x^u) \in Q^*\mathcal{F}$. The canonical projection $\pi : Q^*\mathcal{F} \to M$ given by $\pi(x^{\bar{u}}, p_{\bar{u}}, x^u) = (x^{\bar{u}}, x^u)$ induces an another projection $\pi_* : T(Q^*\mathcal{F}) \to TM$ which maps the tangent vectors to $\widetilde{\mathcal{F}}$ in the vectors tangent to $\mathcal{F}$. Thus π_* induces a mapping $\widetilde{\pi}_* : Q\widetilde{\mathcal{F}} \to Q\mathcal{F}$ and denote by $V(Q^*\mathcal{F}) = \ker \widetilde{\pi}_*$ which is a vertical bundle spanned by the vectors $\left\{\frac{\partial}{\partial p_{\bar{u}}}\right\}$, $\bar{u} = 1, \ldots, n$.

Lemma 1.1. *Let $o : M \to Q^*\mathcal{F}$ be the zero section of the conormal bundle $Q^*\mathcal{F}$. Then the set $o(M)$ is saturated on $Q^*\mathcal{F}$ with leaves of the foliation $\widetilde{\mathcal{F}}$.*

2 Foliated vector bundles

Given a foliated manifold $(M, \mathcal{F})$, we say that a vector bundle $p : E \to M$ of rank $E = k$ is a *foliated vector bundle* if there is a foliated vector bundle atlas on E (i.e. the transition matrices are basic functions as components). There is a foliation $\mathcal{F}_E$ on E such that the canonical projection π is a foliated map that induces a local diffeomorphism on leaves. For example, the transverse bundles, as well as the vertical transverse bundles are foliated vector bundles. The projection p of the slashed bundle $p : \tilde{E} \to M$ is a foliated map.

A foliated vector bundle map $F : E' \to E$ over the foliated map $f : M' \to M$ is defined in a similar way, asking that F be covered by foliated vector bundle maps in a foliated vector bundle atlas on E, M, E' and M' (i.e. the local matrices are basic functions as components). Analogous one can consider the definition of a foliated vector subbundle etc.

For a foliated vector bundle $p : E \to M$, considering the slashed bundle $p : \tilde{E} \to M$, where $\tilde{E} = E - \{\text{zero section}\}$, the differential map $p_* : T\tilde{E} \to TM$ induces a foliated vector bundle map $\tilde{p}_* : Q\mathcal{F}_{\tilde{E}} \to Q\mathcal{F}$ that is surjection on fibers. We denote by $V(Q\mathcal{F}_{\tilde{E}}) = \ker \tilde{p}_*$; it is a foliated vector subbundle of $Q\mathcal{F}_{\tilde{E}}$ (over the base $\tilde{E}$), we call it as the *foliated vertical bundle* of E, and we denote by $I : V(Q\mathcal{F}_{\tilde{E}}) \to Q\mathcal{F}_{\tilde{E}}$ the foliated inclusion. A *transverse non-linear connection* on E is a foliated subbundle

$H(Q\mathcal{F}_{\tilde{E}}) \subset Q\mathcal{F}_{\tilde{E}}$ such that

$$Q\mathcal{F}_{\tilde{E}} = V(Q\mathcal{F}_{\tilde{E}}) \oplus H(Q\mathcal{F}_{\tilde{E}}). \tag{2.1}$$

As in the non-foliated case, a transverse non-linear connection is equivalently defined by $H(Q\mathcal{F}_{\tilde{E}}) = \ker \bar{C}$, where $\bar{C}$ is a left splitting of the inclusion $I : V(Q\mathcal{F}_{\tilde{E}}) \to Q\mathcal{F}_{\tilde{E}}$, i.e. a foliated epimorphism $\bar{C} : Q\mathcal{F}_{\tilde{E}} \to V(Q\mathcal{F}_{\tilde{E}})$ such that $\bar{C} \circ I = 1_{V(Q\mathcal{F}_{\tilde{E}})}$.

Using local coordinates

– $(x^u, x^{\bar{u}})$, $u = 1, \ldots, m, \bar{u} = 1, \ldots, n$ on M,

– $(x^u, x^{\bar{u}}, y^{\bar{a}})$, $u = 1, \ldots, m, \bar{u} = 1, \ldots, n, \bar{a} = 1, \ldots, k$ on E,

– $(x^u, x^{\bar{u}}, y^{\bar{a}}, X^{\bar{u}}, Y^{\bar{a}})$, $u = 1, \ldots, m, \bar{u} = 1, \ldots, n, \bar{a} = 1, \ldots, k$ on $Q\mathcal{F}_{\tilde{E}}$,

then some coordinates $(x^u, x^{\bar{u}}, y^{\bar{a}}, Y^{\bar{a}})$, $u = 1, \ldots, m, \bar{u} = 1, \ldots, n, \bar{a} = 1, \ldots, k$ follow on $V(Q\mathcal{F}_{\tilde{E}})$ and the foliated epimorphism $\bar{C}$ has the local form

$$(x^u, x^{\bar{u}}, y^{\bar{a}}, X^{\bar{u}}, Y^{\bar{a}}) \xrightarrow{\bar{C}} (x^u, x^{\bar{u}}, y^{\bar{a}}, N^{\bar{a}}_{\bar{u}}(x^{\bar{u}}, y^{\bar{a}})X^{\bar{u}} + Y^{\bar{a}}). \tag{2.2}$$

Notice that the local correspondences

$$(x^{\bar{u}}, y^{\bar{a}}, X^{\bar{u}}, Y^{\bar{a}}) \xrightarrow{\bar{C}} (x^{\bar{u}}, y^{\bar{a}}, N^{\bar{a}}_{\bar{u}}(x^{\bar{u}}, y^{\bar{a}})X^{\bar{u}} + Y^{\bar{a}})$$

are non-linear connections on the transverse model; we call them as the *local projected (non-linear) connections*.

We say that a transverse non-linear connection is

– 1–*homogeneous*, if the local functions $((x^{\bar{u}}, y^{\bar{a}}) \to N^{\bar{a}}_{\bar{u}}(x^{\bar{u}}, y^{\bar{a}})$ are 1–homogeneous in the second group of variables, i.e.

$$N^{\bar{a}}_{\bar{u}}(x^{\bar{u}}, \lambda y^{\bar{a}}) = \lambda N^{\bar{a}}_{\bar{u}}(x^{\bar{u}}, y^{\bar{a}}),\ (\forall)\ \lambda > 0;$$

– *linear*, if the local functions $(x^{\bar{u}}, y^{\bar{a}}) \to N^{\bar{a}}_{\bar{u}}(x^{\bar{u}}, y^{\bar{a}})$ are linear in the second group of variables, i.e.

$$N^{\bar{a}}_{\bar{u}}(x^{\bar{u}}, y^{\bar{a}}) = \Gamma^{\bar{a}}_{\bar{u}\bar{b}}(x^{\bar{u}})y^{\bar{b}}.$$

Considering a transverse non-linear connection, the local Berwald connections associated with the local projected non-linear connections glue together to a transverse linear connection on the foliated vector bundle $V(Q\mathcal{F}_{\tilde{E}})$, that we call as the *transverse Berwald connection* associated with the given transverse non-linear connection.

Using local coordinates as above, if a transverse non-linear connection that the left splitting $\bar{C}$ has the local form (2.2), has its transverse Berwald linear connection ∇ given by

$$(x^u, x^{\bar{u}}, y^{\bar{a}}, Y^{\bar{a}}, X^{\bar{u}}, Z^{\bar{a}}, W^{\bar{a}}) \xrightarrow{\nabla} (x^u, x^{\bar{u}}, y^{\bar{a}}, Y^{\bar{a}}, Z^{\bar{a}}, \tfrac{\partial N^{\bar{a}}_{\bar{u}}}{\partial y^{\bar{b}}}(x^{\bar{v}}, y^{\bar{c}})X^{\bar{u}}Y^{\bar{b}} + W^{\bar{a}}).$$

Let us denote by $\Pi : T\tilde{E} \to Q\mathcal{F}_{\tilde{E}}$ the canonical projection on the normal bundle of $\mathcal{F}_{\tilde{E}}$; it induces also a vector bundle isomorphism $\tilde{\Pi} : V(T\tilde{E}) \to V(Q\mathcal{F}_{\tilde{E}})$, that corresponds canonically to the identity map of $p^*_{\tilde{E}}E$. A non-linear connection C :

$T\tilde{E} \to V(T\tilde{E})$ is *projectable* if there is a foliated map $\tilde{C} : Q\mathcal{F}_{\tilde{E}} \to V(Q\mathcal{F}_{\tilde{E}})$ such that the following diagram is commutative.

$$\begin{array}{ccc} T\tilde{E} & \stackrel{C}{\longrightarrow} & VT\tilde{E} \\ \Pi\downarrow & & \downarrow\tilde{\Pi} \\ Q\mathcal{F}_{\tilde{E}} & \stackrel{\tilde{C}}{\longrightarrow} & V(Q\mathcal{F}_{\tilde{E}}) \end{array} \tag{2.3}$$

It is easy to see that $\tilde{C}$ is unique and it is a left splitting of the inclusion $\tilde{I} : V(Q\mathcal{F}_{\tilde{E}}) \to T\tilde{E}$, thus a transverse non-linear connection on $\tilde{E}$.

We say that a non-linear connection $C : T\tilde{E} \to V(T\tilde{E})$ is of *Bott type* if $X^h \in \Gamma(T\mathcal{F}_{\tilde{E}})$, $(\forall)X \in \Gamma(T\mathcal{F})$, where X^h is the horizontal lift and $T\mathcal{F}$ is the tangent bundle to the leaves of $\mathcal{F}$.

Proposition 2.1. *If $\tilde{C}$ is a transverse non-linear connection, then there is a unique Bott type nonlinear connection that is projectable on $\tilde{C}$.*

We say that a Bott type nonlinear connection on $Q\mathcal{F}$ or on $Q^*\mathcal{F}$ is a *projectable Bott nonlinear connection* if it obtained as in the above Proposition 2.1.

Let us use some local coordinates $(x^u, x^{\bar{u}}, y^{\bar{a}})$ on a foliated vector bundle E. If the transverse non-linear connection $\tilde{C} : Q\mathcal{F}_{\tilde{E}} \to V(Q\mathcal{F}_{\tilde{E}})$ has the local form $(x^u, x^{\bar{u}}, y^{\bar{a}}, X^{\bar{u}}, Y^{\bar{a}}) \stackrel{\tilde{C}}{\to} (x^u, x^{\bar{u}}, y^{\bar{a}}, X^{\bar{u}}\tilde{N}^{\bar{a}}_{\bar{u}}(x^{\bar{v}}, y^{\bar{b}}) + Y^{\bar{a}})$, then its Bott connection has the local form

$$(x^u, x^{\bar{u}}, y^{\bar{a}}, X^u, X^{\bar{u}}, Y^{\bar{a}}) \stackrel{C}{\to} (x^u, x^{\bar{u}}, y^{\bar{a}}, X^{\bar{u}}\tilde{N}^{\bar{a}}_{\bar{u}}(x^{\bar{v}}, y^{\bar{b}}) + Y^{\bar{a}}). \tag{2.4}$$

This is also the general form of a Bott non-linear connection. A simple characterization of a Bott connection is as follows.

Proposition 2.2. *A projectable non-linear connection is a Bott connection iff $T\mathcal{F}_{\tilde{E}} \subset H(T\tilde{E})$.*

In the case when $\tilde{C}$ comes from a transverse linear connection on E, denoted by $\tilde{\nabla}$, then the Bott type (non-linear) connection C comes from a linear connection ∇ on E and conditions on ∇ reads:

– if $s(Y')$ is a locally transverse vector field to $\mathcal{F}_{\tilde{E}}$ and A is a local foliated section in $\Gamma(E)$, then $\nabla_{s(Y')}A$ is a local foliated section in $\Gamma(E)$ (projectability condition) and

– if X is tangent to $\mathcal{F}_{\tilde{E}}$ and $A \in \Gamma(E)$ is foliated, then $\nabla_X A = 0$ (Bott condition).

If $E = Q\mathcal{F}$ and $\Pi_0 : TM \to Q\mathcal{F}$ is the canonical projection, then the Bott condition becomes the classical one:

– if X is tangent to $\mathcal{F}$ and $A = \Pi_0(Y) \in \Gamma(Q(\mathcal{F}))$, then $\nabla_X \Pi_0(Y) = \Pi_0\,[X, Y]$.

Using formula (2.4) in the case of a linear connection, we obtain that the following statement is true.

Proposition 2.3. *If $\bar{\nabla} : \Gamma\left(Q\mathcal{F}_{\tilde{E}}\right) \to \Gamma\left(Q^*\mathcal{F}_{\tilde{E}} \otimes Q\mathcal{F}_{\tilde{E}}\right)$ is a transverse linear connection on a foliated bundle E, then there is a unique Bott connection $\nabla : \Gamma(Q\mathcal{F}_{\tilde{E}}) \to \Gamma\left(T^*\tilde{E} \otimes Q\mathcal{F}_{\tilde{E}}\right)$ that locally projects on $\bar{\nabla}$.*

In particular if $E = Q^*\mathcal{F}$, we say that a Bott connection on $Q\widetilde{\mathcal{F}}$ is a *projectable Bott connection* if it obtained as in Proposition 2.3.

The link between the Bott condition for a non-linear connection and a linear connection is given by the Berwald linear connection.

Proposition 2.4. *If C is a Bott non-linear connection on a foliated vector bundle, then its Berwald linear connection is a Bott linear connection.*

We can use in the proof that the Berwald linear connection ∇ of C has the property that $\nabla_{X^h} A = v\left[X^h, A\right]$, for any $X \in \mathcal{X}(M)$ and $A \in \Gamma(V(T\tilde{E}))$, where the projector v and the lift h are according to C.

Notice that, in general, the converse of Proposition 2.4 does not hold.

3 Foliations and connections

In this section we consider a connection adapted to a Hamilton foliation and we show also that the lifted foliation of a Hamilton foliation in the conormal bundle is a Riemannian foliation. Using [5], it follows that any Cartan foliation coming from a transverse Finsler metric is a Riemannian foliation.

We say that a foliation $\mathcal{F}$ is a *transverse Hamiltonian* one if there is a basic function $H : Q^*\mathcal{F} \to I\!R$ that has a non-degenerate vertical Hessian h, called a *transverse Hamiltonian.* For every $X \in \mathcal{X}\left(T(Q^*\mathcal{F})\right)$ we have $\Pi\left(X\right) = \bar{X} \in \Gamma\left(Q\widetilde{\mathcal{F}}\right)$, where $\Pi : T(Q^*\mathcal{F}) \to Q\widetilde{\mathcal{F}}$ is the canonical projection.

The inverse h^{-1} of Hessian h induces a vector bundle isomorphism $J_0 : \pi^*_{Q^*\widetilde{\mathcal{F}}} Q\widetilde{\mathcal{F}} \to \pi^*_{Q^*\widetilde{\mathcal{F}}} Q^*\widetilde{\mathcal{F}} \cong V(Q^*\mathcal{F}) = V(Q\widetilde{\mathcal{F}})$, called the *musical isomorphism.* We can consider now the vector bundle map $J : \Gamma(Q\widetilde{\mathcal{F}}) \to \Gamma(V(Q^*\mathcal{F})) \subset \Gamma(Q\widetilde{\mathcal{F}})$, $J(X) = J_0\left(\widetilde{\pi}_*(X)\right)$, where $\widetilde{\pi}_* : Q\widetilde{\mathcal{F}} \to Q\mathcal{F}$ is the canonical transverse projection. It is easy to see that $J \circ J = 0$ and $\operatorname{Im} J = V(Q^*\mathcal{F})$, thus J is a vector bundle epimorphism. A transverse non-linear connection can be given by an almost product endomorphism P in the fibers of $Q\widetilde{\mathcal{F}}$ (i.e. $P^2 = 1_{Q\widetilde{\mathcal{F}}}$) such that the vectors in the fibers of $V(Q^*\mathcal{F})$ are exactly the eigenvectors corresponding to the eigenvalue -1 of P. The link between P and the transverse map $C : Q\widetilde{\mathcal{F}} \to V(Q^*\mathcal{F})$ is $C = \dfrac{1}{2}\left(1_{Q\widetilde{\mathcal{F}}} - P\right)$. We denote by $\bar{L}$ the transverse Lie derivation.

Proposition 3.1. *Let $H : Q^*\mathcal{F} \to I\!R$ be a transverse Hamiltonian, $\bar{X}$ and J_0 be a transverse vector field for $\widetilde{\mathcal{F}}$ and the musical isomorphism, respectively. Then $P = -\bar{L}_{\bar{X}} J : \Gamma(Q\widetilde{\mathcal{F}}) \to \Gamma(Q\widetilde{\mathcal{F}})$ is an almost product endomorphism giving a transverse non-linear connection.*

Let now $\nabla^v : \Gamma\left(V(Q^*\mathcal{F})\right) \to \Gamma\left(Q^*\widetilde{\mathcal{F}} \otimes V(Q^*\mathcal{F})\right)$ be a transverse linear connection, that we call a *transverse vertical connection.*

In the sequel we will use the basis $\left\{\frac{\delta}{\delta x^{\bar{u}}}, \frac{\partial}{\partial p_{\bar{u}}}\right\}$, called *adapted*, as well as its dual $\{dx^{\bar{u}}, \delta p_{\bar{u}} = dp_{\bar{u}} + N_{\bar{u}\bar{v}} dx^{\bar{v}}\}$, accordingly to the decomposition (2.1). Using this coframe we can define the local connection forms by

$$\nabla^v \frac{\partial}{\partial p_{\bar{v}}} = \omega^{\bar{v}}_{\bar{u}} \otimes \frac{\partial}{\partial p_{\bar{u}}}, \tag{3.1}$$

where

$$\omega_{\bar u}^{\bar v}=\Gamma_{\bar\gamma\bar u}^{\bar v}dx^{\bar\gamma}+\Gamma_{\bar u}^{\bar v\bar\gamma}dp_{\bar\gamma}=\left(\Gamma_{\bar\gamma\bar u}^{\bar v}-\Gamma_{\bar u}^{\bar v\bar\delta}N_{\bar\delta\bar\gamma}\right)dx^{\bar\gamma}+\Gamma_{\bar u}^{\bar v\bar\gamma}\delta p_{\bar\gamma}=H_{\bar\gamma\bar u}^{\bar v}dx^{\bar\gamma}+\Gamma_{\bar u}^{\bar v\bar\gamma}\delta p_{\bar\gamma}. \tag{3.2}$$

But $V(Q^*\mathcal{F})$ and $H(Q^*\mathcal{F})$ are dual vector bundles, thus the linear connection ∇^v on $V(Q^*\mathcal{F})$ give rise to a dual linear connection ∇^h on $H(Q^*\mathcal{F})$. Thus we can construct a linear connection ∇ in $Q\widetilde{\mathcal{F}}$

$$\nabla_X Y=\nabla_X^v\left(v(Y)\right)+\nabla_X^h\left(h(Y)\right), \tag{3.3}$$

where $Y\in\Gamma(Q\widetilde{\mathcal{F}})$, $X\in\Gamma(T(Q^*\mathcal{F}))$ and $v:Q\widetilde{\mathcal{F}}\to V(Q^*\mathcal{F})$ and $h:Q\widetilde{\mathcal{F}}\to H(Q^*\mathcal{F})$ are the vertical and horizontal projector respectively from decomposition (2.1). In particular we have

$$\nabla\frac{\delta}{\delta x^{\bar u}}=-\omega_{\bar u}^{\bar v}\otimes\frac{\delta}{\delta x^{\bar v}}, \tag{3.4}$$

where $\omega_{\bar u}^{\bar v}$ is given in (3.2).

Let us remark that we can consider a transverse linear connection $\nabla^h:\Gamma\left(H(Q^*\mathcal{F})\right)\to\Gamma\left(Q^*\widetilde{\mathcal{F}}\otimes H(Q^*\mathcal{F})\right)$, that we call a *transverse horizontal connection* and then associate a dual transverse linear connection ∇^v, that is a vertical connection. The constructions of ∇^h from ∇^v and of ∇^v from ∇^h are mutually inverse, giving rise to a same transverse linear connection ∇.

If $\varphi\in\Gamma\left(Q^*\widetilde{\mathcal{F}}\otimes Q\widetilde{\mathcal{F}}\right)$ is an 1–form with values in $Q\widetilde{\mathcal{F}}$ locally given by

$$\varphi=\varphi^{\bar u}\otimes\frac{\delta}{\delta x^{\bar u}}+\varphi_{\bar v}\otimes\frac{\partial}{\partial p_{\bar v}} \tag{3.5}$$

then following [1], [2], we can define an exterior differential $D\varphi$ putting

$$D\varphi=\left(d\varphi^{\bar u}+\varphi^{\bar v}\wedge\omega_{\bar v}^{\bar u}\right)\otimes\frac{\delta}{\delta x^{\bar u}}+\left(d\varphi_{\bar\gamma}-\varphi_{\bar v}\wedge\omega_{\bar\gamma}^{\bar v}\right)\otimes\frac{\partial}{\partial p_{\bar\gamma}}. \tag{3.6}$$

A straightforward calculus show that the above formula is well-defined.

The bundle $Q^*\widetilde{\mathcal{F}}\otimes Q\widetilde{\mathcal{F}}$ admits a natural section η given by

$$\eta=dx^{\bar u}\otimes\frac{\partial}{\partial x^{\bar u}}+dp_{\bar v}\otimes\frac{\partial}{\partial p_{\bar v}}=dx^{\bar u}\otimes\frac{\delta}{\delta x^{\bar u}}+\delta p_{\bar v}\otimes\frac{\partial}{\partial p_{\bar v}}. \tag{3.7}$$

It is clear that the form η is well-defined.

The form $\theta=D\eta$ is called the torsion form *of the connection* ∇^h or its dual ∇^v.

Locally the form θ can be expressed as follows:

$$D\eta=(dx^{\bar u}\wedge\omega_{\bar u}^{\bar\gamma})\otimes\frac{\delta}{\delta x^{\bar\gamma}}+(d(\delta p_{\bar v})-\delta p_{\bar\gamma}\wedge\omega_{\bar v}^{\bar\gamma})\otimes\frac{\partial}{\partial p_{\bar v}}=\theta^{\bar\gamma}\otimes\frac{\delta}{\delta x^{\bar\gamma}}+\theta_{\bar v}\otimes\frac{\partial}{\partial p_{\bar v}}, \tag{3.8}$$

where

$$\theta^{\bar\gamma}=\frac{1}{2}\left(H_{\bar\delta\bar u}^{\bar\gamma}-H_{\bar u\bar\delta}^{\bar\gamma}\right)dx^{\bar u}\wedge dx^{\bar\delta}-\Gamma_{\bar u}^{\bar\gamma\bar\delta}dx^{\bar u}\wedge\delta p_{\bar\delta}, \tag{3.9}$$

$$\theta_{\bar{v}} = -dN_{\bar{\gamma}\bar{v}} \wedge dx^{\bar{\gamma}} - H^{\bar{u}}_{\bar{\delta}\bar{v}}\delta p_{\bar{u}} \wedge dx^{\bar{\delta}} - \frac{1}{2}\left(\Gamma^{\bar{u}\bar{\delta}}_{\bar{v}} - \Gamma^{\bar{\delta}\bar{u}}_{\bar{v}}\right)\delta p_{\bar{u}} \wedge \delta p_{\bar{\delta}}. \tag{3.10}$$

The first term and the last one in formulas (3.9) and (3.10) respectively give two global transverse tensors that we call *horizontal torsion* and *vertical torsion* respectively.

Using formulas (3.9) and (3.10), it is easy to check that

a) the horizontal torsion vanishes iff $\theta\,(V,W) = 0$, $(\forall)V,W \in \Gamma\,(H(Q^*\mathcal{F}))$ and

b) the vertical torsion vanishes iff $\theta\,(V,W) = 0$, $(\forall)V,W \in \Gamma\,(V(Q^*\mathcal{F}))$.

If ∇^h and ∇^v are dual and a horizontal and a vertical transverse connection respectively, then, according to Proposition 2.3, they project to two projectable Bott connections $\bar{\nabla}^h$ and $\bar{\nabla}^v$ respectively.

We say that the *horizontal and vertical torsions* of ∇^h and ∇^v are just the horizontal and vertical torsions of $\bar{\nabla}^h$ and $\bar{\nabla}^v$ respectively.

Proposition 3.2. *If $\tilde{g}$ is a non-degenerated and symmetric transverse bilinear form in the fibers of $H(Q^*\mathcal{F})$ and $\tilde{N}$ is a Bott type nonlinear connection of $Q\widetilde{\mathcal{F}}$, then there is a unique projectable Bott linear connection ∇^h, in the horizontal bundle $H(Q^*\mathcal{F})$, such that*

1) ∇^h has null horizontal and vertical torsions and

2) $\tilde{g}$ is parallel with respect to ∇^h, i.e. $\nabla^h_X\tilde{g} = 0$, $(\forall)X \in \mathcal{X}(Q\widetilde{\mathcal{F}})$.

Proposition 3.3. *If g is a non-degenerated and symmetric transverse bilinear form in the fibers of $V(Q^*\mathcal{F})$ and N is a Bott type nonlinear connection of $Q\widetilde{\mathcal{F}}$, then there is a unique projectable Bott linear connection ∇^v, in the vertical bundle $V(Q^*\mathcal{F})$, such that*

1) ∇^v has null horizontal and vertical torsions and

2) g is parallel with respect to ∇^v, i.e. $\nabla^v_X g = 0$, $(\forall)X \in \mathcal{X}(Q\widetilde{\mathcal{F}})$.

Proof. It can be easily inferred that the hypothesis above imply that all the hypothesis of Proposition 3.2 are in fact fulfilled for $\tilde{g}$; thus, using its conclusion by duality, the final conclusions of our statement follow for g. □

The Propositions 3.2 and 3.3 have special forms in the case of a regular transverse Hamiltonian H or a Cartan metric K^2.

Let us suppose that the foliation $\mathcal{F}$ has a regular transverse Hamiltonian $H : Q^*\mathcal{F} \to I\!R$; it reads that H is a basic function for $\widetilde{\mathcal{F}}$ and its transverse Hessian h is a transverse non-degenerated bilinear form in the fibers of $V(Q^*\mathcal{F})$. The Hessian of H on $H(Q^*\mathcal{F})$ is, by its definition, the inverse h^{-1} of the Hessian h on $V(Q^*\mathcal{F})$.

Then, according to Proposition 3.1, H gives rise to a transverse nonlinear connection $\bar{N}$ in $Q^*\mathcal{F}$.

The Propositions 3.2 and 3.3 become in this case as follows, improving [2, Theorem 3.1].

Theorem 3.4. *If H is a regular transverse Hamiltonian, then there are unique projectable Bott linear connections ∇^h and ∇^v, in the horizontal bundle $H(Q^*\mathcal{F})$ and the vertical bundle $V(Q^*\mathcal{F})$ respectively, such that*

1) ∇^v and ∇^h have null horizontal and vertical torsions and

2) the Hessians of H are parallel with respect to ∇^h and ∇^v, i.e. $\nabla^h_X h^{-1} = 0$ and $\nabla^v_X h = 0$, $(\forall) X \in \mathcal{X}(Q\widetilde{\mathcal{F}})$, in the fibers of $H(Q^*\mathcal{F})$ and $V(Q^*\mathcal{F})$, respectively.

If the vertical hessian h is positively defined on $Q^*\mathcal{F}$, and H is differentiable on $Q^*\mathcal{F}$ or on the slashed $Q^*_*\mathcal{F} = Q^*\mathcal{F}\backslash o(M)$, we obtain a transverse Riemannian metric for the foliation $\widetilde{\mathcal{F}}$ on the manifold $Q^*\mathcal{F}$ or for the foliation $\widetilde{\mathcal{F}}_*$ on the manifold $Q^*_*\mathcal{F}$ respectively.

Proposition 3.5. *If H is differentiable on $Q^*\mathcal{F}$ or on the slashed $Q^*_*\mathcal{F}$ and the vertical hessian h is positively defined, then the foliated manifold $\left(Q^*\mathcal{F}, \widetilde{\mathcal{F}}\right)$ or $\left(Q^*_*\mathcal{F}, \widetilde{\mathcal{F}}_*\right)$ respectively is Riemannian.*

According to [8], we say that H is allowed if:

1) H is continuous on $Q^*\mathcal{F}$, differentiable on the slashed $Q^*_*\mathcal{F}$, positively defined (i.e. its vertical hessian is positively defined) and $H(x,p) \geq 0 = H(x,0)$, $(\forall)x \in M$ and $p \in Q^*_x\mathcal{F}$;

2) H is locally projectable on a transverse Hamiltonian;

3) there is a basic function $\varphi : M \to (0,\infty)$, such that for every $x \in M$ there is $p \in Q^*_x\mathcal{F}$ such that $H(x,p) = \varphi(x)$.

If a positively transverse Hamiltonian H is 2–homogeneous (i.e. $H(x,\lambda p) = \lambda^2 H(x,p)$, $(\forall)\lambda > 0$), then H is called a *transverse Cartan form*; it is also a positively admissible Hamiltonian, taking $\varphi \equiv 1$, or any positive constant.

Using the results in [8] we have that the following statement is true.

Proposition 3.6. *If there is an allowed $H : Q^*\mathcal{F} \to I\!R$ (in particular a transverse Cartan form), then the foliation $\mathcal{F}$ is Riemannian.*

Notice that the lagrangian version of the above result was proved in [5], improving [2, Theorem 3.2]. Proposition 3.6 follows by duality from the lagrangian form only in the case when the dual lagrangian of H is also allowed; for example, in the case of a transverse Cartan form, when its dual is a Finslerian. In the general case, the dual Hamiltonian (lagrangian) of an allowed lagrangian (Hamiltonian) does not follows to be allowed; we have not yet a example to prove this statement, so we leave it as an open question.

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