Generalized convexity in the affine differential setting

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Abstract. Generalized convexity on differentiable manifolds was defined ([4]) in order to provide tools for optimization algorithms, working for smooth functions which are not convex from the viewpoint of Euclidean geometry or of some other Riemannian geometry. In particular, we proved ([5]) that any real valued smooth function on a differentiable manifold with *strictly convex* critical points is generalized convex (i.e. is convex with respect to some appropriate "affine differential geometry" on the manifold).

In this paper, we extend that result for *convex* critical points, using the (partial) classification of degenerate minimum points from [1]: for each parameter type in this seven-families classification, we establish if the respective function is generalized convex or not.

M.S.C. 2010: 53A15,90C25, 90C26.
Key words: affine differential convexity; generalized convexity; degenerate critical points of minimum type.

1 Introduction

For (classic) convex functions, Optimization Theory developed simple and powerful algorithms to reach a global minimum point (when there exists one). Unfortunately, the classic convexity is rare among smooth functions having a global minimum point. This was one of the reasons which led to a generalization of the (strictly) convexity of functions, from the Euclidean to the Riemannian setting (cf. Udriste [7], Rapcszak [6] for further details and references up to 2000; for newer papers, see Baker & all [2], Kristaly [3], Yang [9]). Despite the increase of the difficulty of the technical formalism, this new theory of convexity offered also pleasant surprises: many functions which are not (classic) convex become convex with respect to some properly chosen Riemannian metric. One famous example is provided by the "Rosenbrock banana function" ([7]; see also §2). The price to pay for this more general setting is the search for an appropriate Riemannian metric, which sometimes may be tedious.

BSG Proceedings 21. The International Conference "Differential Geometry - Dynamical Systems" DGDS-2013, October 10-13, 2013, Bucharest-Romania, pp. 156-166.

In [4], we extended the Riemannian convexity of functions, in the more general affine differential setting: the Hessian operator is constructed using an arbitrary linear connection (instead of the Levi-Civita one) and the geodesic links are replaced by auto-parallel curves links. Given a real valued smooth function on a differentiable manifold, one looks for a linear connection which "makes" it generalized (strictly) convex.

For the sake of completeness, we expose the main affine differential tools needed to understand the (affine differential) generalized (strictly) convexity (§2). We recall several examples of (classic) non-convex functions which are generalized convex ([4]). These examples were the starting point for the following result, proved by us in [5]: *Let f be a real valued smooth function on a differentiable manifold M.Suppose f is regular or has only strictly convex critical points. Then there exists a linear connection with respect to which f is convex.*

In this paper, we prove an extension of this result (Theorem 3.3): for all real valued smooth functions with a degenerate minimum point (up to a specific stability condition rank), we give necessary and sufficient conditions for the general convexity. The proof rests on the classification of degenerate minimum points given by Vasiliev in [8]; moreover, under the reduction operated by that quoted classification (modulo a coordinate change and up to a positive definite quadratic form), we find - for each generalized convex function f - *an explicit linear connection* with respect to which f is convex.

2 Generalized convexity in Affine Differential Geometry

Consider a differentiable manifold M and $\mathcal{F}(M)$ the algebra of the real valued smooth (i.e. C^∞-differentiable) functions on M. Denote by $\mathcal{X}(M)$ the $\mathcal{F}(M)$-module of vector fields on M and by $\mathcal{C}(M)$ the set of linear connections on M. We recall that a linear connection $\nabla \in \mathcal{C}(M)$ is an operator from $\mathcal{X}(M) \times \mathcal{X}(M)$ to $\mathcal{X}(M)$, $\mathcal{F}(M)$-linear in the first argument, $\mathbb{R}$-linear in the second argument and, for each function $f \in \mathcal{F}(M)$ and for each vector fields X, $Y \in \mathcal{X}(M)$, we have

$$\nabla_X fY = f\nabla_X Y + df(X)Y.$$

Each linear connection ∇ defines an *affine differentiable structure* on M. For $f \in \mathcal{F}(M)$, the Hessian operator with respect to ∇ is a (0,2)-tensor field, defined by

$$H_f^\nabla(X,Y) = (\nabla_X df)(Y).$$

We say $f \in \mathcal{F}(M)$ is ∇-*generalized convex* (respectively ∇-*generalized strictly convex*) in $x_0 \in M$ if its Hessian H_f^∇ is positive semidefinite (respectively positive definite) in x_0. In this case, we say x_0 is a *convex* (respectively *strictly convex) point of* f. A ∇-*generalized convex function on* M must be ∇-generalized convex in all points of M.

A function $f \in \mathcal{F}(M)$ is called *generalized convex* (respectively *generalized strictly convex*) if there exists a linear connection ∇ on M, with respect to which f becomes ∇-*generalized convex* (respectively ∇-*generalized strictly convex*) on M. (These definitions of convexity differ slightly from those in [7]). A function f on M is called

Riemannian convex if it is generalized convex with respect to some (Levi-Civita connection associated to a) Riemannian metric on M.

Remarks 2.1. (i) Any real valued smooth convex function on $\mathbb{R}^n$ is obviously generalized convex, with respect to the canonical (Euclidean) linear connection. Any regular function f on a differentiable manifold is generalized convex, with respect to some linear connection. Moreover, this linear connection may be chosen such that the Hessian of f identically vanishes ([7]).

In local coordinates (x^1,...,x^n), the components of a linear connection are smooth functions Γ^i_{jk}, $i, j, k \in \{1, ..., n\}$, given by

$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} = \Gamma^i_{jk} \frac{\partial}{\partial x^i}.$$

The components of the Hessian (with respect to ∇) of a smooth function f are

$$H^{\nabla}_{ij} = \frac{\partial f}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial f}{\partial x^k}.$$

(ii) There are many functions which are generalized convex without being Riemanian convex. For example, any generalized convex and non-constant function on compact manifolds cannot be Riemannian convex. The explanation is that the Riemannian convexity implies the sub-harmonicity, property forbidden by the theorem of H. Hopf (or, more generally, by the maximum principle from harmonic analysis).

(iii) Consider $\nabla \in C(M)$ and ∇^s its symmetric connection. A function f is ∇-*generalized convex* (respectively ∇-*generalized strictly convex*) if and only if it is ∇^s-*generalized convex* (respectively ∇^s-*generalized strictly convex*). Therefore, it suffices to study the generalized convexity (and the generalized strictly convexity) with respect to symmetric linear connections only.

(iv) If a function f on a differentiable manifold M is *locally* generalized convex (i.e. with respect to locally defined linear connections), then a standard argument, using the partition of unity, ensures the global generalized convexity on M: a global linear connection is obtained by gluing the local connections and its Hessian is a linear combination of the local Hessians. This transition from the local convexity to the global one does not work for the Riemannian (and, in particular, the classic Euclidian) convexity.

(v) In [4] we considered several functions admitting only critical points which are global minimum ones and proved that their lack of convexity was only apparent, because (even if they were not classic convex) they were generalized convex in an appropriate affine differential geometry. We recall briefly the respective examples (for details, including also some pictures, see [4]).

The function $f_1 : \mathbb{R}^2 \to \mathbb{R}$, $f_1(x, y) = x^4 + y^4 - 6(x^3 + y^3) + 14(x^2 + y^2)$ is convex (in a classic sense) and, obviously, has only one (global) minimum point (0,0).

We slightly modify f_1 to $f_2(x, y) = x^4 + y^4 - 6(x^3 + y^3) + 12(x^2 + y^2)$, which is not (classic) convex anymore. However, the minimum point property remains the same as for f_1. In global coordinates on $\mathbb{R}^2$, we found ([4]) a linear connection ∇, with respect to which f_2 is (affine differential) strictly convex.

The (classic) convexity loss becomes more evident for $f_3(x, y) = x^4 + y^4 - 6.3(x^3 + y^3) + 12(x^2 + y^2)$ whose graph has a bigger "bump" on one side. Of course, the (global) minimum point property is stable with respect to these "bumps".

With respect to the same connection ∇ as for f_2, the function f_3 is also strictly convex. (Interestingly, when the "bump" grows, as for 6.54 instead of 6.3, the perturbed function cannot be made generalized convex, no matter how we choose the linear connection; this situation occurs because the new function also gets local maximum points).

The smooth functions having all the critical points of minimum type are likely to be "made convex", by choosing an appropriate linear connection. This fact is justified by the

Theorem 3.1. ([5]) *Let f be a real valued smooth function on a differentiable manifold M. Suppose f is regular or has only critical points which are all strictly convex. Then f is generalized convex.*

The hypothesis of the theorem cannot be weakened: in fact, let consider the functions $f_5, f_6, f_7 : \mathbb{R}^2 \to \mathbb{R}$, given by $f_5(x, y) = x^3 + y^3$; $f_6(x, y) = sin(x^2 + y^2)$; $f_7(x, y) = x^4 + ax^2y^2 + y^4$, with $a \in \mathbb{R}$.

The first function has an unique critical non-extremum point (0,0); the function f_6 has both minimum and maximum points; for $a \in (-2, 0)$, f_7 has a degenerate minimum point (0,0). All these functions fail to be generalized convex ([5]). (In the next section, we will see how the convexity properties of f_7 vary with respect to other values of the parameter a).

3 Necessary conditions for convex critical points to ensure the (global) generalized convexity

We are looking for an invariant able to decide whether the degeneracy of a critical point forbids or enables the generalized convexity. In order to obtain some hints, we shall use the classification of the degenerate critical points of Vasiliev ([8], [1]) and analyse each class in detail.

Consider a (germ of a) differentiable function in a neighborhood of a minimum point. By means of a (local differentiable) change of coordinates, it is possible to reduce the function to a normal form (modulo the addition of a constant and of a positive definite quadratic form); the classification of the normal forms is the following ([1], p.234):

(I) A_{2k-1}; $f(x, y) = x^{2k}$, $k \geq 1$; $(l = 2k - 2)$;

(II) $X_{1,0}$; $f(x, y) = x^4 + ax^2y^2 + y^4$, $a > -2$, $a \neq 2$; $(l = 7)$;

(III) $X_{1,2r}$; $f(x, y) = x^4 + x^2y^2 + ay^{4+r}$, $a > 0$, $r \geq 1$; $(l = 7 + 2r)$;

(IV) $Y^1_{2r,2q}$; $f(x, y) = x^{4+2r} + ax^2y^2 + y^{4+2q}$, $a > 0$, $r, q \geq 1$;$(l = 7 + 2r + 2q)$;

(V) $\tilde{Y}^1_{r,r}$; $f(x, y) = (x^2 + y^2)^2 + ay^{4+r}$, $a \neq 0$, $r \geq 1$;$(l = 7 + 2r)$;

(VI) $W_{1,0}$; $f(x, y) = x^4 + (a + by)x^2y^3 + y^6$, $a^2 < 4$;$(l = 12)$;

(VII) $Y^{\#}_{1,2q}$; $f(x, y) = (x^2 + y^3)^2 + (a + by)x^2y^{3+q}$, $a(-1)^q < 0$, $q > 1$;$(l = 12 + 2q)$.

This classification works for $l < 16$, where l is an invariant related to some kind of stability: it shows the number of parameters to start with, such that the minimum points of the given class cannot be supressed by using a small deformation of the family.

Lemma 3.2. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function having (0,0) as degenerate minimum point; suppose f belongs to one of the classes from Vasiliev's classification. Then there exists a small enough open neighbourhood of (0,0) such that on it:*

(i) $f \in \mathrm{I}$ *is classic convex.*

(ii) $f \in \mathrm{II}$ *is: classic convex for $a \in [0, 2) \bigcup (2, 6]$; not generalized convex for $a \in (-2, 0)$ and for $a > 6$.*

(iii) $f \in \mathrm{III}$ *is generalized convex but not classic convex.*

(iv) $f \in \mathrm{IV}$ *is not generalized convex.*

(v) $f \in \mathrm{V}$ *is classic convex.*

(vi) $f \in \mathrm{VI}$ *is: classic convex for $a = b = 0$; not generalized convex for a=0 and $b < 0$, or $a \neq 0$; generalized convex but not classic convex for $a = 0$ and $b > 0$.*

(vii) $f \in \mathrm{VII}$ *is not generalized convex.*

Proof. We consider successively the classic and/or affine differential Hessians of f; the proof is quite elementary, by a case-by-case check. Some arguments may sound redundant, but they are intended to stress the difference between the classic convexity and the generalized one.

We saw that, without restaining the generality, we may suppose all the considered linear connections being symmetric, hence the resulting Hessians are also symmetric.

i) The function $f(x, y) = x^{2k}$ $(k \geq 1)$ is obviously classic convex on $\mathbb{R}^2$.

ii) Consider $f(x, y) = x^4 + ax^2y^2 + y^4$ with $a > -2, a \neq 2$; its (classic) Hessian is

$$H_f(x, y) = \begin{pmatrix} 12x^2 + 2ay^2 & 4axy \\ 4axy & 12y^2 + 2ax^2 \end{pmatrix}$$

For $a \in (-2, 0)$, let ∇ be a linear connection around (0,0), with the Hessian H_f^∇; the coefficients of H_f^∇ are:

$$(H_f^\nabla)_{11}(x, y) = 12x^2 + 2ay^2 - \Gamma_{11}^1(x, y)(4x^3 + 2axy^2) - \Gamma_{11}^2(x, y)(4y^3 + 2ax^2y)$$

$$(H_f^\nabla)_{12}(x, y) = 4axy - \Gamma_{12}^1(x, y)(4x^3 + 2axy^2) - \Gamma_{12}^2(x, y)(4y^3 + 2ax^2y) \tag{3.1}$$

$$(H_f^\nabla)_{22}(x, y) = 12y^2 + 2ax^2 - \Gamma_{22}^1(x, y)(4x^3 + 2axy^2) - \Gamma_{22}^2(x, y)(4y^3 + 2ax^2y).$$

Suppose ad absurdum that H_f^∇ is positive semidefinite. Then, its determinant must be non-negative in all points. In particular $det(H_f^\nabla)(0, y) \geq 0$; for $y \neq 0$, we simplify by y^4 and get

$$3a \geq 6\Gamma_{11}^2(0, y)y + a\Gamma_{22}^2(0, y)y - 2\Gamma_{11}^2(0, y)\Gamma_{22}^2(0, y)y^2 + 2\Gamma_{12}^2(0, y)\Gamma_{21}^2(0, y)y^2.$$

The leftside part is negative and the rightside is converging to 0 as y converges to 0, an obvious contradiction ! Therefore, in this case, f is not generalized convex around the point (0,0).

For $a \in [0,2) \bigcup (2,6]$, the function is classic convex on $\mathbb{R}^2$, as $H_{11} = 12x^2 + 2ay^2 \geq 0$, $H_{22} = 12y^2 + 2ax^2 \geq 0$ and $det H_f(x,y) = 12[2a(x^2-y^2)^2 + (-a^2+4a+12)x^2y^2] \geq 0$.

For $a > 6$, we have $H_{11} \geq 0$ and $det H_f(x,y) = 12[2a(x^2 - y^2)^2 + (-a^2 + 4a + 12)x^2y^2]$, with $-a^2 + 4a + 12 < 0$. We make $y := x$ and derive $det H_f(x,x) = 12(-a^2 + 4a + 12)x^4 \leq 0$, so f is not classic convex around (0,0).

We will see that f is not even generalized convex. Suppose ad absurdum there exists a linear connection ∇ on $\mathbf{R^2}$, such that (H_f^{∇}) is positive definite, at least locally around $(0,0)$. In particular, $det(H_f^{\nabla})(x,x) \geq 0$, for every x in a small enough neighborhood U of $0 \in \mathbb{R}$.

From (3.1), this is equivalent to

$$(6 + a - \Gamma_{11}^1(x,x)(2+a)x - \Gamma_{11}^2(x,x)(2+a)x)(6 + a - \Gamma_{22}^1(x,x)(2+a)x -$$

$$-\Gamma_{22}^2(x,x)(2+a)x) - (2a - \Gamma_{12}^1(x,x)(2+a)x - \Gamma_{12}^2(x,x)(2+a)x)^2 \geq 0.$$

When x converges to 0, we get $a^2 - 4a - 12 \leq 0$, which is impossible for $a > 6$. So, for $a > 6$, f is not generalized convex around $(0,0) \in \mathbb{R}^2$.

iii) A simple calculation leads to

$$H_f(x,y) = \begin{pmatrix} 12x^2 + 2y^2 & 4xy \\ 4xy & a(4+r)(3+2r)y^{2+r} + 2x^2 \end{pmatrix}.$$

Denote $\alpha = 2a(4+r)(3+r) > 0$ and obtain

$$det H_f(x,y) = 24x^4 - 12x^2y^2 + 6\alpha \cdot x^2y^{2+r} + \alpha \cdot y^{4+r}.$$

We try to find points (x,y) where $det H_f(x,y) < 0$, so we are looking for solutions of the form (x_0, y_0), with $y_0 = bx_0$ and $\mid x_0 \mid$, $\mid y_0 \mid$ sufficiently small.

In order to solve the inequation

$$24x^4 - 12b^2x^4 + 6\alpha x^2 b^{2+r} x^{2+r} + \alpha x^{r+4} b^{r+4} < 0,$$

we divide by x^4 and obtain

$$x^r < \frac{12b^2 - 24}{\alpha b^{r+2}(6 + b^2)}.$$

Choosing $b > \sqrt{2}$, there exists a solution x_0 of this inequation, with absolute value arbitrarily small. We take $y_0 := bx_0$. Hence we conclude that:

Every neighborhood of $(0,0)$ *contains a point* (x_0, y_0) *such that* $det H_f(x_0, y_0) < 0$. Moreover, as a consequence, we obtain:

$(0,0)$ *is a degenerate critical point for* f*, which does not admit any neighborhood where* f *is classic convex.*

Nevertheless, we shall prove that f *is generalized convex on* $\mathbb{R}^2$. For a linear conection ∇, the coefficients of the Hessian are

$$(H_f^{\nabla})_{11}(x,y) = 12x^2 + 2y^2 - \Gamma_{11}^1(x,y)(4x^3 + 2xy^2) -$$
$$-\Gamma_{11}^2(x,y)[a(4+2r)y^{3+2r} + 2x^2y]$$
$$(H_f^{\nabla})_{12}(x,y) = 4xy - \Gamma_{12}^1(x,y)(4x^3 + 2xy^2) - \Gamma_{12}^2(x,y)[a(4+2r)y^{3+2r} + 2x^2y]$$
$$(H_f^{\nabla})_{22}(x,y) = a(4+2r)(3+2r)y^{2+2r} + 2x^2 - \Gamma_{22}^1(x,y)(4x^3 + 2xy^2) -$$
$$-\Gamma_{22}^2(x,y)[a(4+2r)y^{3+2r} + 2x^2y].$$

We choose $\Gamma_{12}^1 = \Gamma_{12}^2 = \Gamma_{21}^1 = \Gamma_{21}^2 = 0$, $\Gamma_{11}^1(x,y) = \Gamma_{22}^1(x,y) = -x$, $\Gamma_{11}^2(x,y) = \Gamma_{22}^2(x,y) = -y$.
Then

$$det(H_f^{\nabla}) = (H_f^{\nabla})_{11}(x,y)(H_f^{\nabla})_{22}(x,y) - 16x^2y^2$$

and $(H_f^{\nabla})_{11}(x,y) \geq 4x^2$, $(H_f^{\nabla})_{22}(x,y) \geq 4y^2$. Therefore, $detH^{\nabla}{}_f(x,y) \geq 0$.

iv) We calculate the classic Hessian

$$H_f(x,y) = \begin{pmatrix} (4+2r)(3+2r)x^{2+2r} + 2ay^2 & 4axy \\ 4axy & (4+2q)(3+2q)y^{2+2q} + 2ax^2 \end{pmatrix}.$$

Obviously, $H_{11} \geq 0$; for $\mid x \mid$ sufficiently small,

$$detH_f(x,x) = x^4[2a(4+2r)(3+2r)x^{2r} + 2a(4+2q)(3+2q)x^{2q} +$$
$$+(4+2r)(3+2r)(4+2q)(3+2q)x^{2r+2q} - 12a^2]$$

becomes negative. We conclude that f is not classic convex around (0,0).

Moreover, we shall prove that f *is not generalized convex* around (0,0). Assume ad absurdum that there exists a linear connection ∇, such that $(H_f^{\nabla})_{11} \geq 0$, $(H_f^{\nabla})_{22} \geq 0$ and $det(H_f^{\nabla})(x,y) \geq 0$ for all $(x,y) \in \mathbb{R}^2$. We divide $det(H^{\nabla}{}_f)(x,x)$ by x^4(for $x \neq 0$) and get

$$\{(4+2r)(3+2r)x^{2r} + 2a - \Gamma_{11}^1[(4+2r)x^{1+2r} + 2ax] -$$
$$-\Gamma_{11}^2[(4+2q)x^{1+2q} + 2ax]\} \cdot \{(4+2q)(3+2q)x^{2q} + 2a -$$
$$-\Gamma_{22}^1[(4+2r)x^{1+2r} + 2ax] - \Gamma_{22}^2[(4+2q)x^{1+2q} + 2ax]\} \geq$$
$$\geq \{4a - \Gamma_{12}^1[(4+2r)x^{1+2r} + 2ax] - \Gamma_{12}^2[(4+2q)x^{1+2q} + 2ax]\}^2.$$

Now, for $x \to 0$, we obtain $4a^2 \geq 16a^2$, which is impossible. Therefore, in this case, f is not generalized convex.

v) The classic Hessian of f is

$$H_f(x,y) = \begin{pmatrix} 12x^2 + 4y^2 & 8xy \\ 8xy & 4x^2 + 12y^2 + (4+r)(3+r)ay^{2+r} \end{pmatrix}$$

and has the determinant

$$detH_f = 48x^4 + 48y^4 + 96x^2y^2 + 4a(4+r)(3+r)(3x^2+y^2)y^{2+r}.$$

For $a > 0$ and r even, f is obviously classic convex on $\mathbb{R}^2$.

Consider $a > 0$ and r odd; then f has as critical points: (0,0) and (0,y_0), where $y_0 = -\sqrt[r]{\frac{4}{(4+r)a}}$). Then

$$detH_f(0, y_0) = -16r \left[\frac{4}{(4+r)a}\right]^{\frac{4}{r}} < 0,$$

so f is not classic convex on $\mathbb{R}^2$. Moreover, the Hessian in the critical point (0,y_0) does not depend on the connection used to construct it; this implies that f is not generalized convex on $\mathbb{R}^2$ either.

However, $(H_f)_{11} \geq 0$, $(H_f)_{22} \geq 0$ and $detH_f \geq 0$ for small enough x^2 and y^2, so f *is classic convex* around (0,0).

A similar argument proves that, for $a < 0$, f *is classic convex* around (0,0).

vi) The classic Hessian is

$$H_f(x, y) = \begin{pmatrix} 12x^2 + 2(a + by)y^3 & 2(3a + 4by)xy^2 \\ 2(3a + 4by)xy^2 & 30y^4 + 6(a + 2by)x^2y \end{pmatrix}.$$

In particular, we have

$$H_f(0, y) = \begin{pmatrix} 2(a + by)y^3 & 0 \\ 0 & 30y^4 \end{pmatrix}.$$

For $a \neq 0$, f is not classic convex on $\mathbb{R}^2$ (and neither on an arbitrarly small neighborhood of (0,0)).

For $a = 0, b < 0$, $f = x^4 + bx^2y^4 + y^6$ and it is not classic convex around (0,0), because

$$H_f(0, y) = \begin{pmatrix} 2by^4 & 0 \\ 0 & 30y^4 \end{pmatrix}$$

is, obviously, not positive definite.

For $a = 0, b = 0$, $f = x^4 + y^6$ is classic convex on $\mathbb{R}^2$.

If $a = 0, b > 0$, then $\frac{1}{4}H_f(x, y) = (36by^2)x^4 + 10y^4(9 - b^2y^2)x^2 + 15by^8$ is a quadratic form in x, with

$$\Delta = 4y^8[5(9 - b^2y^2) + 6\sqrt{15}by][5(9 - b^2y^2) - 6\sqrt{15}by],$$

which has not constant sign on $\mathbb{R}$; so, $H_f(x, y)$ is not positive definite on $\mathbb{R}^2$.
We conclude that: for $a \in (-2, 2)$ and $b \in \mathbb{R}$, f is not classic convex; for $a = b = 0$, f is classic convex.

Studying the generalized convexity of f, we distinguish the following cases:

For $a = 0, b \geq 0$: we choose the linear connection with the coefficients ${\Gamma^1}_{12} = {\Gamma^2}_{12} = 0$, ${\Gamma^1}_{11} = {\Gamma^2}_{11} = -x$, ${\Gamma^2}_{11} = {\Gamma^2}_{22} = -y$ and f is ∇-convex.

For $a = 0, b < 0$: we assume that exists a linear connection $\nabla \in C(\mathbb{R}^2)$ so that f is ∇- convex. The condition $detH_f^{\nabla}(x, y) \geq 0$ leads to

$$\{12x^2 + 2by^4 - \Gamma^1_{11}[4x^3 + 2bxy^4] - \Gamma^2_{11}[6y^5 + 4bx^2y^3]\} \cdot \{30y^4 +$$

$$+12bx^2y^2 - \Gamma^1_{22}[4x^3 + 2bxy^4] - \Gamma^2_{22}[6y^5 + 4bx^2y^3]\} \geq$$
$$\geq \{8bxy^3 - \Gamma^1_{12}[4x^3 + 2bxy^4] - \Gamma^2_{12}[6y^5 + 4bx^2y^3]\}^2, \ \forall (x,y) \in \mathbb{R}^2.$$

Taking $x = 0$ and dividing by y^8(for $y \neq 0$), we obtain

$$[2b - \Gamma^2_{11} \cdot 6y] \cdot [30 - \Gamma^2_{22} \cdot 6y] \geq (\Gamma^1_{12})^2 \cdot 36y^2.$$

Making $y \to 0$, we reach to $2b \cdot 30 \geq 0$, which is false($b < 0$). Therefore, in this case, f is not generalized convex around (0,0).

For $a \neq 0$: we assume that exists $\nabla \in C(\mathbb{R}^2)$ so that f ∇- convex. Then, $detH^\nabla_f(x,y) \geq 0$ for all $(x,y) \in \mathbb{R}^2$. But

$$(H^\nabla_f(0,y))_{11} = y^3(2a + by) - 6\Gamma^2_{11}y^5.$$

Consider a small enough $\epsilon > 0$. Then $(H^\nabla_f(0,\epsilon))_{11}$ has the same sign as a and $(H^\nabla_f(0,-\epsilon))_{11}$ has opposite sign. This tells us that there is no neighborhood of $(0,0)$, however small, where $(H^\nabla_f)_{11} \geq 0$. Therefore, f is not generalized convex around (0,0).

vii) The classic Hessian of f is

$$(H_f(x,y))_{11} = 12x^2 + 4y^3 + 2(a + by)y^{3+q}$$
$$(H_f(x,y))_{12} = 12xy^2 + 2xa(3 + q)y^{2+q} + 2xb(4 + q)y^{3+q}$$
$$(H_f(x,y))_{22} = 12x^2y + 30y^4 + x^2a(3 + q)(2 + q)y^{1+q} + x^2b(4 + q)(3 + q)y^{2+q}.$$

In particular,

$$H_f(0,y) = \begin{pmatrix} 4y^3 + 2(a + by)y^{3+q} & 0 \\ 0 & 30y^4 \end{pmatrix}.$$

The function $(H_f)_{11}(0,y) = y^3(4 + 2ay^q + 2by^{q+1})$ has different sign for $y > 0$ and $y < 0$, sufficiently close to 0. So, f is not classic convex around (0,0).

We prove that f is not generalized convex: suppose a linear connection ∇ exists, such that f has H^∇_f positive definite. But

$$(H^\nabla_f)_{11}(0,y) = y^3(4 + 2(a + by)y^q - 6y^2\Gamma^2_{11}).$$

For $y \to 0$, $(detH^\nabla_f)(0,y)$ takes different sign values. We conclude that f is not generalized convex around (0,0). □

Theorem 3.3. *Let M be an n-dimensional differentiable manifold ($n \geq 2$) and $f \in F(M)$, with an unique degenerate critical point of minimum type x_0, satisfying the stability condition $l < 16$. If the reduction of f through the classification of Vasiliev belongs to the families I, II (for $a \in [0,6]$), III, V (for $a > 0$ and r even), VI (for a=0 and $b \geq 0$), then f is generalized convex.*

Proof. Consider an open neighborhood U of x_0 in M and a (local) diffeomorphism ϕ from U onto $V = (-1,1) \times (-1,1) \times \mathbb{R}^{n-2}$, such that $\phi(x_0) = 0$ and the reduction $\tilde{f}$ of f in the classification of Vasiliev has one of the normal forms I-VII , modulo a positive definite quadratic form. Without restraining the generality, we may suppose that $f \circ \phi^{-1}(x^1, x^2, ..., x^n) = \tilde{f}(x^1, x^2) + (x^3)^2 + ... + (x^n)^2$. By hypothesis, $\tilde{f}$ is generalized convex on V. The (function associated with the) quadratic form is convex on $\mathbb{R}^{n-2}$; the product of two generalized convex functions is generalized convex, so $f \circ \phi^{-1}$ is generalized convex on V. Unlike the classic or the Riemannian convexity, the generalized (affine differential) convexity is invariant under local diffeomorphisms; it follows that f is generalized convex on U.

Consider now an open neighborhood W of x_0, whose closure is included in U. On $M \setminus \{x_0\}$, f is regular, so f is generalized convex on $M \setminus W$. Using a suitable partition of unity, we obtain that f is generalized convex on M. □

Combining Theorem 3.3. and Theorem 3.1., we get the

Corollary 3.4. *Let M be an n-dimensional differentiable manifold ($n \geq 2$) and $f \in F(M)$, regular or having (only) critical points of at least one of the following two kinds:*
(i) *strictly convex;*
(ii) *degenerate of minimum type, satisfying the stability condition $l < 16$ and (modulo the reduction through the classification of Vasiliev) belong to the families I, II (for $a \in [0,6]$), III, V (for $a > 0$ and r even), VI (for a=0 and $b \geq 0$).*
Then, f is generalized convex.

Proof. Around each critical point, f is generalized convex, with respect to appropriate locally defined linear connections. Using the partition of unity, we glue all these connections to a global one, making f generalized convex on M. □

4 Conclusions

It is not surprising that the generalized convexity "availability" differs very much from strictly convex critical points to convex (minimum) critical points; in fact, Morse theory distinguishes very well the "order" in the behavior of non-degenerate critical points from the "jungle" of degenerate ones. In our paper, this difference is exemplified by the Theorems 3.1 and 3.3.

The next step will be a deeper journey into "the jungle" of convex (minimum) critical points, attempting to find *an invariant* able to decide which points are "convexificable" and which are not. The examples studied until now suggest an important role played by the trace of the Hessian. But, even if this object is covariant and does not depend on the choice of the linear connection in the critical points, it might fluctuate in the regular points (as an operator on $C(M)$). The invariant we are looking for must be a differential one (even if the search emerges from geometric reasons), but we believe that analytic techniques alone will not be sufficient for its discovery.

References

[1] V. Arnold, A. Varchenko, S. Goussein-Zade, *Singularities of Differentiable Applications* (in Russian), Mir, 1982.

[2] C. G. Baker, P. A. Absil, K. A. Gallivan, *An implicit trust-region method on Riemannian manifolds*, IMA Journal of Numerical Analysis **28**, 4 (2008), 665-689.

[3] A. Kristaly, *Economic Optimization Problems via Riemann-Finsler Geometry*, Ph.D. thesis, Budapest, 2010.
[4] C. L. Pripoae, G. T. Pripoae, *Differential and affine differential invariants for convexity on differentiable manifolds*, Proc. Balkan Conference on Oper. Research, Constanta May 25-28 2005, Bucuresti 2007, 165–170.
[5] C. L. Pripoae, G. T. Pripoae, *Invexity vs. generalized convexity*, BSG Proc. 18, Geometry Balkan Press, Bucharest 2011, 86-93.
[6] T. Rapcsak, *Smooth Nonlinear Optimization in* $\mathbb{R}^n$, Kluwer Acad. Publ. 1997.
[7] C. Udrişte, *Convex Functions and Optimization Methods on Riemannian Manifolds*, Kluwer Acad. Publ., 1994.
[8] V. A. Vasiliev: *Asymptotics of exponential integrals. Newton's diagram and the classification of minimal points.* Funktsional'nyi Analiz i ego Prilozhenyia **11**, 3 (1977),1-11; engl.transl. FAA 11,163-172.
[9] Y. Yang, *Globally convergent optimization algorithms on Riemannian manifolds*, J. Optim. Theory Appl. **132** (2007), 245-265.

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