On tensor products of nonnegative linear relations

Marcel Roman, Adrian Sandovici

Abstract. Certain characterizations of the Friedrichs and the Kre˘r n von–Neumann extensions of the tensor product of two nonnegative linear relations A and B in terms of the Friedrichs and the Kre˘r n–von Neumann of A and B are provided. A characterization of the extremal extensions of the tensor product of A and B is also given.

M.S.C. 2010: Primary 47A06; Secondary 47A80, 46M05.
Key words: Hilbert space, linear relation, tensor product.

1 Introduction

This note is devoted to tensor products of multi–valued linear operators (linear relations) in Hilbert spaces. The main objective is to investigate the links between the extremal extensions of the tensor product of two nonnegative linear relations and the extremal extensions of the linear relations themselves. In particular, the cases of Friedrichs and Kre˘r n–von Neumann extensions are considered.

More precisely, assume that A and B are nonnegative linear relations in the Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$, respectively. Then the tensor product $A \otimes B$ of A and B is a nonnegative linear relation in the Hilbert space $\mathfrak{H}\hat{\otimes}\mathfrak{H}$. Its closure is denoted by $A\hat{\otimes}B$. It will be proven that the Friedrichs and the Kre˘r n von–Neumann extensions of $A\hat{\otimes}B$ are given by:

$$\left(A\hat{\otimes}B\right)_F = A_F\hat{\otimes}B_F, \quad \text{and} \quad \left(A\hat{\otimes}B\right)_N = A_N\hat{\otimes}B_N.$$

Furthermore, it will be shown that, if $\widetilde{A}$ is an extremal extension of A and $\widetilde{B}$ is an extremal extension of B, then $\widetilde{A}\hat{\otimes}\widetilde{B}$ is an extremal extension of $A\hat{\otimes}B$.

The results obtained in this note extend and complete the corresponding ones in [9, 10] and they are strongly related to concepts from various concrete problems in differential geometry (see for instance [3, 6, 15, 16]). The full proofs of the obtained results will be presented elsewhere. Also, should be pointed out that these results will be applied to study the composition and the tensor product of Dirac structures on infinite dimensional spaces, cf. [12].

BSG Proceedings 21. The International Conference "Differential Geometry - Dynamical Systems" DGDS-2013, October 10-13, 2013, Bucharest-Romania, pp. 167-174.

2 Preliminary results

2.1 Some terminology

A linear relation from a Hilbert space $(\mathfrak{H}, \langle\cdot,\cdot\rangle)$ to a Hilbert space $(\mathfrak{K}, \langle\cdot,\cdot\rangle)$ is a linear subspace of the Cartesian product $\mathfrak{H} \times \mathfrak{K}$. The following self-explanatory notions domain, range, kernel, and multi-valued part of A will be used throughout the paper:

$$\operatorname{dom} A = \{\, f \in \mathfrak{H} : \, (f, f') \in A \,\}, \quad \operatorname{ran} A = \{\, f' \in \mathfrak{K} : \, (f, f') \in A \,\},$$

$$\ker A = \{\, f \in \mathfrak{H} : \, (f, 0) \in A \,\}, \quad \operatorname{mul} A = \{\, f' \in \mathfrak{K} : \, (0, f') \in A \,\}.$$

The closures of $\operatorname{dom} A$ and $\operatorname{ran} A$ in $\mathfrak{H}$ and $\mathfrak{K}$, respectively, will be denoted by $\overline{\operatorname{dom}}\, A$ and $\overline{\operatorname{ran}}\, A$. The formal inverse A^{-1} is defined as $A^{-1} = \{\, (f', f) : \, (f, f') \in A \,\}$; it is a linear relation from $\mathfrak{K}$ to $\mathfrak{H}$. Observe the following formal identities $\operatorname{dom} A^{-1} = \operatorname{ran} A$ and $\ker A^{-1} = \operatorname{mul} A$. The relation A is closed if it is closed as a subspace of $\mathfrak{H} \times \mathfrak{K}$; the closure of the relation A is the closure of the subspace A in $\mathfrak{H} \times \mathfrak{K}$. If A is closed then the subspaces $\ker A$ and $\operatorname{mul} A$ are closed. A linear relation A is the graph of an operator if and only if $\operatorname{mul} A = \{0\}$. In the present context a linear operator A from $\mathfrak{H}$ to $\mathfrak{K}$ is identified with its graph. It is said to be closable if its closure is the graph of an operator.

The adjoint of a linear relation A from $\mathfrak{H}$ to $\mathfrak{K}$ is the closed linear relation A^* from $\mathfrak{K}$ to $\mathfrak{H}$ defined by

$$A^* = \{\, (f, f') \in \mathfrak{K} \times \mathfrak{H} : \, \langle f', h \rangle = \langle f, h' \rangle \text{ for all } (h, h') \in A \,\}.$$

Observe that $(A^{-1})^* = (A^*)^{-1}$, so that $(\operatorname{dom} A)^\perp = \operatorname{mul} A^*$ and $(\operatorname{ran} A)^\perp = \ker A^*$. Clearly the double adjoint A^{**} is the closure of the relation A. A linear relation A in a Hilbert space $\mathfrak{H}$ (i.e., from $\mathfrak{H}$ to $\mathfrak{H}$) is said to be symmetric if $A \subset A^*$, or equivalently, if $\langle f', f \rangle \in \mathbb{R}$ for all $(f, f') \in A$. A linear relation A in a Hilbert space $\mathfrak{H}$ is said to be nonnegative if $\langle f', f \rangle \geq 0$ for all $(f, f') \in A$. A linear relation A in a Hilbert space $\mathfrak{H}$ is said to be selfadjoint if $A^* = A$ (so that it is automatically closed); it is said to be essentially selfadjoint if its closure is equal to A^*. For any closed linear relation A the relation $A_\infty = \{0\} \times \operatorname{mul} A$ is selfadjoint in the Hilbert space $\operatorname{mul} A$. A selfadjoint relation can always be orthogonally decomposed as $A = A_s \oplus A_\infty$ where A_s is a (densely defined) selfadjoint operator in $\mathfrak{H} \ominus \operatorname{mul} A = \overline{\operatorname{dom}}\, A$.

2.2 The core of a linear relation

Assume that A is a linear relation from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}$. A subspace $\mathfrak{D} \subset \operatorname{dom} A$ it said to be a core of A if $\overline{A \upharpoonright \mathfrak{D}} = \overline{A}$. The following result gives a characterization of the core of a linear relation.

Lemma 2.1. *Let $\mathfrak{H}$ and $\mathfrak{K}$ be Hilbert spaces and let A be a linear relation from $\mathfrak{H}$ to $\mathfrak{K}$. Assume that $\mathfrak{D}$ is a subspace of* $\operatorname{dom} A$ *having the following property: for any $(f, f') \in A$ there exists a sequence $((f_n, f'_n))_n \subset A$, $(f_n)_n \subset \mathfrak{D}$ such that $f_n \to f$, $f'_n \to f'$ when $n \to \infty$. Then $\mathfrak{D}$ is a core of $\overline{A}$.*

This result will be used in Lemma 3.1 when describing the core of tensor products of linear relations.

2.3 Friedrichs, Kreĭn–von Neumann and extremal extensions of nonnegative linear relations

Let A be a nonnegative linear relation in a Hilbert space $\mathfrak{H}$. There exist two nonnegative selfadjoint extensions A_F, the Friedrichs extension, and A_N, the Kreĭn–von Neumann extension, such that all nonnegative selfadjoint extensions $\widetilde{A}$ of A satisfy the following inequality

$$A_N \preceq \widetilde{A} \preceq A_F,$$

which is understood in the sense of the associated forms (see for instance [7]). In the language of linear relations A_F consists of all pairs $(f, f') \in A^*$ for which there exists a sequence $((f_n, f_n'))_n \subset A$ such that

$$f_n \to f, \quad \langle f_n' - f_m', f_n - f_m \rangle \to 0, \quad m, n \to \infty.$$

Analogously, the Kreĭn–von Neumann extension A_N consists of all pairs $(f, f') \in A^*$ for which there exists a sequence $((f_n, f_n'))_n \subset A$ such that

$$f_n' \to f', \quad \langle f_n' - f_m', f_n - f_m \rangle \to 0, \quad m, n \to \infty,$$

which is equivalent to $A_N = \left((A^{-1})_F \right)^{-1}$, cf. [4].

The class of extremal extensions was introduced by Yu. Arlinskiĭ in [2]. By definition, a nonnegative selfadjoint extension $\widetilde{A}$ of A is called *extremal* when

$$\inf\{ \langle f' - h', f - h \rangle : (h, h') \in A \} = 0 \quad \text{for all} \quad (f, f') \in \widetilde{A}, \tag{2.1}$$

see [2]. Clerly, the Kreĭn–von Neumann extension A_N and the Friedrichs extension A_F are extremal extensions.

2.4 A factorization of extremal extensions

Let A be a nonnegative linear relation in a Hilbert space $\mathfrak{H}$. Provide the linear space $\operatorname{ran} A$ with the semi-inner product $\langle \cdot, \cdot \rangle_R$ defined by

$$\langle f', g' \rangle_R := \langle f', g \rangle = \langle f, g' \rangle, \quad (f, f'), (g, g') \in A. \tag{2.2}$$

Define the linear space $\mathfrak{R}_0 \subset \operatorname{ran} A$ by

$$\mathfrak{R}_0 = \{ f' : \langle f', f \rangle = 0 \text{ for some} (f, f') \in A \}. \tag{2.3}$$

Then $\mathfrak{R}_0 = \operatorname{ran} A \cap \operatorname{mul} A^*$, cf. [8, Lemma 4.1]. Clearly, the quotient space $\operatorname{ran} A/\mathfrak{R}_0$ is a pre-Hilbert space with the inner product

$$\langle [f'], [g'] \rangle_S := \langle f', g \rangle = \langle f, g' \rangle, \quad (f, f'), (g, g') \in A, \tag{2.4}$$

where $[f']$, $[g']$ denote the equivalence classes containing f' and g'. Let $\mathfrak{H}_A$ be the Hilbert space completion of $\operatorname{ran} A/\mathfrak{R}_0$, whose inner product is again denoted by $\langle \cdot, \cdot \rangle_S$. The linear relation Q from $\mathfrak{H}$ to $\mathfrak{H}_A$ is defined by

$$Q = \{ (f, [f']) : (f, f') \in A \}, \tag{2.5}$$

so that actually Q is (the graph of) an operator. Define the linear relation J from $\mathfrak{H}_A$ to $\mathfrak{H}$ by

$$J = \{ ([f'], f') : (f, f') \in A \}, \tag{2.6}$$

so that $\operatorname{mul} J = \mathfrak{R}_0$. Since J is densely defined in $\mathfrak{H}_A$, J^* is an operator. The definitions (2.5) and (2.6) imply that $J \subset Q^*$ and $Q \subset J^*$. In particular Q^{**}, the closure of Q, is an operator. Now one can state the following factorization result established in the general case in [8, Theorem 4.3]; in the case that A is densely defined, see [1, Proposition 3.1].

Theorem 2.2. *[1, 8] Let A be a nonnegative relation in a Hilbert space $\mathfrak{H}$ and let J and Q be defined by* (2.5) *and* (2.6)*. Then the Kreĭn–von Neumann extension A_N of A is given by $A_N = J^{**}J^*$ and the corresponding closed form $\mathfrak{t}_N$ is given by*

$$\mathfrak{t}_N[f, g] = \langle J^* f, J^* g \rangle_R, \quad f,\, g \in \operatorname{dom} J^* = \operatorname{dom} A_N^{\frac{1}{2}}.$$

Furthermore, the Friedrichs extension A_F of A is given by $A_F = Q^* Q^{**}$ and the corresponding closed form $\mathfrak{t}_F$ is given by and

$$\mathfrak{t}_F[f, g] = \langle Q^{**} f, Q^{**} g \rangle_R, \quad f,\, g \in \operatorname{dom} Q^{**} = \operatorname{dom} A_F^{\frac{1}{2}}.$$

Let $\mathfrak{L}$ be any subspace such that

$$\operatorname{dom} A \subset \mathfrak{L} \subset \operatorname{dom} J^* = \operatorname{dom} A_N^{\frac{1}{2}}, \tag{2.7}$$

and associate with $\mathfrak{L}$ the restriction operator $R_{\mathfrak{L}}$ from $\mathfrak{H}$ to $\mathfrak{H}_A$ by

$$R_{\mathfrak{L}} := J^* \upharpoonright \mathfrak{L} = \{ (f, f') \in J^* \, : \, f \in \mathfrak{L} \}. \tag{2.8}$$

Since J^* is a closed operator from $\mathfrak{H}$ to $\mathfrak{H}_A$, it is clear that $R_{\mathfrak{L}}$ is a closable operator. The definition of $R_{\mathfrak{L}}$ and Theorem 2.2 lead to

$$\langle R_{\mathfrak{L}} f, R_{\mathfrak{L}} g \rangle_R = \langle J^* f, J^* g \rangle_R = \mathfrak{t}_N[f, g], \quad f,\, g \in \mathfrak{L}.$$

Hence, $R_{\mathfrak{L}}$ is closed if and only if the restriction to $\mathfrak{L}$ of the form $\mathfrak{t}_N[\cdot, \cdot]$ is closed. Clearly, operators of the form $R_{\mathfrak{L}}$ induce nonnegative selfadjoint relations $R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**}$ and the corresponding closed nonnegative forms $\mathfrak{t}_{\mathfrak{L}}$ are given by

$$\mathfrak{t}_{\mathfrak{L}}[f, g] = \langle R_{\mathfrak{L}}^{**} f, R_{\mathfrak{L}}^{**} g \rangle_R \quad f, g \in \operatorname{dom} R_{\mathfrak{L}}^{**}.$$

This yields the following useful characterization of extremal extensions of S; again the general case is established in [8, Theorem 6.1] and the densely defined case in [1, Proposition 4.1].

Theorem 2.3. *[1, 8] Let A be a nonnegative relation in a Hilbert space $\mathfrak{H}$. Then the following statements are equivalent:*

(i) *$\widetilde{A}$ is an extremal extension of A;*

(ii) *$\widetilde{A} = R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**}$ for some subspace $\mathfrak{L}$ such that* $\operatorname{dom} A \subset \mathfrak{L} \subset \operatorname{dom} A_N^{\frac{1}{2}}$*;*

(iii) *$\widetilde{A}$ is a nonnegative selfadjoint extension of A whose corresponding form $\widetilde{\mathfrak{t}}$ satisfies $\widetilde{\mathfrak{t}} \subset \mathfrak{t}_N$.*

2.5 Tensor product of Hilbert spaces

Assume that $\mathfrak{H}$ and $\mathfrak{K}$ are two separable complex Hilbert spaces. For $f \in \mathfrak{H}$ and $g \in \mathfrak{K}$ define the conjugate bilinear form $f \otimes g$ on $\mathfrak{H} \times \mathfrak{K}$ by:

$$(f \otimes g)(h,k) = \langle h, f\rangle_{\mathfrak{H}} \langle k, g\rangle_{\mathfrak{K}}, \quad (h,k) \in \mathfrak{H} \times \mathfrak{K}.$$

The set of finite linear combinations of such forms is denoted by $\mathfrak{H} \otimes \mathfrak{K}$. By extending the map

$$(f \otimes g, h \otimes k)_{\mathfrak{H}\otimes\mathfrak{K}} := \langle f, h\rangle_{\mathfrak{H}} \langle g, k\rangle_{\mathfrak{K}}, \quad f,\, h \in \mathfrak{H},\, g,\, k \in \mathfrak{K},$$

sesqui-linearly to $\mathfrak{H} \otimes \mathfrak{K}$, one obtains an inner product which turns $\mathfrak{H} \otimes \mathfrak{K}$ into a pre–Hilbert space. Its completion is denoted by $\mathfrak{H}\hat{\otimes}\mathfrak{K}$ and is called the tensor product of $\mathfrak{H}$ and $\mathfrak{K}$, cf. [11].

The following simple remark will be used several times from now on: if $f_n,\, f \in \mathfrak{H}$, $g_n,\, g \in \mathfrak{K}$ then

$$f_n \to f,\, g_n \to g \quad \text{implies} \quad f_n \otimes g_n \to f \otimes g,\, n \to \infty. \tag{2.9}$$

3 The tensor product of linear relations in Hilbert spaces

Assume that A and B are linear relations in the Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$, respectively. Define the linear relation $A \otimes B$ as the set of all finite linear combinations of the following pairs of the conjugate bilinear forms $(f \otimes g, f' \otimes g')$, when $(f, f') \in A$ and $(g, g') \in B$.

Remark 3.1. Some basic properties of the tensor product of linear relations are listed below; their proofs are similar to the corresponding properties of linear operators.

(i) For linear relations A and B in $\mathfrak{H}$ and $\mathfrak{K}$, respectively, $A \otimes B$ is a linear relation in the Hilbert space $\mathfrak{H}\hat{\otimes}\mathfrak{K}$ i.e. $A \otimes B$ is a linear subset of $\big(\mathfrak{H}\hat{\otimes}\mathfrak{K}\big) \times \big(\mathfrak{H}\hat{\otimes}\mathfrak{K}\big)$.

(ii) It is easily seen that:

$$A^* \otimes B^* \subset (A \otimes B)^*. \tag{3.1}$$

(iii) The closure $A\hat{\otimes}B$ of $A \otimes B$ is called the tensor product of A and B (in the sense of relations). In particular, when A and B are closable (their closures are graphs of linear operators), so is $A \otimes B$ and then $A\hat{\otimes}B$ is a closed operator in the Hilbert space $\mathfrak{H}\hat{\otimes}\mathfrak{K}$.

(iv) If both relations A and B are symmetric, so is $A \otimes B$. Furthermore, in the case that A and B are essentially selfadjoint relations the same is true for $A \otimes B$ which implies that $A\hat{\otimes}B$ is a selfadjoint relation in $\mathfrak{H}\hat{\otimes}\mathfrak{K}$, cf. [17, pag. 264].

The next lemma will be used in the proof of the main results of this note.

Lemma 3.1. *Assume that A and B are closed linear relations in the Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$, respectively. Let $\mathfrak{M}$ be a core of A and let $\mathfrak{N}$ be a core of B. Then $\mathfrak{M} \otimes \mathfrak{N}$ is a core of $A\hat{\otimes}B$.*

3.1 Tensor products of nonnegative linear relations

In what follows some preparatory results concerning the tensor products of nonnegative linear relations are presented.

Lemma 3.2. *Let A and B be two nonnegative linear relations in the Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$, respectively. Then $A\hat{\otimes}B$ is a closed nonnegative relation in the Hilbert space $\mathfrak{H}\hat{\otimes}\mathfrak{K}$.*

The statement of Lemma 3.2 can be shown using some nonnegative selfadjoint extensions as follows. Let $\widetilde{A}$ be a nonnegative selfadjoint extension of A and let $\widetilde{B}$ be a nonnegative selfadjoint extension of B. It follows immediately by Remark 3.1 (iv) and the definition of a nonnegative linear relation that $\widetilde{A}\hat{\otimes}\widetilde{B}$ is a nonnegative selfadjoint extension of $A\hat{\otimes}B$. Hence $A\hat{\otimes}B \subset \widetilde{A}\hat{\otimes}\widetilde{B}$ is a nonnegative linear relation as well.

The next result describes the tensor product of the square root of two nonnegative selfadjoint relations by means of the square root of their tensor product.

Lemma 3.3. *Let A and B be two nonnegative selfadjoint linear relations in the Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$, respectively. Then:*

$$\left(A\hat{\otimes}B\right)^{\frac{1}{2}} = A^{\frac{1}{2}}\hat{\otimes}B^{\frac{1}{2}} \tag{3.2}$$

3.2 The extremal extensions of the tensor product of two nonnegative linear relations

The first main result of this note offers some links between the extreme extensions of the tensor product of two linear relations and the extreme extensions of the linear relations themselves, respectively. The proof is based on the characterizations of the Friedrichs and the Kre˘ı n–von Neumann extensions given in Subsection 2.3.

Theorem 3.4. *Let A and B be two nonnegative linear relations in the Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$, respectively. Then one has:*
(i) $A_F\hat{\otimes}B_F = \left(A\hat{\otimes}B\right)_F$;
(ii) $A_N\hat{\otimes}B_N = \left(A\hat{\otimes}B\right)_N$.

The next result shows that the tensor product of two extremal extensions is an extremal extension as well.

Theorem 3.5. *Let A and B be two nonnegative linear relations in the Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$, respectively. Denote by J the linear relation associated to $A \otimes B$ given by:*

$$J = \{\, ([e'], e') : (e, e') \in A \otimes B \,\}.$$

Then the following statements are valid.

(i) Let $\widetilde{A}$ and $\widetilde{B}$ be two extremal extensions of A and B, respectively. Then $\widetilde{A}\hat{\otimes}\widetilde{B}$ is an extremal extension of $A \otimes B$. In particular, for every subspace $\mathfrak{L}$ of $\mathfrak{H}\hat{\otimes}\mathfrak{K}$ satisfying

$$\operatorname{dom} \widetilde{A} \otimes \operatorname{dom} \widetilde{B} \subset \mathfrak{L} \subset \operatorname{dom} \widetilde{A}^{\frac{1}{2}} \otimes \operatorname{dom} \widetilde{B}^{\frac{1}{2}}$$

one has

$$\widetilde{A}\hat{\otimes}\widetilde{B} = (J^* \restriction_{\mathfrak{L}})^* (J^* \restriction_{\mathfrak{L}})^{**}.$$

(*ii*) *Let* $\widetilde{E}$ *be an extremal extension of* $A \otimes B$. *Then there exists a subspace* $\mathfrak{L}$ *of* $\mathfrak{H}\hat{\otimes}\mathfrak{K}$ *with* $\operatorname{dom} A_F^{\frac{1}{2}} \otimes \operatorname{dom} B_F^{\frac{1}{2}} \subset \mathfrak{L} \subset \operatorname{dom} A_N^{\frac{1}{2}} \otimes \operatorname{dom} B_N^{\frac{1}{2}}$ *such that*

$$\widetilde{E} = (J^* \restriction_{\mathfrak{L}})^* (J^* \restriction_{\mathfrak{L}})^{**}.$$

(*iii*) *Let* $\mathfrak{M}$ *and* $\mathfrak{N}$ *be linear subspaces of* $\mathfrak{H}$ *and* $\mathfrak{K}$*, respectively, such that* $\operatorname{dom} A_F^{\frac{1}{2}} \subset \mathfrak{M} \subset \operatorname{dom} A_N^{\frac{1}{2}}$ *and* $\operatorname{dom} B_F^{\frac{1}{2}} \subset \mathfrak{N} \subset \operatorname{dom} B_N^{\frac{1}{2}}$*, respectively. Then one has:*

$$\widetilde{A}_{\mathfrak{M}}\hat{\otimes}\widetilde{B}_{\mathfrak{N}} = (J^* \restriction_{\mathfrak{M}\otimes\mathfrak{N}})^* (J^* \restriction_{\mathfrak{M}\otimes\mathfrak{N}})^{**}.$$

References

[1] Yu. M. Arlinskiĭ, S. Hassi, Z. Sebestyén, and H. S. V. de Snoo, *On the class of extremal extensions of a nonnegative operator*, Oper. Theory: Adv. Appl. (B. Sz.-Nagy memorial volume), 127 (2001), 41–81.

[2] Yu. M. Arlinskiĭ and E. R. Tsekanovskiĭ, *Quasi selfadjoint contractive extensions of Hermitian contractions*, Teor. Funkts., Funkts. Anal. Prilozhen, 50 (1988), 9–16.

[3] C. Bernard, F. Grandin and L. Vrancken, *Minimal tensor product immersions*, Balkan Jour. Geom. Appl. 14, 2 (2009), 21-27.

[4] E. A. Coddington, H. S. V. de Snoo, *Positive selfadjoint extensions of positive symmetric subspaces*, Math. Z. 159, 203–214 (1978).

[5] B. Farkas and M. Matolcsi, *Commutation properties of the form sum of positive, symmetric operators*, Acta Sci. Math. (Szeged), 6 (2001), 777–790.

[6] W. Goemans, I. Van de Woestyne and L. Vrancken, *Minimal tensor product surfaces of two pseudo-Euclidean curves*, Balkan Jour. Geom. Appl. 16, 2 (2011), 62-69.

[7] S. Hassi, A. Sandovici, H. S. V. de Snoo, and H. Winkler, *Form sums of nonnegative selfadjoint operators*, Acta Math. Hungar. 111 (2006), 81–105.

[8] S. Hassi, A. Sandovici, H. S. V. de Snoo, and H. Winkler, *A general factorization approach to the extension theory of nonnegative operators and relations*, J. Operator Theory, 58, 2 (2007), 351-386.

[9] M. M. Nafalska, *Extremal extensions of the tensor product of nonnegative operators*, PAMM, Proc. Appl. Math. Mech. 9, 659–660 (2009).

[10] M. M. Nafalska, *Extremal Extensions of Nonnegative operators with Applications*, Doctoral Thesis, Technische Universitat Berlin, 2008.

[11] M. Reed, B. Simon, *Methods of modern mathematical physics, I Functional Analisys*, (revised and enlarged edition), Academic Press Inc., San Diego, 1980.

[12] M. Roman, A. Sandovici, *The composition and the tensor product of Dirac structures on infinite–dimensional spaces* (in preparation).

[13] Z. Sebestyén and J. Stochel, *Restrictions of positive self-adjoint operators*, Acta Sci. Math. (Szeged), 55 (1991), 149–154.

[14] Z. Sebestyén and J. Stochel, *On products of unbounded operators*, Acta Math. Hungar., 100 (1–2) (2003), 105–129.
[15] C. Udriste, V. Arsinte, C. Cipu, *Von Neumann analysis of linearized discrete Tzitzeica PDE*, Balkan Jour. Geom. Appl. 15, 2 (2010), 108-120.
[16] C. Udriste, S. Dinu, I. Tevy, *Multitime optimal control for linear PDEs with curvilinear cost functional*, Balkan Jour. Geom. Appl. 18, 1 (2013), 87–100.
[17] J. Weidmann, *Linear operators in Hilbert spaces*, Springer–Verlag, 1980.

Authors' address:

Marcel Roman and Adrian Sandovici
Department of Mathematics, "Gheorghe Asachi" Technical University,
B-dul Carol I nr. 11, 700506, Iaşi, Romania.
E-mail: marcelroman@gmail.com , adrian.sandovici@luminis.ro