Optimal control problems on higher order jet bundles

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Abstract. The main purpose of this paper is to formulate and prove necessary optimality conditions for a class of optimal control problems subject to distribution-type constraints on the higher order jet bundles, using a geometrical language and Variational Calculus techniques under simplified hypothesis. Also, the main original results relate to the form of variational and adjoint equations.

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1 Curvilinear integral cost functional with distribution-type constraints

In the last few years, the multi-time optimization and optimal control problems have been intensively studied because of the infinite dimensional nature (see [7], [9], [12]-[14]), and both from theoretical and applied reasonings. Most of the times, the PDEs constraints (even of first order) represent significant challenges in optimization problems and principles (see [4], [5]). The present work can be seen as a natural continuation and extension of some recent works (see [3], [8]), where just unitemporal ingredients have been considered. The main results of this work are new and they complement the previously known results (see [1]-[3], [11]). For other different but connected viewpoints to this subject, the reader is addressed to [10] and [15]. Very interesting ideas, related to this subject, were given to me by Professor C. Udrişte (see [12]-[14]).

Let start with two Riemannian manifolds, (T, h) and (M, g), of dimensions m, respectively n, denoting by $t = (t^\alpha)$, $\alpha = \overline{1, m}$, and $x = (x^i)$, $i = \overline{1, n}$, the local coordinates on (T, h) and (M, g), respectively. These two Riemannian manifolds determine the $(s-1)$-th order jet bundle $J^{s-1}(T, M)$, where $s \geq 2$ is a fixed natural number. Also, consider the hyperparallelepiped $\Omega_{t_0,t_1} \subset \mathbb{R}^m_+$, with the diagonal

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opposite points $t_0 = \left(t_0^1, \ldots, t_0^m\right)$ and $t_1 = \left(t_1^1, \ldots, t_1^m\right)$. The C^1-class functions

$$X_\beta = \left(X_\beta^i\right) : J^{s-1}(T, M) \times \mathcal{U} \to \mathbb{R}^n, \quad \beta = \overline{1, m},\ i = \overline{1, n}$$

determine the following *distribution-type equations*

$$dx^i_{\alpha_1\alpha_2\ldots\alpha_{s-1}}(t) = X^i_\beta \left(t, x(t), x_{\alpha_1}(t), \ldots, x_{\alpha_1\alpha_2\ldots\alpha_{s-1}}(t), u(t)\right) dt^\beta, \quad i = \overline{1, n}$$

with the solution $x(t)$.

Thanks to their physical meaning, the cost functionals of mechanical work type become very important in applications. Here we initiate an optimization theory on the higher order jet bundles considering as objective function a curvilinear integral cost functional involving higher order partial derivatives. Our study is strongly motivated by its applications, especially in Mechanical Engineering and Analytical Mechanics (see [1]), where partial derivatives of order higher than one are often involved. In this direction, let analyze the following multi-time optimal control problem, formulated using as cost functional a curvilinear integral with distribution-type constraints:

$$\text{(1.1)} \qquad \max_{u(\cdot)} \left\{ J\left(u(\cdot)\right) = \int_{\Gamma_{t_0,t_1}} X_\beta \left(t, x(t), x_{\alpha_1}(t), \ldots, x_{\alpha_1\alpha_2\ldots\alpha_{s-1}}(t), u(t)\right) dt^\beta \right\}$$

subject to

$$\text{(1.2)} \qquad dx^i_{\alpha_1\alpha_2\ldots\alpha_{s-1}}(t) = X^i_\beta \left(t, x(t), x_{\alpha_1}(t), \ldots, x_{\alpha_1\alpha_2\ldots\alpha_{s-1}}(t), u(t)\right) dt^\beta$$

$$\text{(1.3)} \qquad u(t) \in \mathcal{U},\ \forall t \in \Omega_{t_0,t_1}; \quad x(t_\xi) = x_\xi,\ x_{\alpha_1\ldots\alpha_j}(t_\xi) = \tilde{x}_{\alpha_1\ldots\alpha_j\xi}$$

$$i = \overline{1, n}, \quad \alpha_\zeta \in \{1, \ldots, m\}, \quad \zeta, j = \overline{1, s-1}, \quad \xi = 0, 1.$$

The ingredients used are: $t = (t^\alpha) \in \Omega_{t_0,t_1}$ is a *multi-parameter of evolution* or a *multi-time*; Γ_{t_0,t_1} is a C^1-class curve joining the points t_0 and t_1; $x(t) = (x^i(t)),\ i = \overline{1, n}$, is a C^{s+1}-class function, called *state vector*; $u(t) = (u^a(t)),\ a = \overline{1, k}$, is a *continuous control vector*; the *running cost* $X_\beta \left(t, x(t), x_{\alpha_1}(t), \ldots, x_{\alpha_1\alpha_2\ldots\alpha_{s-1}}(t), u(t)\right) dt^\beta$ is a *non-autonomous Lagrangian* 1*-form*; the equations in (1.2) are *distribution-type equations*; the functions $X^i_\beta \left(t, x(t), x_{\alpha_1}(t), \ldots, x_{\alpha_1\alpha_2\ldots\alpha_{s-1}}(t), u(t)\right)$ are of C^1-class; we accept the notations $x_{\alpha_1}(t) := \dfrac{\partial x}{\partial t^{\alpha_1}}(t), \ldots,\ x_{\alpha_1\ldots\alpha_{s-1}}(t) := \dfrac{\partial^{s-1} x}{\partial t^{\alpha_1} \ldots \partial t^{\alpha_{s-1}}}(t)$, $\alpha_j \in \{1, 2, \ldots, m\}$, $j = \overline{1, s-1}$. We assume summation over the repeated indices.

Remark 1.1. (i) Like any functions, the previous running cost and the vector fields $\left(X^i_\beta\right)$, $\beta = \overline{1, m}$, depend on independent variables. For instance, the partial derivative $x_{123}(t)$ is the same with the partial derivatives $x_{231}(t)$, $x_{312}(t)$, $x_{321}(t)$, $x_{213}(t)$, $x_{132}(t)$ and, consequently, only one of these partial derivatives will appear as variable for X_β (for more details, see [6], [10]).

(ii) The constraints in (1.2), using some notations, can be rewritten under the next equivalent form: $x(t) = v_1(t)$, $(x_{\alpha_1}(t)) = v_2(t)$, $\cdots$, $\left(x_{\alpha_1\alpha_2\ldots\alpha_{s-1}}(t)\right) = v_s(t)$, $dv^i_s(t) = X^i_\beta \left(t, v_1(t), v_2(t), \ldots, v_s(t), u(t)\right) dt^\beta$, $i = \overline{1, n}$, $\beta = \overline{1, m}$, or, equivalently,

$$\text{(1.2}'\text{)} \qquad (v_{1\alpha_1}(t)) = v_2(t), \quad (v_{2\alpha_2}(t)) = v_3(t),\ \cdots,\ \left(v_{s-1\alpha_{s-1}}(t)\right) = v_s(t)$$

$$v^i_{s\beta}(t) = X^i_\beta\left(t, v_1(t), v_2(t), \dots, v_s(t), u(t)\right), \quad i = \overline{1,n},$$

where we have denoted $v_{\gamma\eta}(t) := \dfrac{\partial v_\gamma}{\partial t^\eta}(t),\ \gamma = \overline{1,s},\ \eta \in \{1, \dots, m\}$.

Introduce the *co-state* 1*-forms* or *Lagrange multiplier* 1*-forms* $p_\gamma(t) = p_{i\gamma}(t)dv^i_\gamma = p_{i\gamma}(t)v^i_{\gamma\alpha_\gamma}(t)dt^{\alpha_\gamma},\ \gamma = \overline{1,s},\ \alpha_s := \beta$, and build a new Lagrangian 1-form

$$L\left(t, v_1(t), v_2(t), \dots, v_s(t), dv_1(t), dv_2(t), \dots, dv_s(t), u(t), p_1(t), \dots, p_s(t)\right)$$
$$= X_\beta\left(t, v_1(t), v_2(t), \dots, v_s(t), u(t)\right) dt^\beta$$
$$+ p_{i1}(t)\left[v^i_{1\alpha_1}(t)dt^{\alpha_1} - dv^i_1(t)\right] + \dots + p_{is-1}(t)\left[v^i_{s-1\alpha_{s-1}}(t)dt^{\alpha_{s-1}} - dv^i_{s-1}(t)\right]$$
$$+ p_{is}(t)\left[X^i_\beta\left(t, v_1(t), v_2(t), \dots, v_s(t), u(t)\right) dt^\beta - dv^i_s(t)\right].$$

The solutions of the foregoing multi-time control problem are between the solutions of the following free maximization problem

$$\max_{u(\cdot)} \int_{\tilde\Gamma} L\left(t, V^T(t), dV^T(t), u(t), p_1(t), \dots, p_s(t)\right),$$

with

$$u(t) \in \mathcal{U},\ \{p_1(t), \dots, p_s(t)\} \subseteq \mathcal{P},\ \forall t \in \Omega_{t_0,t_1}$$

$$(\star) \qquad \tilde\Gamma := \left(\Gamma_{t_0,t_1}, x(\Gamma_{t_0,t_1}), \dots, x_{\alpha_1\dots\alpha_{s-1}}(\Gamma_{t_0,t_1})\right)$$
$$x(t_\xi) = x_\xi, \quad x_{\alpha_1\dots\alpha_j}(t_\xi) = \tilde x_{\alpha_1\dots\alpha_j\xi}, \quad \xi = 0,1$$
$$\alpha_\zeta \in \{1, \dots, m\}, \quad \zeta, j = \overline{1, s-1}$$

(see $V(t) := \left(v_1(t), v_2(t), \dots, v_s(t)\right)^T,\ dV(t) := \left(dv_1(t), dv_2(t), \dots, dv_s(t)\right)^T$) and the set $\mathcal{P}$ of co-state 1-forms will be defined later.

Remark 1.2. The above curve $\tilde\Gamma$ satisfies

$$\tilde\Gamma \subset \mathbb{R}^m_+ \times \mathbb{R}^n \times \dots \times \mathbb{R}^{[nm(m+1)\cdots(m+s-2)]/(s-1)!}.$$

The *control Hamiltonian* 1*-form*,

$$H\left(t, v_1(t), v_2(t), \dots, v_s(t), u(t), p_1(t), \dots, p_s(t)\right)$$
$$= X_\beta\left(t, v_1(t), v_2(t), \dots, v_s(t), u(t)\right) dt^\beta + p_{i1}(t)v^i_{1\alpha_1}(t)dt^{\alpha_1}$$
$$+ \dots + p_{is-1}(t)v^i_{s-1\alpha_{s-1}}(t)dt^{\alpha_{s-1}} + p_{is}(t)X^i_\beta\left(t, v_1(t), v_2(t), \dots, v_s(t), u(t)\right) dt^\beta,$$

or, equivalently,

$$H = L + p_{i1}dv^i_1 + \dots + p_{is}dv^i_s = L + p_{i1}v^i_{1\alpha_1}dt^{\alpha_1} + \dots + p_{is}v^i_{s\beta}dt^\beta$$
$$= L + p_{i1}x^i_{\alpha_1}dt^{\alpha_1} + \dots + p_{is}x^i_{\alpha_1\dots\alpha_{s-1}\beta}dt^\beta,$$

(*modified Legendrian duality*) permits us rewriting the previous optimization problem as

$$\max_{u(\cdot)}\Big\{ \int_{\tilde\Gamma} H\left(t, x(t), x_{\alpha_1}(t), \dots, x_{\alpha_1\dots\alpha_{s-1}}(t), u(t), p_1(t), \dots, p_s(t)\right)$$
$$- \int_{\tilde\Gamma}\left[p_{i1}(t)x^i_{\alpha_1}(t)dt^{\alpha_1} + \dots + p_{is}(t)x^i_{\alpha_1\dots\alpha_{s-1}\beta}(t)dt^\beta\right]\Big\}$$

such that the conditions $(\star)$ are fulfilled.

1.1 Adjointness on higher order jet bundles

Consider the differential relations in (1.2′). Let fix the control vector $u(t)$ and a corresponding solution $(v_1(t), \dots, v_s(t))$ in (1.2′). Using the differentiable variations $(v_1(t,\varepsilon), \dots, v_s(t,\varepsilon))$, satisfying

$$(v_{1\alpha_1}(t,\varepsilon)) = v_2(t,\varepsilon),\ (v_{2\alpha_2}(t,\varepsilon)) = v_3(t,\varepsilon),\ \cdots,\ \left(v_{s-1\alpha_{s-1}}(t,\varepsilon)\right) = v_s(t,\varepsilon)$$

$$v^i_{s\beta}(t,\varepsilon) = X^i_\beta\left(t, v_1(t,\varepsilon), v_2(t,\varepsilon), \dots, v_s(t,\varepsilon), u(t)\right),\ v_\gamma(t,0) = v_\gamma(t),\ i = \overline{1,n},\ \gamma = \overline{1,s}$$

and considering the derivative with respect to ε, at $\varepsilon = 0$, we find the following *variational equations*

$$(\omega_{1\alpha_1}(t)) := \omega_2(t),\ (\omega_{2\alpha_2}(t)) := \omega_3(t),\ \cdots,\ \left(\omega_{s-1\alpha_{s-1}}(t)\right) := \omega_s(t)$$

$$\omega^i_{s\beta}(t) = \frac{\partial X^i_\beta}{\partial v_1}\left(t, v_1(t), \dots, v_s(t), u(t)\right)\omega_1(t) + \dots + \frac{\partial X^i_\beta}{\partial v_s}\left(t, v_1(t), \dots, v_s(t), u(t)\right)\omega_s(t),$$

for $i = \overline{1,n}$, where we used the notations: $v_{\gamma\varepsilon}(t,0) := \omega_\gamma(t),\ \gamma = \overline{1,s}$, with $v_{\gamma\varepsilon}(t,0)$ as the derivative of $v_\gamma(t,\varepsilon)$ with respect to ε, evaluated at $\varepsilon = 0$. Taking into account the co-state variables $\{p_1(t), \dots, p_s(t)\}$, we find the following equations

$$dp_{j1}(t) = -p_{ls}(t)\frac{\partial X^l_\beta}{\partial v^j_1}\left(t, v_1(t), v_2(t), \dots, v_s(t), u(t)\right)dt^\beta$$

$$dp_{j2}(t) = \left[-p_{j1}(t)\delta^{\alpha_1}_\beta - p_{ls}(t)\frac{\partial X^l_\beta}{\partial v^j_2}\left(t, v_1(t), v_2(t), \dots, v_s(t), u(t)\right)\right]dt^\beta$$

$$\vdots$$

$$dp_{js}(t) = \left[-p_{js-1}(t)\delta^{\alpha_{s-1}}_\beta - p_{ls}(t)\frac{\partial X^l_\beta}{\partial v^j_s}\left(t, v_1(t), v_2(t), \dots, v_s(t), u(t)\right)\right]dt^\beta,$$

called the *adjoint equations* associated to the previous variational equations, that is the following differential condition is fulfilled: $\sum_{\gamma=1}^{s} d\left[p_{j\gamma}(t)\omega^j_\gamma(t)\right] = 0,\ t \in \Omega_{t_0,t_1}$.

1.2 Necessary optimality conditions for a feasible solution

Let assume there exists a continuous control $\hat{u}(t)$ defined on Ω_{t_0,t_1}, with $\hat{u}(t) \in Int\mathcal{U}$, which is an optimum point in the previous optimization problem. Consider a variation, $u(t,\varepsilon) = \hat{u}(t) + \varepsilon h(t)$, with h an arbitrary continuous vector function. Since $\hat{u}(t) \in Int\mathcal{U}$, there exists a number $\varepsilon_h > 0$ such that $u(t,\varepsilon) = \hat{u}(t) + \varepsilon h(t) \in Int\mathcal{U}$, $\forall|\varepsilon| < \varepsilon_h$. This ε will be used in our variational arguments.

The variation of the control vector $u = \hat{u} + \varepsilon h$ determines the variation of the state variable $x = x(t,\varepsilon)$, i.e., for $i = \overline{1,n}$ and $\forall t \in \Omega_{t_0,t_1}$, we have

$$dx^i_{\alpha_1\alpha_2\dots\alpha_{s-1}}(t,\varepsilon) = X^i_\beta\left(t, x(t,\varepsilon), x_{\alpha_1}(t,\varepsilon), \dots, x_{\alpha_1\alpha_2\dots\alpha_{s-1}}(t,\varepsilon), u(t,\varepsilon)\right)dt^\beta$$

and $x(t_0,\varepsilon) = x_0,\ x_{\alpha_1\dots\alpha_j}(t_0,\varepsilon) = \tilde{x}_{\alpha_1\dots\alpha_j 0},\ j = \overline{1,s-1}$.

For $|\varepsilon| < \varepsilon_h$, let consider the function

$$J(\varepsilon) := \int_{\tilde{\Gamma}(\varepsilon)} X_\beta \left(t, x(t,\varepsilon), x_{\alpha_1}(t,\varepsilon), \ldots, x_{\alpha_1\alpha_2\ldots\alpha_{s-1}}(t,\varepsilon), u(t,\varepsilon)\right) dt^\beta.$$

On the other hand, the control vector $\hat{u}(t)$ is an optimal control vector. Consequently, we must have $J(0) \geq J(\varepsilon),\ \forall |\varepsilon| < \varepsilon_h$. We have

$$\int_{\tilde{\Gamma}(\varepsilon)} p_{i1}(t) \left[v^i_{1\alpha_1}(t,\varepsilon) dt^{\alpha_1} - dv^i_1(t,\varepsilon)\right] = 0$$

$$\vdots$$

$$\int_{\tilde{\Gamma}(\varepsilon)} p_{is-1}(t) \left[v^i_{s-1\alpha_{s-1}}(t,\varepsilon) dt^{\alpha_{s-1}} - dv^i_{s-1}(t,\varepsilon)\right] = 0$$

$$\int_{\tilde{\Gamma}(\varepsilon)} p_{is}(t) \left[X^i_\beta \left(t, v_1(t,\varepsilon), v_2(t,\varepsilon), \ldots, v_s(t,\varepsilon), u(t,\varepsilon)\right) dt^\beta - dv^i_s(t,\varepsilon)\right] = 0,$$

for any continuous co-state 1-forms $p_\gamma = (p_{i\gamma})\colon \Omega_{t_0,t_1} \to \mathbb{R}^n,\ \gamma = \overline{1,s}$.

Necessarily, the variations involve the Lagrangian 1-form

$$\begin{aligned} &L\left(t, V^T(t,\varepsilon), dV^T(t,\varepsilon), u(t,\varepsilon), p_1(t), \ldots, p_s(t)\right) \\ &= X_\beta\left(t, V^T(t,\varepsilon), u(t,\varepsilon)\right) dt^\beta + p_{i1}(t)\left[v^i_{1\alpha_1}(t,\varepsilon) dt^{\alpha_1} - dv^i_1(t,\varepsilon)\right] \\ &+ \ldots + p_{is-1}(t)\left[v^i_{s-1\alpha_{s-1}}(t,\varepsilon) dt^{\alpha_{s-1}} - dv^i_{s-1}(t,\varepsilon)\right] \\ &+ p_{is}(t)\left[X^i_\beta\left(t, V^T(t,\varepsilon), u(t,\varepsilon)\right) dt^\beta - dv^i_s(t,\varepsilon)\right] \end{aligned}$$

and the associated function

$$J(\varepsilon) = \int_{\tilde{\Gamma}(\varepsilon)} L\left(t, V^T(t,\varepsilon), dV^T(t,\varepsilon), u(t,\varepsilon), p_1(t), \ldots, p_s(t)\right).$$

Further, we assume that the co-state 1-forms $p_\gamma(t) = \left(p_{i\gamma}(t)\right),\ \gamma = \overline{1,s}$, are C^1-class functions. The corresponding control Hamiltonian 1-form with variations is

$$\begin{aligned} &H\left(t, x(t,\varepsilon), x_{\alpha_1}(t,\varepsilon), \ldots, x_{\alpha_1\alpha_2\ldots\alpha_{s-1}}(t,\varepsilon), u(t,\varepsilon), p_1(t), \ldots, p_s(t)\right) \\ &= L\left(t, V^T(t,\varepsilon), dV^T(t,\varepsilon), u(t,\varepsilon), p_1(t), \ldots, p_s(t)\right) \\ &+ p_{i1}(t) dx^i(t,\varepsilon) + \ldots + p_{is}(t) dx^i_{\alpha_1\ldots\alpha_{s-1}}(t,\varepsilon) \end{aligned}$$

and the above curvilinear integral $J(\varepsilon)$ can be rewritten as follows

$$\begin{aligned} J(\varepsilon) = &\int_{\tilde{\Gamma}(\varepsilon)} H\left(t, x(t,\varepsilon), x_{\alpha_1}(t,\varepsilon), \ldots, x_{\alpha_1\ldots\alpha_{s-1}}(t,\varepsilon), u(t,\varepsilon), p_1(t), \ldots, p_s(t)\right) \\ &- \int_{\tilde{\Gamma}(\varepsilon)} \left[p_{i1}(t) dx^i(t,\varepsilon) + \ldots + p_{is}(t) dx^i_{\alpha_1\ldots\alpha_{s-1}}(t,\varepsilon)\right]. \end{aligned}$$

Computing the derivative with respect to ε, evaluated at $\varepsilon = 0$, and using the formula of integration by parts, with $\tilde{\Gamma}(0) := \tilde{\Gamma}$, we get

$$J'(0) = \int_{\tilde{\Gamma}} \left[H_{x^j}\left(t, x(t), \dots, x_{\alpha_1 \dots \alpha_{s-1}}(t), \hat{u}(t), p(t)\right) + dp_{j1}(t)\right] x^j_{\varepsilon}(t, 0)$$

$$+ \int_{\tilde{\Gamma}} \left[H_{x^j_{\alpha_1}}\left(t, x(t), \dots, x_{\alpha_1 \dots \alpha_{s-1}}(t), \hat{u}(t), p(t)\right) + dp_{j2}(t)\right] x^j_{\alpha_1 \varepsilon}(t, 0)$$

$$+ \dots + \int_{\tilde{\Gamma}} \left[H_{x^j_{\alpha_1 \dots \alpha_{s-1}}}\left(t, x(t), \dots, x_{\alpha_1 \dots \alpha_{s-1}}(t), \hat{u}(t), p(t)\right) + dp_{js}(t)\right] x^j_{\alpha_1 \dots \alpha_{s-1} \varepsilon}(t, 0)$$

$$+ \int_{\tilde{\Gamma}} H_{u^a}\left(t, x(t), \dots, x_{\alpha_1 \dots \alpha_{s-1}}(t), \hat{u}(t), p(t)\right) h^a(t)$$

$$- \left[p_{j1}(t) x^j_{\varepsilon}(t, 0) + p_{j2}(t) x^j_{\alpha_1 \varepsilon}(t, 0) + \dots + p_{js}(t) x^j_{\alpha_1 \dots \alpha_{s-1}, \varepsilon}(t, 0)\right] |^{t_1}_{t_0},$$

where $x(t)$ is the m-sheet of the state variable corresponding to the optimal control $\hat{u}(t)$ (see $p(t) := \{p_1(t), \dots, p_s(t)\}$). Also, the following Cauchy problem is fulfilled

$$x^i_{\alpha_1 \dots \alpha_{s-1} \beta \varepsilon}(t, 0) = X^i_{\beta x}\left(t, x(t), x_{\alpha_1}(t), \dots, x_{\alpha_1 \dots \alpha_{s-1}}(t), u(t)\right) x_{\varepsilon}(t, 0)$$

$$+ \dots + X^i_{\beta x_{\alpha_1 \dots \alpha_{s-1}}}\left(t, x(t), x_{\alpha_1}(t), \dots, x_{\alpha_1 \dots \alpha_{s-1}}(t), u(t)\right) x_{\alpha_1 \dots \alpha_{s-1} \varepsilon}(t, 0)$$

$$+ X^i_{\beta u}\left(t, x(t), x_{\alpha_1}(t), \dots, x_{\alpha_1 \dots \alpha_{s-1}}(t), u(t)\right) h(t)$$

$$\forall t \in \Omega_{t_0, t_1}, \quad x_{\varepsilon}(t_0, 0) = 0, \quad x_{\alpha_1 \dots \alpha_j \varepsilon}(t_0, 0) = 0, \quad j = \overline{1, s-1}.$$

We impose $J'(0) = 0$, for any continuous vector function $h(t) = (h^a(t))$. Therefore, using the adjoint equations, we define the set $\mathcal{P}$ of co-state 1-forms as the set of solutions of the following problem

$$dp_{j1}(t) = -H_{x^j}\left(t, x(t), \dots, x_{\alpha_1 \dots \alpha_{s-1}}(t), \hat{u}(t), p(t)\right), \quad p_{j1}(t_1) = 0 \tag{1.4}$$

$$dp_{j2}(t) = -H_{x^j_{\alpha_1}}\left(t, x(t), \dots, x_{\alpha_1 \dots \alpha_{s-1}}(t), \hat{u}(t), p(t)\right), \quad p_{j2}(t_1) = 0$$

$$\vdots$$

$$dp_{js}(t) = -H_{x^j_{\alpha_1 \dots \alpha_{s-1}}}\left(t, x(t), \dots, x_{\alpha_1 \dots \alpha_{s-1}}(t), \hat{u}(t), p(t)\right), \quad p_{js}(t_1) = 0$$

$$\forall t \in \Omega_{t_0, t_1}.$$

Also, we get

$$H_{u^a}\left(t, x(t), \dots, x_{\alpha_1 \dots \alpha_{s-1}}(t), \hat{u}(t), p(t)\right) = 0, \quad \forall t \in \Omega_{t_0, t_1}. \tag{1.5}$$

Moreover,

$$dx^i_{\alpha_1 \dots \alpha_{r-1}}(t) = \frac{\partial H}{\partial p_{ir}}\left(t, x(t), \dots, x_{\alpha_1 \dots \alpha_{s-1}}(t), \hat{u}(t), p(t)\right), \quad \forall t \in \Omega_{t_0, t_1} \tag{1.6}$$

$$x(t_0) = x_0, \ x_{\alpha_1 \dots \alpha_j}(t_0) = \tilde{x}_{\alpha_1 \dots \alpha_j 0}, \ j = \overline{1, s-1}, \ i = \overline{1, n}, \ r = \overline{1, s}$$

$$\left(see \ dx^i_{\alpha_1 \dots \alpha_0}(t) := dx^i(t)\right).$$

In summary, considering all the previous computations and reasonings, we can formulate the main result of this section which provides necessary optimality conditions for the feasible point $\hat{u} = (\hat{u}^a)$. The word "simplified" used below signifies that the principle is obtained by variational calculus techniques, under simplified hypothesis.

Theorem 1.1. *(***Simplified multi-time maximum principle; necessary conditions***) Consider* $\hat{u}(t) \in Int\mathcal{U}$ *an interior optimal solution which determines the optimal evolution* $x(t)$ *in* (1.1)*, subject to* (1.2) *and* (1.3)*. Then, there exist the* C^1*-class co-state* 1*-forms,* $p_r = (p_{ir})$, $r = \overline{1,s}$*, defined on* Ω_{t_0,t_1}*, such that the relations* (1.4)*,* (1.5)*,* (1.6) *are fulfilled.*

2 Multiple integral cost functional with distribution-type constraints

Next, we consider a multi-time optimal control problem that involves as basic tools a multiple integral cost functional and constraints of distribution type:

$$\max_{u(\cdot)} \left\{ J(u(\cdot)) = \int_{\Omega_{t_0,t_1}} X\left(t, x(t), x_{\alpha_1}(t), \ldots, x_{\alpha_1\alpha_2\ldots\alpha_{s-1}}(t), u(t)\right)\omega \right\} \tag{2.1}$$

subject to

$$dx^i_{\alpha_1\alpha_2\ldots\alpha_{s-1}}(t) = X^i_\beta\left(t, x(t), x_{\alpha_1}(t), \ldots, x_{\alpha_1\alpha_2\ldots\alpha_{s-1}}(t), u(t)\right)dt^\beta \tag{2.2}$$

$$u(t) \in \mathcal{U},\ \forall t \in \Omega_{t_0,t_1};\quad x(t_\xi) = x_\xi,\ x_{\alpha_1\ldots\alpha_j}(t_\xi) = \tilde{x}_{\alpha_1\ldots\alpha_j\xi} \tag{2.3}$$
$$i = \overline{1,n},\quad \alpha_\zeta \in \{1,\ldots,m\},\quad \zeta, j = \overline{1,s-1},\quad \xi = 0,1.$$

This kind of problems may appear when we want to describe the torsion of prismatic bars in the elastic or elastic-plastic case. As method of investigation we shall use some variational calculus techniques which are adequate in the study of multi-time optimal control problems. Therefore, we change the optimal control problem into a variational problem.

We generally used here the same mathematical data as in the previous section. The *running cost* $X\left(t, x(t), x_{\alpha_1}(t), \ldots, x_{\alpha_1\alpha_2\ldots\alpha_{s-1}}(t), u(t)\right)\omega$ is a *non-autonomous Lagrangian m-form*; $\omega = dt^1 \wedge \ldots \wedge dt^m$ is the *volume form* in $\mathbb{R}^m_+$; the vector fields $X_\beta = \left(X^i_\beta\right)$, $\beta = \overline{1,m}$, are functions of C^1-class which have independent variables (see *Remark* 1.1, (i)). Also, we accept the notations introduced in *Remark* 1.1 (see (ii)) and consider summation over the repeated indices.

Keeping in mind some ideas suggested by Professor C. Udrişte, let consider the *Lagrange multiplier tensors* or *co-state tensors*, $p_\gamma(t) = p^\alpha_{i\gamma}(t)\dfrac{\partial}{\partial t^\alpha} \otimes dx^i$, $\gamma = \overline{1,s}$, and the $(m-1)$-forms $\omega_\lambda = \dfrac{\partial}{\partial t^\lambda}\rfloor\omega$ (see $Y\rfloor\omega$ as the *contraction* between Y and ω). Now, we build a new Lagrangian m-form

$$\begin{aligned} &L(t, v_1(t), v_2(t), \ldots, v_s(t), dv_1(t), dv_2(t), \ldots, dv_s(t), u(t), p_1(t), \ldots, p_s(t)) \\ &= X(t, v_1(t), v_2(t), \ldots, v_s(t), u(t))\,\omega + p^\lambda_{i1}(t)\left[v^i_{1\alpha_1}(t)dt^{\alpha_1} - dv^i_1(t)\right] \wedge \omega_\lambda \\ &\quad + \ldots + p^\lambda_{is-1}(t)\left[v^i_{s-1\alpha_{s-1}}(t)dt^{\alpha_{s-1}} - dv^i_{s-1}(t)\right] \wedge \omega_\lambda \\ &\quad + p^\lambda_{is}(t)\left[X^i_\beta(t, v_1(t), v_2(t), \ldots, v_s(t), u(t))\,dt^\beta - dv^i_s(t)\right] \wedge \omega_\lambda. \end{aligned}$$

Also, introduce the *control Hamiltonian m-form*

$$H\left(t, v_1(t), v_2(t), \ldots, v_s(t), u(t), p_1(t), \ldots, p_s(t)\right)$$
$$= X\left(t, v_1(t), v_2(t), \ldots, v_s(t), u(t)\right)\omega + p^\lambda_{i1}(t) v^i_{1\alpha_1}(t) dt^{\alpha_1} \wedge \omega_\lambda$$
$$+ \ldots + p^\lambda_{is-1}(t) v^i_{s-1\alpha_{s-1}}(t) dt^{\alpha_{s-1}} \wedge \omega_\lambda + p^\lambda_{is}(t) X^i_\beta\left(t, v_1(t), v_2(t), \ldots, v_s(t), u(t)\right) dt^\beta \wedge \omega_\lambda$$
$$= \left[X(\cdot) + p^{\alpha_1}_{i1}(t) v^i_{1\alpha_1}(t) + \ldots + p^\beta_{is}(t) X^i_\beta(\cdot)\right]\omega$$
$$= H_1\left(t, v_1(t), v_2(t), \ldots, v_s(t), u(t), p_1(t), \ldots, p_s(t)\right)\omega,$$

or, equivalently, $H = L + p^\lambda_{i1}(t) dv^i_1(t) \wedge \omega_\lambda + \ldots + p^\lambda_{is}(t) dv^i_s(t) \wedge \omega_\lambda$, (*modified Legendrian duality*) that permits us rewriting the foregoing multi-time optimal control problem into the next equivalent form

$$\max_{u(\cdot)} \left\{ \int_{\tilde{\Omega}} H\left(t, x(t), x_{\alpha_1}(t), \ldots, x_{\alpha_1 \ldots \alpha_{s-1}}(t), u(t), p_1(t), \ldots, p_s(t)\right) \right.$$
$$\left. - \int_{\tilde{\Omega}} \left[p^\lambda_{i1}(t) dv^i_1(t) \wedge \omega_\lambda + \ldots + p^\lambda_{is}(t) dv^i_s(t) \wedge \omega_\lambda\right] \right\}$$

with

$$\tilde{\Omega} := \left(\Omega_{t_0,t_1}, x(\Omega_{t_0,t_1}), \ldots, x_{\alpha_1 \ldots \alpha_{s-1}}(\Omega_{t_0,t_1})\right)$$
$$u(t) \in \mathcal{U}, \quad \{p_1(t), \ldots, p_s(t)\} \subseteq \mathcal{P}, \quad \forall t \in \Omega_{t_0,t_1}$$
$$x(t_\xi) = x_\xi, \quad x_{\alpha_1 \ldots \alpha_j}(t_\xi) = \tilde{x}_{\alpha_1 \ldots \alpha_j \xi}, \quad \xi = 0, 1$$
$$\alpha_\zeta \in \{1, \ldots, m\}, \quad \zeta, j = \overline{1, s-1}$$

and the set $\mathcal{P}$ of co-state tensors will be defined later.

Remark 2.1. The relation $\tilde{\Omega} \subset \mathbb{R}^m_+ \times \mathbb{R}^n \times \cdots \times \mathbb{R}^{[nm(m+1)\cdots(m+s-2)]/(s-1)!}$ is satisfied.

2.1 Variational and adjoint equations

As in Section 1.1 of this paper, we consider the following equations

$$(w_{1\alpha_1}(t)) := w_2(t),\ (w_{2\alpha_2}(t)) := w_3(t),\ \cdots,\ \left(w_{s-1\alpha_{s-1}}(t)\right) := w_s(t)$$

$$w^i_{s\beta}(t) = \frac{\partial X^i_\beta}{\partial v_1}\left(t, v_1(t), \ldots, v_s(t), u(t)\right) w_1(t) + \ldots + \frac{\partial X^i_\beta}{\partial v_s}\left(t, v_1(t), \ldots, v_s(t), u(t)\right) w_s(t),$$

for $i = \overline{1, n}$, called *variational equations*. We used the same style of notation as in Section 1.1. Considering the co-state tensors $\{p_1(t), \ldots, p_s(t)\}$, we find the following equations

$$dp^\lambda_{j1}(t) \wedge \omega_\lambda = -p^\lambda_{ls}(t) \frac{\partial X^l_\beta}{\partial v^j_1}\left(t, v_1(t), \ldots, v_s(t), u(t)\right) dt^\beta \wedge \omega_\lambda$$

$$dp^\lambda_{j2}(t) \wedge \omega_\lambda = \left[-p^\lambda_{j1}(t)\delta^{\alpha_1}_\beta - p^\lambda_{ls}(t) \frac{\partial X^l_\beta}{\partial v^j_2}\left(t, v_1(t), \ldots, v_s(t), u(t)\right)\right] dt^\beta \wedge \omega_\lambda$$

$$\vdots$$

$$dp^{\lambda}_{js}(t) \wedge \omega_{\lambda} = \left[-p^{\lambda}_{js-1}(t)\delta^{\alpha_{s-1}}_{\beta} - p^{\lambda}_{ls}(t)\frac{\partial X^l_{\beta}}{\partial v^j_s}\left(t, v_1(t), \ldots, v_s(t), u(t)\right)\right] dt^{\beta} \wedge \omega_{\lambda},$$

called the *adjoint equations* associated to the previous variational equations. Also, via formula $d\left(p^{\lambda}_{j\gamma} w^j_{\gamma} \omega_{\lambda}\right) = \left(w^j_{\gamma} dp^{\lambda}_{j\gamma} + p^{\lambda}_{j\gamma} dw^j_{\gamma}\right) \wedge \omega_{\lambda}$, the following differential condition is satisfied: $\sum_{\gamma=1}^{s} d\left[p^{\lambda}_{j\gamma}(t) w^j_{\gamma}(t)\omega_{\lambda}\right] = 0,\ t \in \Omega_{t_0,t_1}$.

2.2 Simplified multi-time maximum principle

Now, we have all the necessary mathematical tools to establish the main result of this section.

Theorem 2.1. (Simplified multi-time maximum principle; necessary conditions) *Assume that the problem of maximizing the functional* (2.1)*, constrained by* (2.2) *and* (2.3)*, has an interior optimal solution* $\hat{u}(t) \in Int\mathcal{U}$*, which determines the optimal evolution* $x(t)$*. Then, there exist the* C^1*-class co-state tensors,* $p_r = \left(p^{\alpha}_{ir}\right),\ r = \overline{1,s}$*, defined on* Ω_{t_0,t_1}*, such that*

$$dx^i_{\alpha_1\ldots\alpha_{r-1}}(t) \wedge \omega_{\lambda} = \frac{\partial H}{\partial p^{\lambda}_{ir}}\left(t, \ldots, x_{\alpha_1\ldots\alpha_{s-1}}(t), \hat{u}(t), p(t)\right), \quad \forall t \in \Omega_{t_0,t_1}, \tag{2.4}$$

the function $p = (p_r),\ r = \overline{1,s}$*, is the unique solution for the following Pfaff system*

$$\begin{cases} dp^{\lambda}_{j1}(t) \wedge \omega_{\lambda} = -H_{x^j}\left(t, x(t), \ldots, x_{\alpha_1\ldots\alpha_{s-1}}(t), \hat{u}(t), p(t)\right) \\ dp^{\lambda}_{j2}(t) \wedge \omega_{\lambda} = -H_{x^j_{\alpha_1}}\left(t, x(t), \ldots, x_{\alpha_1\ldots\alpha_{s-1}}(t), \hat{u}(t), p(t)\right) \\ \vdots \\ dp^{\lambda}_{js}(t) \wedge \omega_{\lambda} = -H_{x^j_{\alpha_1\ldots\alpha_{s-1}}}\left(t, x(t), \ldots, x_{\alpha_1\ldots\alpha_{s-1}}(t), \hat{u}(t), p(t)\right), \quad \forall t \in \Omega_{t_0,t_1} \\ \delta_{\alpha\beta} p^{\alpha}_{j1}(t)\eta^{\beta}(t) = 0, \ldots, \delta_{\mu\nu} p^{\mu}_{js}(t)\zeta^{\nu}(t) = 0 \quad (\textit{orthogonality / tangency}) \end{cases} \tag{2.5}$$

and satisfies the critical point conditions

$$H_{u^a}\left(t, x(t), \ldots, x_{\alpha_1\ldots\alpha_{s-1}}(t), \hat{u}(t), p(t)\right) = 0, \quad \forall t \in \Omega_{t_0,t_1}. \tag{2.6}$$

Proof. For proving the previous result, we use the control Hamiltonian m-form H. Suppose that there exists a continuous control $\hat{u}(t)$ defined on Ω_{t_0,t_1}, with $\hat{u}(t) \in Int\mathcal{U}$, which is an optimum point in our optimization problem. Now we consider a control variation $u(t,\varepsilon) = \hat{u}(t) + \varepsilon h(t)$, where h is an arbitrary continuous vector function, and a state variation $x(t,\varepsilon),\ t \in \Omega_{t_0,t_1}$, connected by

$$dx^i_{\alpha_1\alpha_2\ldots\alpha_{s-1}}(t,\varepsilon) = X^i_{\beta}\left(t, x(t,\varepsilon), x_{\alpha_1}(t,\varepsilon), \ldots, x_{\alpha_1\alpha_2\ldots\alpha_{s-1}}(t,\varepsilon), u(t,\varepsilon)\right) dt^{\beta},$$

for $i = \overline{1,n},\ \forall t \in \Omega_{t_0,t_1}$, with $x(t_0,\varepsilon) = x_0,\ x_{\alpha_1\ldots\alpha_j}(t_0,\varepsilon) = \tilde{x}_{\alpha_1\ldots\alpha_j 0},\ j = \overline{1,s-1}$. We have $\hat{u}(t) \in Int\mathcal{U}$ and it is well-known that any continuous function over a compact set Ω_{t_0,t_1} is bounded. Thus, there exists a number $\varepsilon_h > 0$ such that $u(t,\varepsilon) = \hat{u}(t) + \varepsilon h(t) \in$

$Int\mathcal{U}$, $\forall|\varepsilon| < \varepsilon_h$. The previous ε will be considered in our variational arguments. At the same time, the following Cauchy problem is satisfied

$$x^i_{\alpha_1\ldots\alpha_{s-1}\beta\varepsilon}(t,0) = X^i_{\beta x}\left(t, x(t), x_{\alpha_1}(t), \ldots, x_{\alpha_1\ldots\alpha_{s-1}}(t), u(t)\right) x_\varepsilon(t,0)$$

$$+ \ldots + X^i_{\beta x_{\alpha_1\cdots\alpha_{s-1}}}\left(t, x(t), x_{\alpha_1}(t), \ldots, x_{\alpha_1\ldots\alpha_{s-1}}(t), u(t)\right) x_{\alpha_1\ldots\alpha_{s-1}\varepsilon}(t,0)$$

$$+ X^i_{\beta u}\left(t, x(t), x_{\alpha_1}(t), \ldots, x_{\alpha_1\ldots\alpha_{s-1}}(t), u(t)\right) h(t)$$

$$\forall t \in \Omega_{t_0,t_1}, \quad x_\varepsilon(t_0,0) = 0, \quad x_{\alpha_1\ldots\alpha_j\varepsilon}(t_0,0) = 0, \quad j = \overline{1, s-1}.$$

Consider the co-state tensors $p_\gamma(t) = \left(p^\alpha_{i\gamma}(t)\right)$, $\gamma = \overline{1,s}$, of C^1-class and, for $|\varepsilon| < \varepsilon_h$, define the function

$$J(\varepsilon) = \int_{\tilde{\Omega}(\varepsilon)} H\left(t, x(t,\varepsilon), x_{\alpha_1}(t,\varepsilon), \ldots, x_{\alpha_1\ldots\alpha_{s-1}}(t,\varepsilon), u(t,\varepsilon), p_1(t), \ldots, p_s(t)\right)$$

$$- \int_{\tilde{\Omega}(\varepsilon)} \left[p^\lambda_{i1}(t) dx^i(t,\varepsilon) \wedge \omega_\lambda + \ldots + p^\lambda_{is}(t) dx^i_{\alpha_1\ldots\alpha_{s-1}}(t,\varepsilon) \wedge \omega_\lambda\right].$$

To evaluate the multiple integrals,

$$\int_{\tilde{\Omega}(\varepsilon)} p^\lambda_{i1}(t) dx^i(t,\varepsilon) \wedge \omega_\lambda, \ \ldots, \ \int_{\tilde{\Omega}(\varepsilon)} p^\lambda_{is}(t) dx^i_{\alpha_1\ldots\alpha_{s-1}}(t,\varepsilon) \wedge \omega_\lambda,$$

we shall use the following formula

$$d\left(p^\lambda_i x^i \omega_\lambda\right) = \left(p^\lambda_i dx^i + x^i dp^\lambda_i\right) \wedge \omega_\lambda$$

and the Stokes integral formula (see $\left(\eta^\beta(t)\right)$ as the unit normal vector to the boundary $\partial\tilde{\Omega}(\varepsilon)$),

$$\int_{\tilde{\Omega}(\varepsilon)} d\left(p^\lambda_i x^i \omega_\lambda\right) = \int_{\partial\tilde{\Omega}(\varepsilon)} \delta_{\alpha\beta} p^\alpha_i x^i \eta^\beta d\sigma.$$

We find

$$J(\varepsilon) = \int_{\tilde{\Omega}(\varepsilon)} H\left(t, x(t,\varepsilon), x_{\alpha_1}(t,\varepsilon), \ldots, x_{\alpha_1\ldots\alpha_{s-1}}(t,\varepsilon), u(t,\varepsilon), p_1(t), \ldots, p_s(t)\right)$$

$$+ \int_{\tilde{\Omega}(\varepsilon)} \left[dp^\lambda_{j1}(t) x^j(t,\varepsilon) \wedge \omega_\lambda + \ldots + dp^\lambda_{js}(t) x^j_{\alpha_1\ldots\alpha_{s-1}}(t,\varepsilon) \wedge \omega_\lambda\right]$$

$$- \int_{\partial\tilde{\Omega}(\varepsilon)} \delta_{\alpha\beta} p^\alpha_{i1}(t) x^i(t,\varepsilon) \eta^\beta(t) d\sigma - \ldots - \int_{\partial\tilde{\Omega}(\varepsilon)} \delta_{\mu\nu} p^\mu_{is}(t) x^i_{\alpha_1\ldots\alpha_{s-1}}(t,\varepsilon) \zeta^\nu(t) d\theta.$$

Differentiating with respect to ε, evaluating at $\varepsilon = 0$, with $\tilde{\Omega}(0) = \tilde{\Omega}$, we obtain

$$J'(0) = \int_{\tilde{\Omega}} \left[H_{x^j}\left(t, x(t), \ldots, x_{\alpha_1\ldots\alpha_{s-1}}(t), \hat{u}(t), p(t)\right) + dp^\lambda_{j1}(t) \wedge \omega_\lambda\right] x^j_\varepsilon(t,0)$$

$$+ \int_{\tilde{\Omega}} \left[H_{x^j_{\alpha_1}}\left(t, x(t), \ldots, x_{\alpha_1\ldots\alpha_{s-1}}(t), \hat{u}(t), p(t)\right) + dp^\lambda_{j2}(t) \wedge \omega_\lambda\right] x^j_{\alpha_1\varepsilon}(t,0)$$

$$\dots + \int_{\tilde{\Omega}} \left[H_{x^j_{\alpha_1\dots\alpha_{s-1}}} \left(t, \dots, x_{\alpha_1\dots\alpha_{s-1}}(t), \hat{u}(t), p(t)\right) + dp^{\lambda}_{js}(t) \wedge \omega_\lambda \right] x^j_{\alpha_1\dots\alpha_{s-1}\varepsilon}(t,0)$$
$$+ \int_{\tilde{\Omega}} H_{u^a} \left(t, x(t), \dots, x_{\alpha_1\dots\alpha_{s-1}}(t), \hat{u}(t), p(t)\right) h^a(t)$$
$$- \int_{\partial\tilde{\Omega}} \delta_{\alpha\beta} p^{\alpha}_{i1}(t) x^i_{\varepsilon}(t,0) \eta^{\beta}(t) d\sigma - \dots - \int_{\partial\tilde{\Omega}} \delta_{\mu\nu} p^{\mu}_{is}(t) x^i_{\alpha_1\dots\alpha_{s-1}\varepsilon}(t,0) \zeta^{\nu}(t) d\theta,$$

where $x(t)$ is the state vector corresponding to the optimal control vector $\hat{u}(t)$ (see $p(t) := \{p_1(t), \dots, p_s(t)\}$). We need $J'(0) = 0$, for any continuous vector function $h(t) = (h^a(t))$. Now, using the adjoint equations, we select the set $\mathcal{P}$ of co-state tensors as the set of solutions for the next problem (see (2.5))

$$\begin{cases} dp^{\lambda}_{j1}(t) \wedge \omega_\lambda = -H_{x^j} \left(t, x(t), \dots, x_{\alpha_1\dots\alpha_{s-1}}(t), \hat{u}(t), p(t)\right) \\ dp^{\lambda}_{j2}(t) \wedge \omega_\lambda = -H_{x^j_{\alpha_1}} \left(t, x(t), \dots, x_{\alpha_1\dots\alpha_{s-1}}(t), \hat{u}(t), p(t)\right) \\ \vdots \\ dp^{\lambda}_{js}(t) \wedge \omega_\lambda = -H_{x^j_{\alpha_1\dots\alpha_{s-1}}} \left(t, x(t), \dots, x_{\alpha_1\dots\alpha_{s-1}}(t), \hat{u}(t), p(t)\right), \quad \forall t \in \Omega_{t_0,t_1} \\ \delta_{\alpha\beta} p^{\alpha}_{j1}(t) \eta^{\beta}(t) = 0, \; \dots, \; \delta_{\mu\nu} p^{\mu}_{js}(t) \zeta^{\nu}(t) = 0 \quad (\textit{orthogonality / tangency}). \end{cases}$$

Since the variation h is arbitrary, we get the *critical point conditions* (see (2.6))

$$H_{u^a} \left(t, x(t), \dots, x_{\alpha_1\dots\alpha_{s-1}}(t), \hat{u}(t), p(t)\right) = 0, \quad a = \overline{1,k}, \; \forall t \in \Omega_{t_0,t_1}.$$

Moreover, we have

$$dx^i_{\alpha_1\dots\alpha_{r-1}}(t) \wedge \omega_\lambda = \frac{\partial H}{\partial p^{\lambda}_{ir}} \left(t, \dots, x_{\alpha_1\dots\alpha_{s-1}}(t), \hat{u}(t), p(t)\right), \quad \forall t \in \Omega_{t_0,t_1}$$
$$x(t_0) = x_0, \; x_{\alpha_1\dots\alpha_j}(t_0) = \tilde{x}_{\alpha_1\dots\alpha_j 0}, \; j = \overline{1, s-1}, \; i = \overline{1,n}, \; r = \overline{1,s},$$

where $(dx^i_{\alpha_1\dots\alpha_0}(t) := dx^i(t))$, (see (2.4)), and the proof is complete. □

3 Conclusions

The present paper introduced a study of two multi-time optimal control problems subject to distribution-type constraints. Using a geometrical language, the notion of adjointness (see *variational* and *adjoint equations*) on the higher order jet bundles, and variational calculus techniques (under simplified hypotheses), we formulated the main results of this paper (see Theorems 1.1 and 2.1).

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