Torsion field and dark energy

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Abstract. The Absolute Parallelism geometry is used, instead of Riemannian geometry, to investigate the effect of torsion on the dynamics of the Friedmann-Roberttson-Walker model. We show that the torsion field can be considered as the source of the present cosmic acceleration. Moreover, depending on the possible evolution of the torsion field with time, different world models for accelerating Universe, including the dark energy model, can be deduced from our approach.

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1 Introduction

It is well known that physical theories are usually constructed for the following two main aims:
(i) to interpret, qualitatively and quantitatively, observed and/or experimental phenomena.
(ii) to predict new phenomena and/or new physics, if any.
The geometrization philosophy is advocated by Einstein during the second decade of the twentieth century. In the context of the geometrization philosophy Einstein, in his theory of General Relativity (GR), has proved that the 4-dimensional Riemannian space provides a complete representation of the physical world including space and time. It should be noted that:

- GR is a geometric theory for gravity.
- Gravity is nothing but a geometric phenomenon, the curvature of space.
- Riemannian space of 4-dimension is just sufficient to describe the gravitation.

Einstein field equations of GR are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\kappa T_{\mu\nu}.$$

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The left-hand side (L.H.S.) of these equations is purely geometric while the right-hand side (R.H.S.) is a physical object defined from outside the geometry. There is no room in Riemannian geometry to accommodate the material distribution or any other interaction except the gravitational field.

The accelerating expansion of the Universe, discovered in 1998 [10, 7], represents a big problem for theoretical physists. Existing gravity theories, including GR, cannot account for this phenomenon in a satisfactory manner. A solution suggested is the assumption of the existence of an exotic type of energy, dark energy, giving rise to a negative pressure. This suggestion leads to an exponential solution of the Friedmann-Roberttson-Walker (FRW-) dynamical equations of standard cosmology.

The simplest candidate that may be responsible for dark energy is the cosmological constant Λ, inserted in the cosmological term in General Relativity (cf. [18]). A lot of work has been done to obtain the value of Λ, consistent with observations, in order to show that the cosmological constant is responsible for the present cosmic acceleration. There are other approaches to construct models of dark energy. One of these models is based on modifying the R.H.S. of Einstein field equations, by considering specific forms of the energy-momentum tensor $T_{\mu\nu}$, in order to get a negative pressure (eg. quintessence model (cf.[1]), k-essence (cf.[8]), Chaplygin gas model (cf.[4])). Another approach is to modify the L.H.S. of Einstein field equations, called in the literature "modified gravity", by constructing alternative theories depending on the curvature scalar R, f(R)-theories (cf. [6]). All these attempts have been done within the context of Riemannian geometry. A third class of attempts has been done by constructing other type of theories, alternative to GR, in the context of geometries with non-vanishing torsion T [17]. This class is known in the literature as f(T)-theories (cf. [3, 9]). Also, there are theories that deal with both R and T, called F(R,T)-gravity [5].

In the present work, we investigate, within Absolute Parallelism (AP-) geometry, the effect of torsion on the solution of the FRW-dynamical equations. This may throw some light on the nature of dark energy. The article is organized as follows. The next section gives a brief review of the main features of AP-geometry, a geometry with non-vanishing torsion. Section 3 is concerned with the investigation of an AP-structure satisfying the cosmological principle and the evaluation of tensors necessary for the present application. In Section 4, the effect of the presence of a non-vanishing torsion on the FRW-dynamics is investigated. The paper ends, in Section 5, with some concluding remarks.

2 A geometry with non-vanishing torsion

In this section we use a simple type of geometry with non-vanishing torsion, the AP-geometry (cf.[13, 19, 16]). We review briefly some of its properties and the geometric objects necessary for the present work. It is worth mentioning that calculations in AP-geometry are much more easier compared to other types of geometries with non-vanishing torsion.

In 4-dimensions, the structure of an AP-space $(M, \underset{i}{\lambda})$ is defined by a tetrad vector fields $\underset{i}{\lambda}^{\mu}$, where i $(= 0, 1, 2, 3)$ denotes vector numbering and μ $(= 0, 1, 2, 3)$ denotes coordinate components of the tetrad vector fields. It is assumed that the vector fields $\underset{i}{\lambda}$ are independent and globally defined on M. Consequently, the matrix $(\underset{i}{\lambda}^{\mu})$ is nondegenerate so that[1]

$$\underset{i}{\lambda}^{\mu}\underset{i}{\lambda}_{\nu}=\delta^{\mu}_{\nu},$$
$$\underset{i}{\lambda}^{\nu}\underset{j}{\lambda}_{\nu}=\delta_{ij}.$$

Using the tetrad vector fields, a Riemannian metric on M is defined by

$$g_{\alpha\beta}\stackrel{def}{=}\underset{i}{\lambda}_{\alpha}\underset{i}{\lambda}_{\beta}, \tag{2.1}$$

with inverse

$$g^{\mu\nu}=\underset{i}{\lambda}^{\mu}\underset{i}{\lambda}^{\nu}.$$

Consequently, we have

$$g^{\mu\alpha}g_{\alpha\nu}=\delta^{\mu}_{\nu},\quad g\stackrel{def}{=}\|g_{\mu\nu}\|\neq 0. \tag{2.2}$$

The AP-condition

$$\underset{i}{\lambda}{}^{\mu}{}_{\stackrel{+}{|}\nu}=0 \tag{2.3}$$

implies the existence of a linear connection

$$\Gamma^{\alpha}_{.\ \mu\nu}=\underset{i}{\lambda}^{\alpha}\underset{i}{\lambda}_{\mu,\nu}, \tag{2.4}$$

which is the solution of (2.3). Here we use the strock (|) and the (+) sign to denote covariant differentiation with respect to the connection (2.4), and the comma (,) to denote ordinary partial differentiation.

It is clear that the linear connection (2.4) is non-symmetric and its torsion is thus given by

$$\Lambda^{\alpha}_{.\ \mu\nu}\stackrel{def}{=}\Gamma^{\alpha}_{.\ \mu\nu}-\Gamma^{\alpha}_{.\ \nu\mu}. \tag{2.5}$$

On the other hand, the non-symmetry of (2.4) implies that its dual $\widetilde{\Gamma}^{\alpha}_{.\ \mu\nu}(\stackrel{def}{=}\Gamma^{\alpha}_{.\ \nu\mu})$ and its symmetric part $\Gamma^{\alpha}_{(\mu\nu)}(\stackrel{def}{=}\frac{1}{2}(\Gamma^{\alpha}_{.\ \mu\nu}+\Gamma^{\alpha}_{.\ \nu\mu}))$ are also linear connections. Moreover, we have the Riemannian connection $\{^{\alpha}_{\mu\nu}\}$ associated with (2.1). We define the third order tensor

$$\gamma^{\alpha}_{.\ \mu\nu}\stackrel{def}{=}\underset{i}{\lambda}^{\alpha}\underset{i}{\lambda}_{\mu;\nu}.$$

where (;) denotes the covariant differentiation with respect to the Christoffel symbols $\{^{\alpha}_{\mu\nu}\}$. The tensor $\gamma^{\alpha}_{.\ \mu\nu}$ is called *the contortion* of the space. It is easy to derive the following relations [13]:

$$\begin{cases}\gamma^{\alpha}_{.\ \mu\nu}=\Gamma^{\alpha}_{.\ \mu\nu}-\{^{\alpha}_{\mu\nu}\},\\ \Lambda^{\alpha}_{.\ \mu\nu}=\gamma^{\alpha}_{.\ \mu\nu}-\gamma^{\alpha}_{.\ \nu\mu},\\ C_{\mu}\stackrel{def}{=}\Lambda^{\alpha}_{.\ \mu\alpha}=\gamma^{\alpha}_{.\ \mu\alpha}.\end{cases} \tag{2.6}$$

The 1-form C_{μ} is known as *the basic form of the AP-space.*

[1] In the present article we use Latin (mesh) indices for vector numbering and Greek (world) indices for coordinate components. Einstein summation convention is applied to Greek indices in the usual manner, while for Latin indices it is applied to repeated indices wherever their positions are.

3 AP-structure for cosmological applications

Robertson [11] has derived the most general two AP-structures with homogeneity and isotropy, i.e. satisfying the cosmological principle[1]. Further investigation of the two structures have been done in [12]. In the present work, we use the Robertson second AP-space whose structure is given by the following matrix (coordinate system used are: $x^0 = t$, $x^1 = x$, $x^2 = y$, $x^3 = z$)

(3.1)
$$(\underset{j}{\lambda}{}^{\mu}) = \begin{pmatrix} (1-\frac{k}{4}r^2)h & -ik^{\frac{1}{2}}\frac{x}{a} & -ik^{\frac{1}{2}}\frac{y}{a} & -ik^{\frac{1}{2}}\frac{z}{a} \\ k^{\frac{1}{2}}xh & \frac{i}{a}(1+\frac{k}{4}r^2-\frac{k}{2}x^2) & -ik\frac{xy}{2a} & -ik\frac{xz}{2a} \\ k^{\frac{1}{2}}yh & -ik\frac{yx}{2a} & \frac{i}{a}(1+\frac{k}{4}r^2-\frac{k}{2}y^2) & -ik\frac{yz}{2a} \\ k^{\frac{1}{2}}zh & -ik\frac{zx}{2a} & -ik\frac{zy}{2a} & \frac{i}{a}(1+\frac{k}{4}r^2-\frac{k}{2}z^2) \end{pmatrix},$$

where a is a function of time only, k $(= 0, 1, -1)$ is the sectional curvature of the space, $r = \sqrt{x^2 + y^2 + z^2}$ and

$$h = \left(1 + \frac{1}{4}kr^2\right)^{-1}.$$

Using (2.1), we can compute the metric tensor of the Riemannian structure associated with the AP-structure (3.1). The non-vanishing components of $g_{\mu\nu}$ are:

$$g_{00} = 1,\ g_{11} = g_{22} = g_{33} = -a^2h^2.$$

Using (2.2), the non-vanishing contravariant components of the metric tensor are written in the form

$$g^{00} = 1,\ g^{11} = g^{22} = g^{33} = -a^{-2}h^{-2}. \tag{3.2}$$

Consequently, the covariant components of $\underset{i}{\lambda}{}^{\mu}$ can be written in the form

$$(\underset{j}{\lambda}{}_{\mu}) = \begin{pmatrix} (1-\frac{k}{4}r^2)hc & ik^{\frac{1}{2}}xah^2 & ik^{\frac{1}{2}}ayh^2 & ik^{\frac{1}{2}}zah^2 \\ k^{\frac{1}{2}}xhc & -iXah^2 & ikxyah^2 & \frac{i}{2}kxzah^2 \\ k^{\frac{1}{2}}yhc & \frac{i}{2}yxkah^2 & -iYah^2 & \frac{i}{2}kyzah^2 \\ k^{\frac{1}{2}}zhc & \frac{i}{2}zxkah^2 & \frac{i}{2}zkah^2 & -iZah^2 \end{pmatrix}, \tag{3.3}$$

where

$$X = 1 + \frac{k}{4}r^2 - \frac{k}{2}x^2, \quad Y = 1 + \frac{k}{4}r^2 - \frac{k}{2}y^2, \quad Z = 1 + \frac{k}{4}r^2 - \frac{k}{2}z^2$$

[1]In this article, we use relativistic system of units $c = G = 1$, where c is the speed of light and G is the Newton's gravitational constant.

By (3.1) and (3.3), we get the following non-vanishing coefficients of the linear connection (2.4):

$$(3.4)\qquad \begin{cases} \Gamma^0_{.11} = \Gamma^0_{.22} = \Gamma^0_{.33} = i\ a\ \sqrt{k}\ h^2 \\ \Gamma^1_{.01} = \Gamma^2_{.02} = \Gamma^3_{.03} = \frac{i}{a}\sqrt{k} \\ \Gamma^1_{.10} = \Gamma^2_{.20} = \Gamma^3_{.30} = \frac{\dot{a}}{a} \\ \Gamma^1_{.11} = \Gamma^2_{.12} = \Gamma^2_{.21} = \Gamma^3_{.13} = \Gamma^3_{.31} = -\Gamma^1_{.22} = -\Gamma^1_{.33} = -\frac{h}{2}kx \\ \Gamma^1_{.12} = \Gamma^1_{.21} = \Gamma^2_{.22} = \Gamma^3_{.23} = \Gamma^3_{.32} = -\Gamma^2_{.11} = -\Gamma^2_{.33} = -\frac{h}{2}ky \\ \Gamma^1_{.13} = \Gamma^1_{.31} = \Gamma^2_{.23} = \Gamma^2_{.32} = \Gamma^2_{.33} = -\Gamma^3_{.11} = -\Gamma^3_{.22} = -\frac{h}{2}kz. \end{cases}$$

Using (2.5) and the connection coefficients (3.4), we get the following non-vanishing components of the torsion tensor:

$$(3.5)\qquad \begin{aligned} \Lambda^1_{.01} = \Lambda^2_{.02} = \Lambda^3_{.03} = -\Lambda^1_{.10} = -\Lambda^2_{.20} = -\Lambda^3_{.30} = \frac{i\sqrt{k} - \dot{a}}{a}, \\ \Lambda^1_{.32} = \Lambda^2_{.13} = \Lambda^3_{.21} = -\Lambda^1_{.23} = -\Lambda^2_{.31} = -\Lambda^3_{.12} = 2\sqrt{k}h, \end{aligned}$$

where $\dot{a}$ denotes the derivative of a with respect to time. Consequently, by (2.6) and (3.5), the only non-vanishing component of the basic form C_μ is given by:

$$C_0 = 3\left(\frac{i\sqrt{k} - \dot{a}}{a}\right).$$

To facilitate comparison with FRW-cosmology, we take $k = 0$. So, the above equation takes the form:

$$(3.6)\qquad C_0 = -3\left(\frac{\dot{a}}{a}\right).$$

Now, we define the torsion scalar $\mathcal{T}$ by

$$(3.7)\qquad \mathcal{T} \overset{def}{=} \sqrt{g^{\mu\nu}C_\mu C_\nu},$$

Making use of (3.2), (3.6) and (3.7), $\mathcal{T}$ is written as

$$(3.8)\qquad \mathcal{T} = 3\left(\frac{\dot{a}}{a}\right).$$

This is the value of the torsion scalar characterizing the AP-structure (3.1) (in the case of $k = 0$).

4 Effect of torsion on the dynamics of FRW-model

The application of Einstein field equations to the Robertson structure (3.1) gives rise to the FRW-dynamical equations:

$$(4.1)\qquad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi\rho - \frac{3k}{a^2}.$$

$$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi(\rho + 3P). \tag{4.2}$$

The function a is called the scale factor in the context of FRW-cosmology.

The torsion field (3.7) is related to the scale factor a by (3.8):

$$\mathcal{T} = 3\left(\frac{\dot{a}}{a}\right). \tag{4.3}$$

In order to investigate the effect of the torsion field $\mathcal{T}$ on the solutions of the FRW-dynamical equations (4.1) and (4.2), we proceed as follows. Differentiating (4.3) gives

$$\dot{\mathcal{T}} = 3\left(\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2\right).$$

From which, we can write

$$\frac{\ddot{a}}{a} = \frac{1}{9}\mathcal{T}^2 + \frac{1}{3}\dot{\mathcal{T}}. \tag{4.4}$$

Comparison of the pair of equations (4.3) and (4.4) with the pair (4.1) and (4.2), for $k = 0$, suggests the following definitions for the *torsion density* $\rho_{_\mathcal{T}}$ *and pressure* $P_{_\mathcal{T}}$, respectively:

$$\rho_{_\mathcal{T}} \stackrel{def}{=} \frac{1}{24\pi}\mathcal{T}^2, \tag{4.5}$$

$$P_{_\mathcal{T}} \stackrel{def}{=} -\frac{1}{4\pi}\left(\frac{1}{6}\mathcal{T}^2 + \frac{1}{3}\dot{\mathcal{T}}\right). \tag{4.6}$$

Equations (4.5) and (4.6) enable us to write an equation of state in the form:

$$P_{_\mathcal{T}} = -(1+\epsilon)\rho_{_\mathcal{T}}, \tag{4.7}$$

where

$$\epsilon \stackrel{def}{=} \frac{2\dot{\mathcal{T}}}{\mathcal{T}^2}.$$

Equation (4.7) represents an equation of state of a perfect fluid, where the energy density and pressure here are induced by the torsion field $\mathcal{T}$. The corresponding equation of state in FRW-standard cosmology is given by:

$$P = \omega\rho. \tag{4.8}$$

Comparison of (4.7) and (4.8) enables us to write

$$\omega_{_\mathcal{T}}(t) \stackrel{def}{=} -(1+\epsilon). \tag{4.9}$$

It should be noted here that the equation of state parameter $\omega_{_\mathcal{T}}(t)$ is a function of time. In the present work we are interested in the accelerating expansion of the universe. Our equation of state (4.7) shows that the evolution of the universe depends mainly on the value of the function ϵ. The case $\epsilon = 0$ implies that $P_{_\mathcal{T}} = -\rho_{_\mathcal{T}}$. Consequently, from (4.9), $\omega_\mathcal{T} = -1$, which corresponds to dark energy (solution of the accelerating expansion problems in the context of FRW-cosmology). We may thus conclude that the torsion field can act as a source of dark energy, and this is tempting to believe that the present universe is dominated by torsion field.

5 Concluding remarks

1. In the present work we show that if the Universe is dominated by a torsion field, the problem of the accelerating expansion of the universe is geometrically solved without the need to introduce a scalar field from outside the geometry.

2. The torsion field is inserted into the FRW-dynamical equations which are consequences of the GR field equations in the cosmological domain.

3. It is well known that a Riemannian space is torsion-free. For this reason, to investigate the effect of torsion on the solution of FRW-dynamical equations, we have moved to AP-space which has a non-vanishing torsion (besides an associated Riemannian structure [13, 19, 16]).

4. The general AP-structure (3.1) has the same FRW-metric with homogeneity and isotropy.

5. On one hand, there are no definitions for the energy density and pressure density in the context of the AP-structure (3.1). On the other hand, there is no torsion in the Riemannian space within which the FRW-dynamical equations are written. The comparison between the FRW-dynamical equations of the world model and the values of the torsion scalar in the structure (3.1), enables us to define density and pressure associated with the torsion field.

6. The comparison of the values of the density and pressure has given rise to an equation of state, for the torsion field, leading to a negative pressure.

7. Dark energy maybe interpreted as due to the existence of a non-vanishing torsion field in the Universe. This leads to an accelerating expansion of the Universe. Some similar results have been obtained, but from different points of view [14, 15]

8. Others physical models of accelerating expansion of the universe can be deduced from our model:

- When $\epsilon > \mathbf{0}$, equations (4.8) and (4.9) gives negative pressure ($w < -1$), which leads to the phantom accelerating universe "Super-Negative Equation of State" (cf.[2]).
- When $\epsilon < \mathbf{0}$ and $-\mathbf{1} < \omega < -\frac{\mathbf{1}}{\mathbf{3}}$, the parameter $w(t)$ varies with time, which leads to the tracking model of Dark Energy (cf.[1]).

9. It should finally be noted that the present treatment is purely geometric.

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