

Spanned vector bundles on product varieties

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Abstract. Fix an integer $s \geq 2$ and projective smooth varieties X_i , $1 \leq i \leq s$. For any spanned vector bundle \mathcal{E} on $X := X_1 \times \cdots \times X_s$ let $\rho(\mathcal{E})$ be the minimal integer $\sum_{h=1}^x \text{rank}(U_h)$, where x is a positive integer, each U_h is a spanned vector bundle on X , $U_h \cong A_{1,h} \boxtimes \cdots \boxtimes A_{s,h}$ with $A_{i,h}$ spanned vector bundle on X_i and \mathcal{E} is a quotient of $\bigoplus_{h=1}^x U_h$. For each integer $r \geq \dim(X) - 1$ we prove the existence of spanned and stable vector bundles \mathcal{E} with $\rho(\mathcal{E}) > r$ (and $\rho(\mathcal{E}) = r + 1$ if $r \geq \dim(X)$).

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1 Introduction

Fix an integer $s \geq 2$ and smooth and connected projective varieties X_i , $1 \leq i \leq s$. Set $X := X_1 \times \cdots \times X_s$ and let $\pi_i : X \rightarrow X_i$ denote the projection on the i -th factor. For all vector bundles A_i on X_i , $1 \leq i \leq s$, set $A_1 \boxtimes \cdots \boxtimes A_s := \pi_1^*(A_1) \otimes \cdots \otimes \pi_s^*(A_s)$. A is spanned if and only if each A_i is spanned. Let \mathcal{E} be a spanned vector bundle on X . Let $\rho(\mathcal{E})$ be the minimal integer $\sum_{h=1}^x \text{rank}(U_h)$, where x is a positive integer, each U_h is a spanned vector bundle on X , $U_h \cong A_{1,h} \boxtimes \cdots \boxtimes A_{s,h}$ with $A_{i,h}$ spanned vector bundle on X_i and \mathcal{E} is a quotient of $\bigoplus_{h=1}^x U_h$. Among all (U_1, \dots, U_x) evincing $\rho(\mathcal{E})$ and with each $U_{i,h}$ indecomposable, call $\rho'(\mathcal{E})$ the minimal such integer x . We always have $\rho(\mathcal{E}) \leq \min\{h^0(\mathcal{E}), \text{rank}(\mathcal{E}) + \dim(X)\}$ (take all $A_{i,h}$ the trivial line bundle and apply [2, Theorem 2]). It is easy to give a “tautological” description of all spanned bundles \mathcal{E} with $\rho(\mathcal{E}) \leq \text{rank}(\mathcal{E}) + 1$ (see Remark 2.1).

Let $r(X_1, \dots, X_s)$ be the minimal rank of a spanned vector bundle \mathcal{E} on X with $\rho(\mathcal{E}) > \text{rank}(\mathcal{E})$. We see that the integer $r(X_1, \dots, X_s)$ is well-defined and that $r(X_1, \dots, X_s) \leq \dim(X)$ (Theorem 1.1), and that $r(X_1, \dots, X_s) \leq \dim(X) - 1$ if $\dim(X) \geq 3$ (Theorem 1.2). Recall that two projective varieties A, Y are called *Pic-independent* if $\text{Pic}(A \times Y) \cong \text{Pic}(A) \otimes \text{Pic}(Y)$ ([6]). We say that X_1, \dots, X_s are Pic-independent if $\text{Pic}(X) \cong \text{Pic}(X_1) \otimes \cdots \otimes \text{Pic}(X_s)$. For instance this is the case if $s - 1$ of the varieties X_i are projective spaces ([6]). The varieties X_1, \dots, X_s are Pic-independent if and only if $r(X_1, \dots, X_s) > 1$ (Lemma 2.3).

As an open question we ask to compute $r(X_1, \dots, X_s)$ for some interesting X_1, \dots, X_s . We list our main results:

Theorem 1.1. *Fix X_1, \dots, X_s and any polarization R on X . Fix an integer $r \geq n := \dim(X)$. Then there is a rank r spanned vector bundle \mathcal{E} on X with $\rho(\mathcal{E}) = r + 1$ and slope R -stable.*

Theorem 1.2. *Fix X_1, \dots, X_s and set $n := \dim(X)$. If $n \geq 3$, then $r(X_1, \dots, X_s) \leq n - 1$ and for any polarization R on X there is a slope R -stable spanned bundle \mathcal{E} with $\text{rank}(\mathcal{E}) = n - 1 < \rho(\mathcal{E})$.*

Proposition 1.3. *If $\dim(X) = 2, 3, 4$ (resp. $\dim(X) = 5$), then $r(X_1, \dots, X_s) \leq 2$ (resp. $r(X_1, \dots, X_s) \leq 3$).*

By Lemma 2.3 and Proposition 1.3 when $2 \leq \dim(X) \leq 4$ either $r(X_1, \dots, X_s) = 1$ (case in which X_1, \dots, X_s are Pic-independent) or $r(X_1, \dots, X_s) = 2$ (case in which X_1, \dots, X_s are not Pic-independent).

Another interesting invariant of X_1, \dots, X_s is the maximum $d(X_1, \dots, X_s)$ of all integers $\rho(\mathcal{E}) - \text{rank}(\mathcal{E})$ with \mathcal{E} a spanned vector bundle on X . We have $d(X_1, \dots, X_s) \leq \dim(X)$, but we do not know when this inequality is sharp.

We work over an algebraically closed field with characteristic zero.

2 The proofs

Remark 2.1. Let \mathcal{E} be a rank r spanned vector bundle on X . It is a tautology to say that $\rho(\mathcal{E}) = r$ if and only if $\mathcal{E} \cong \bigoplus_{h=1}^x A_{1,h} \boxtimes \dots \boxtimes A_{s,h}$ for some $x > 0$ and some spanned vector bundles $A_{i,h}$ on X_i , but given \mathcal{E} it seems not to be easy to test if or not the bundles $A_{i,h}$ exists. It is a tautology to say that $\rho(\mathcal{E}) \leq r + 1$ if and only if there is a rank $r + 1$ spanned bundle \mathcal{F} on X , $R \in \text{Pic}(X)$ and a non-zero map $f : R \rightarrow \mathcal{F}$ such that $\rho(\mathcal{F}) \leq r + 1$ and $\text{coker}(f) \cong \mathcal{E}$.

Lemma 2.1. *Set $A := A_1 \boxtimes \dots \boxtimes A_s$. If A has a trivial factor, then each A_i has a trivial factor.*

Proof. Up to renaming the factors it is sufficient to do the case $i = s$. Fix $P_j \in X_j$, $1 \leq j \leq s - 1$, and set $\{O\} := \{P_1\} \times \dots \times \{P_{s-1}\}$. By assumption $A|_{\{O\} \times X_s}$ has a trivial factor. Since $A|_{\{O\} \times X_s}$ is isomorphic to a finite direct sum of A_s (up to the identification of $\{O\} \times X_s$ with X_s), A_s has a trivial factor by the uniqueness of the Krull-Schmidt decomposition ([1, Theorem 3]). \square

Proposition 2.2. *For any spanned vector bundle \mathcal{E} on X and any integer $k > 0$ we have $\rho(\mathcal{E} \oplus \mathcal{O}_X^{\oplus k}) = \rho(\mathcal{E}) + k$*

Proof. It is sufficient to do the case $k = 1$. Take $x > 0$ and spanned vector bundles $A_{i,j}$ on X_i , $1 \leq i \leq s$, $1 \leq j \leq x$, such that $\mathcal{E} \oplus \mathcal{O}_X$ is a quotient of $F := \bigoplus_{j=1}^x F_j$, where $F_j := A_{1,j} \boxtimes \dots \boxtimes A_{s,j}$ and $\text{rank}(F) = \rho(\mathcal{E} \oplus \mathcal{O}_X)$. Call $u : F \rightarrow \mathcal{E} \oplus \mathcal{O}_X$ the quotient map and $u' : F \rightarrow \mathcal{O}_X$ the surjection induced by u . Since F is spanned, u' induces a splitting, i.e. $F = G \oplus \mathcal{O}_X$ with $u'|_{\{0\} \oplus \mathcal{O}_X} \rightarrow \mathcal{O}_X$ an isomorphism. By the uniqueness of the Krull-Schmidt decompositions of a vector bundle on a projective

variety ([1, Theorem 3]), there is $h \in \{1, \dots, x\}$ such that \mathcal{O}_X is a factor of F_h and $u'|_{F_h}$ is surjective. By Lemma 2.1 \mathcal{O}_{X_i} is a factor of each $A_{i,h}$, say $A_{i,h} = B_{i,h} \oplus \mathcal{O}_{X_i}$ with either $B_{i,h}$ a spanned vector bundle on X_i or the trivial sheaf on X_i . Set $G' := \bigoplus_{j \neq h} F_j$ and $G'' := \bigoplus_{i=1}^s M_i$, where $M_i := D_1 \boxtimes \dots \boxtimes D_s$ with $D_e = A_{e,h}$ if $e \neq i$ and $D_i = B_{i,h}$. Since $u'|_{F_h}$ is surjective, we get that the composition of $u|_{G' \oplus G''}$ with the projection $\mathcal{E} \oplus \mathcal{O}_X \rightarrow \mathcal{E}$ is surjective. Hence $\rho(\mathcal{E}) \leq \rho(\mathcal{E} \oplus \mathcal{O}_X) - 1$. The opposite inequality is trivial. \square

Let \mathcal{E}, \mathcal{F} be spanned vector bundles on X . We have $\rho(\mathcal{E} \oplus \mathcal{F}) \leq \rho(\mathcal{E}) + \rho(\mathcal{F})$, but often strict inequality holds (see Example 2.2). Hence Proposition 2.2 seems to be a quite exceptional phenomenon.

Example 2.2. Let F be any spanned vector bundle on X with $\rho(F) > \text{rank}(F)$ (e.g., use Theorem 1.1). Set $r := \text{rank}(F)$ and $y := \dim(X)$. For each integer $t > 0$ we have $\rho(F^{\oplus t}) \leq rt + y$. Hence $\rho(F^{\oplus t}) < t\rho(F)$ if $t > y$.

Lemma 2.3. X_1, \dots, X_s are Pic-independent if and only if $r(X_1, \dots, X_s) > 1$.

Proof. If X_1, \dots, X_s are Pic-independent, then for each spanned line bundle \mathcal{L} on X there are spanned line bundles L_i on X_i , $1 \leq i \leq s$, with $\mathcal{L} \cong L_1 \boxtimes \dots \boxtimes L_s$. If the latter is true we have $\text{Pic}(X) \cong \text{Pic}(X_1) \otimes \dots \otimes \text{Pic}(X_s)$, because (as in any projective variety) every element of $\text{Pic}(X)$ is of the form $\mathcal{L} \otimes \mathcal{R}^\vee$ with \mathcal{L} and \mathcal{R} spanned line bundles. \square

Proof of Theorem 1.1: Fix $O_i \in X_i$, $i = 1, \dots, s$. For each $i = 1, \dots, s$ set $X'_i := \prod_{h=1}^s U_h$ with $U_h = \{O_h\}$ if $h \neq i$ and $U_i := X_i$. Each X'_i is an irreducible subvariety of X and $X'_i \cong X_i$. Fix an integer $t \gg 0$ such that $R^{\otimes t}$ is very ample and take a general $(n+1)$ -dimensional linear subspace $V \subseteq H^0(R^{\otimes t})$. We get a finite and separable morphism $f : X \rightarrow \mathbb{P}^n$ such that no variety $f(X'_i)$, $1 \leq i \leq s$, is contained in a hyperplane of \mathbb{P}^n . Fix an integer $d > 0$ such that $n \leq r \leq \binom{d+2}{2} + n - 3$ and let F be the cokernel of a general map $\alpha := \mathcal{O}_{\mathbb{P}^n}(-d) \rightarrow \mathcal{O}_{\mathbb{P}^n}^{r+1}$. By [5, Theorem 4.2] F is slope stable (one may also use [4], [3] or [14]). Set $\mathcal{E} := f^*(F)$. Since F is spanned, \mathcal{E} is spanned. Since F is slope stable, \mathcal{E} is slope R -polystable ([11]). By construction there is a surjection $\mathcal{O}_X^{\oplus(r+1)} \rightarrow \mathcal{E}$. Hence $\rho(\mathcal{E}) \leq r + 1$.

Claim 1: For $t \gg 0$ and a general V the bundle \mathcal{E} is slope R -stable.

Proof of Claim 1: For $t \gg 0$ and a general V we may assume that the ramification divisor $D \subset X$ of f is an irreducible variety and that $f|_D$ is birational onto its image. Fix a general line $M \subseteq \mathbb{P}^n$ and set $Y := f^{-1}(M)$. For a general M the scheme $Y := f^{-1}(M)$ is an irreducible and smooth curve by the first and second theorem of Bertini ([10]). It is sufficient to prove that $G := \mathcal{E}|_Y$ is slope stable. Set $g := f|_Y$ and $z := \deg(f)$. The morphism g is a finite covering ramified exactly over $f(D) \cap M$ and over each ramification point $f(D) \cap M$ the fiber of f has cardinality $z - 1$, i.e. each ramification point of g is an ordinary ramification point and no two of the ramification points of g have the same image. Hence there are no maps $u : Y \rightarrow T$, $v : T \rightarrow C$ with $g = u \circ v$, v unramified and $\deg(v) > 1$, i.e. g is strongly ramified in the sense of [15, Definition 3.1]. G is stable ([15, Lemma 3.5 (b)]). Hence \mathcal{E} is indecomposable. Since \mathcal{E} is R -polystable, it is slope R -stable.

Set $H := R^{\otimes t}$. We have $H = f^*(\mathcal{O}_{\mathbb{P}^n}(1))$. By the definition of f and V we have an exact sequence

$$0 \rightarrow (H^\vee)^{\otimes d} \xrightarrow{u} V \otimes \mathcal{O}_X \rightarrow \mathcal{E} \rightarrow 0$$

Since H is ample, Kodaira's vanishing gives $H^0(\mathcal{E}) = \dim(V) = r + 1$.

Assume $\rho(\mathcal{E}) = r$. Since \mathcal{E} is indecomposable (Claim 1), there are spanned vector bundles A_i on X_i , $1 \leq i \leq s$, such that $\mathcal{E} \cong A_1 \boxtimes \cdots \boxtimes A_s$. Set $r_i := \text{rank}(A_i)$ and $a_i := h^0(A_i)$. We have $r = \prod_{i=1}^s r_i$ and $r + 1 = \prod_{i=1}^s a_i$ (Künneth). Since each A_i is spanned, we have $a_i \geq r_i$ and $a_i = r_i$ if and only if A_i is trivial. Since $s \geq 2$, we get a contradiction, unless A_i is trivial for at least one index i . This is not possible, because H is ample. \square

Lemma 2.4. *Let C be a smooth and connected projective curve and let A, B spanned vector bundles on C . Set $a := \text{rank}(A)$, $b := \text{rank}(B)$ and assume $a \geq 2$ and $A \otimes B \cong R \oplus \mathcal{O}_C^{\oplus(ab-1)}$ with R an ample line bundle. Then $b = 1$ and $B \cong \mathcal{O}_C$.*

Proof. Write $A \cong A' \oplus \mathcal{O}_C^{\oplus c}$ and $B \cong B' \oplus \mathcal{O}_C^{\oplus d}$ with A' and B' with no trivial factor (but they may be the zero sheaf). By the Krull-Schmidt theorem it is sufficient to prove that either A' or B' is the zero-sheaf. Assume $B' \neq 0$ and $A' \neq 0$. If $d > 0$ (resp. $c > 0$) then we get $A' \cong \mathcal{O}_C^{\oplus a'}$ (resp. $B' \cong \mathcal{O}_C^{\oplus b'}$), a contradiction. Hence $d = c = 0$. We get $A' = A$ and $B' = B$. Since \mathcal{O}_C is not a factor of A and A is spanned, then $h^0(A) \geq a + 1$. Fix $P \in C$. Since $h^0(A) \geq a + 1$, there is $s \in H^0(A)$ such that $s \neq 0$ and s vanishes at P . Hence there is an inclusion $i : \mathcal{O}_C(P) \hookrightarrow A$. Let $\mathcal{O}_C(D)$ be the saturation of $i(\mathcal{O}_C(P))$. By assumption $D - P \geq 0$ and hence $\mathcal{O}_C(D)$ is ample. If $a > 1$, we apply the same observation to the rank $r - 1$ spanned bundle $A/\mathcal{O}_C(D)$. By induction on a we get the existence of a filtration of A by subbundles A_i , $1 \leq i \leq a$ with $A_1 = \mathcal{O}_C(D)$, $A_a = A$ and each A_i/A_{i-1} , $2 \leq i \leq a$, an ample line bundle. By tensoring this filtration with B we get $a = b = 1$, a contradiction. \square

Proof of Theorem 1.2: Let F be a Tango bundle ([16]). By [9] F is spanned, $c_1(F) = n - 1$ and F is stable. Take $t, V, f : X \rightarrow \mathbb{P}^n$ as in the proof of Theorem 1.1. Set $\mathcal{E} := f^*(F)$. As in the proof of Theorem 1.1 we see that \mathcal{E} is slope R -stable. Assume $\rho(\mathcal{E}) = n - 1$. Since \mathcal{E} is indecomposable, there are spanned vector bundles A_i on X_i , $1 \leq i \leq s$, such that $\mathcal{E} \cong A_1 \boxtimes \cdots \boxtimes A_s$. Set $r_i := \text{rank}(A_i)$. For a general Tango bundle F there is a line $T \subset \mathbb{P}^n$ such that $F|_T$ has splitting type $b_1 \geq \cdots \geq b_{n-1}$ with $b_1 = 2$, $b_i = 1$ is $2 \leq i \leq n - 2$ and $b_{n-1} = 0$. The line T is not a general line of \mathbb{P}^n , but since all lines of \mathbb{P}^n are projectively equivalent, we may change f and V (and hence \mathcal{E}) and obtain f such that $C := f^{-1}(T)$ is a smooth and irreducible curve. Set $G := \mathcal{E}|_C$, $\mathcal{O}_C(x) := f^*(\mathcal{O}_{\mathbb{P}^n}(x))$, $B_i := A_i|_C$ and write $B_i = B'_i \oplus \mathcal{O}_C^{k_i}$ with $k_i \geq 0$ and B'_i with no trivial factor. Since $G \cong \mathcal{O}_C(2) \oplus \mathcal{O}_C(1)^{\oplus(n-3)} \oplus \mathcal{O}_C$, Lemma 2.1 gives that $k_i > 0$ for all i and that $\min_i \{k_i\} = 1$. Write $B_i := D_i \oplus \mathcal{O}_C$. We get that $\otimes_{i=1}^s D_i$ has a factor isomorphic to $\mathcal{O}_C(2)$ and all the other factors (if any) isomorphic to $\mathcal{O}_C(1)$, while $\oplus_{i=1}^s D_i$ is a direct sum of copies of $\mathcal{O}_C(1)$. Since $\mathcal{O}_C(1)$ is ample, we get that $s = 2$ and $\text{rank}(D_i) = 1$ for all i , i.e. this case may occur only if $n = 5$. To quote [9] it is easier to use the existence of extremal jumping lines of F , i.e. lines T such that $F|_T$ has splitting type $b_1 \geq \cdots \geq b_{n-1}$ with $b_1 = n - 1$ and $b_i = 0$ if $2 \leq i \leq n - 1$ ([9]). Since $B_1 \otimes \cdots \otimes B_s \cong \mathcal{O}_C(n - 1) \oplus \mathcal{O}_C^{\oplus(n-2)}$, Lemma 2.4 gives $r_i = k_i = 1$ for $s - 1$ indices i , and $r_i = n - 1$, $k_i = n - 2$, for the other index i , say

for $i = 1$. In particular for each $i \geq 2$, $A'_i := \pi_i^*(A_i)$ is a spanned line bundle on X and it is not trivial. Hence $a_i \geq 2$ for all $i \geq 2$. Since $r_1 = n - 1$ and A_1 is not trivial, we have $a_1 \geq n$. Hence $h^0(\mathcal{E}) \geq 2n$. We have an exact sequence ([9, (4.1)]):

$$(2.1) \quad 0 \rightarrow T\mathbb{P}^n(-2) \rightarrow \mathcal{O}_{\mathbb{P}^n}^{2n-1} \rightarrow F \rightarrow 0$$

To get a contradiction it is sufficient to prove that $h^1(f^*(T\mathbb{P}^n(-2))) = 0$. We twist (2.1) by $\mathcal{O}_{\mathbb{P}^n}(-1)$, pull back it by f^* and then we use Kodaira's vanishing. \square

Proof of Proposition 1.3: To prove Proposition 1.3 we may assume that X_1, \dots, X_s are Pic-independent (Lemma 2.3). We use spanned twists of one of the many rank 2 stable bundles on \mathbb{P}^2 and \mathbb{P}^3 with odd c_1 , of the Horrocks-Mumford's bundle on \mathbb{P}^4 ([8]) and the rank 3 Horrocks bundle on \mathbb{P}^5 ([7]) and recycle the proof of Theorem 1.2. Set $n := \dim(X)$. Assume $\rho(\mathcal{E}) = \text{rank}(F)$. Since $\mathcal{E} := f^*(F)$ is R -stable, it is indecomposable and hence there are spanned vector bundles A_i on X_i such that $\mathcal{E} \cong A_1 \boxtimes \dots \boxtimes A_s$. Since 2 and 3 are prime integers, there is an integer $e \in \{1, \dots, s\}$ such that A_i is a line bundle if $i \neq e$. Without losing generality we may assume $e = 1$. Set $L_1 := \det(A_1)$ and $L_i := A_i$ if $i \neq 1$. We have $L_i \in \text{Pic}(X_i)$ and $\det(\mathcal{E}) \cong L_1 \boxtimes L_2^{\otimes 2} \dots \boxtimes L_s^{\otimes 2}$ if $n = 2, 3, 4$, while $\det(\mathcal{E}) \cong L_1 \boxtimes L_2^{\otimes 3} \dots \boxtimes L_s^{\otimes 3}$ if $n = 5$. Write $c := c_1(F)$. Since $\mathcal{E} \cong f^*(F)$, we have $\det(\mathcal{E}) \cong R^{\otimes tc}$. To get a contradiction it is sufficient to take td coprime with 2 and with 3 and $R \cong R_1 \boxtimes \dots \boxtimes R_s$ with R_s very ample and divided neither by 2 nor by 3 in $\text{Pic}(X_s)$. \square

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