Symmetries and conservation laws for the Chaplygin sleigh

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Abstract. The Chaplygin sleigh is a mechanical system subject to one linear nonholonomic constraint enforcing the plane motion. We solve equations of motion and study symmetries and conservation laws for this system after deriving general equations of nonholonomic symmetries of the constraint Lagrangian. Our considerations are based on an efficient geometrical theory on fibred manifolds first presented and developed by Olga Rossi (Krupková). The obtained results are thoroughly discussed from the point of view of physics.

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1. Introduction

The motion of nonholonomic systems is studied by various authors using various methods. Bibliography concerning nonholonomic systems is very rich, besides famous monographs e.g. by Bloch, Bullo, Cortés Monforte, Neimark and Fufaev [1] and others, many papers as e.g. [2]-[8], or recently e.g. [9] (for nonlinear constraints) to mention just a few. A geometrical theory of nonholonomic systems was proposed by Olga Rossi (Krupková) in [10] and elaborated in her later works among which we can emphasize e.g. [11] and [12]. This theory is developed on fibred manifolds and their jet prolongations. It differs from other approaches by the idea that the nonholonomic mechanical system is considered as a *dynamical system on a constraint submanifold* which is its true phase space. The equations of motion called the *reduced equations* are equivalent with the well known Chetaev equations [14] based on the standardly used d'Alembert's principle. In this sense the geometrical model is a generalization of the d'Alembert's principle to nonlinear as well as higher order constraints.

The geometrical theory is efficient for solving general and practical problems connected with nonholonomic systems with linear as well as nonlinear constraints (see e.g. [15]-[18]). Nevertheless, some questions are still not satisfactorily answered. One

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of them is the problem of nonholonomic symmetries and conservation laws. A new concept of a nonholonomic symmetry of Lagrangian and the corresponding constraint Noether theorem was formulated by Olga Rossi [19] within the framework of her geometrical theory. An interesting example was discussed and completely solved in [20].

In this paper we derive general equations of nonholonomic Noether symmetries for a first order mechanical system subjected to a general nonholonomic constraint, and corresponding conservation laws. We illustrate the results on an example interesting from the physical point of view: the Chaplygin sleigh. We present solutions of its reduced equations, and constraint conservation laws and symmetries related with its Lagrangian. We find the Chetaev constraint forces. The physical interpretation of results is emphasized.

We note that, in general, the constraint Noether symmetries need not be symmetries of the corresponding constraint semispray. Symmetries of the constraint semispray for the case of the Chaplygin sleigh are discussed in [8].

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2. A brief review on the geometry of nonholonomic mechanics

We summarize basic concepts of the geometrical theory of first order nonholonomic mechanical systems arising from initially Lagrangian unconstrained ones.

2.1 Structures and notations.

The underlying geometrical structure is a $(m+1)$-dimensional fibred manifold $(Y,\, \pi,\, X)$ with the total space Y, the one-dimensional base X and the projection π. We use the standard notation for jet prolongations of this manifold, $(J^rY,\, \pi_r,\, X)$, $r = 0,\, 1,\, 2$, $Y = J^0Y$, $\pi = \pi_0$ and for fibred manifolds $(J^rY,\, \pi_{r,s},\, J^sY)$, $s = 0,\, 1$. We denote $(V,\, \psi)$ a fibred chart on Y, $V \subset Y$ being an open set, $\psi = (t,\, q^\sigma)$, $1 \leq \sigma \leq m$, $(U,\, \varphi)$, $U = \pi(V)$, $\varphi = (t)$, is the associated chart on X, and $(V_r,\, \psi_r)$, $V_r = \pi_{r,0}^{-1}(V)$, $\psi_1 = (t,\, q^\sigma,\, \dot{q}^\sigma)$, $\psi_2 = (t,\, q^\sigma,\, \dot{q}^\sigma,\, \ddot{q}^\sigma)$, are the associated fibred charts on J^1Y and J^2Y, respectively. Let $U \subset X$ be an open set. A section $\delta : U \ni t \to \delta(t) \in J^rY$, $r = 1,\, 2$, is called *holonomic* if there exists a section $\gamma : U \ni t \to \gamma(t) \in Y$ such that $\delta = J^r\gamma$.

The standard concept of a vector field ξ on J^rY is used, as well as of the jet prolongations $J^r\xi$ of a vector field ξ on Y. Projectable and vertical vector fields are introduced in a usual way. A differential q-form η on J^rY is called π_r-*horizontal* if $i_\xi\eta = 0$ for every π_r vertical vector field ξ on J^rY. A q-form η on J^rY is called $\pi_{r,s}$-*horizontal* if $i_\xi\eta = 0$ for every $\pi_{r,s}$ vertical vector field ξ on J^rY. π_r-horizontal 1-forms have a chart expression $\eta = \eta_0(t,\, q^\sigma,\, \ldots,\, q^\sigma_r)\, \mathrm{d}t$. A q-form η on J^rY is called *contact* if $J^r\gamma^*\eta = 0$ for every section γ of π. Contact forms on J^rY form an ideal $\mathcal{I}_C$ called the *contact ideal*. For expressing differential forms in coordinates we use the basis of 1-forms adapted to the contact structure, $(t,\, \omega^\sigma,\, \mathrm{d}\dot{q}^\sigma)$ and $(t,\, \omega^\sigma,\, \dot{\omega}^\sigma,\, \mathrm{d}\ddot{q}^\sigma)$ on J^1Y and J^2Y, respectively, where $\omega^\sigma = \mathrm{d}q^\sigma - \dot{q}^\sigma\, \mathrm{d}t$, $\dot{\omega}^\sigma = \mathrm{d}\dot{q}^\sigma - \ddot{q}^\sigma\, \mathrm{d}t$. There exists a unique decomposition of a q-form η on J^rY into its $(q-1)$-*contact* and q-*contact component* $\pi^*_{r+1,r}\eta = p_{q-1}\eta + p_q\eta$. The jet prolongations of π-projectable vector fields are closely related to the contact ideal being its symmetries: $\partial_{J^r\xi}\omega \in \mathcal{I}_C$ for every $\omega \in \mathcal{I}_C$. Here $\partial_{J^r\xi}$ denotes the Lie derivative along a vector field $J^r\xi$.

A *distribution* on J^rY is a mapping $\mathcal{D}:\ J^rY \ni x \to \mathcal{D}(x) \subset T_xJ^rY$, where $\mathcal{D}(x)$ is a vector subspace of T_xJ^rY. A distribution $\mathcal{D}$ is annihilated by 1-forms η on J^rY such that $i_\xi\eta = 0$ for every vector field ξ belonging to $\mathcal{D}$.

2.2 Unconstrained Lagrangian systems

Let λ be a first order Lagrangian, i.e a horizontal form on J^1Y, $\lambda = L(t,\, q^\sigma,\, \dot q^\sigma)\,\mathrm{d}t$. The pair $(\pi,\, \lambda)$ is called a *Lagrange structure*. In mechanics the *extremals* of the Lagrange structure are zero points of the variational derivative integral $\int\limits_\Omega J^1\gamma^*\,\partial_{J^1\xi}\lambda$, where $\Omega \subset \operatorname{dom}\gamma$ is compact, and ξ is a π-projectable vector field called the *variation*. This yields the *first variation formula*

$$\int\limits_\Omega J^1\gamma^*\partial_{J^1\xi}\lambda = \int\limits_\Omega J^1\gamma^* i_{J^1\xi}\,\mathrm{d}\theta_\lambda + \int\limits_{\partial\Omega} J^1\gamma^* i_{J^1\xi}\theta_\lambda. \tag{2.1}$$

Here $\theta_\lambda = L\,\mathrm{d}t + \frac{\partial L}{\partial \dot q^\sigma}\,\omega^\sigma$ is the the Poincaré-Cartan form of λ. The condition for extremals leads to *Euler-Lagrange equations* – equations of motion of the system. The coordinate free expression of these equations reads $J^1\gamma^*\, i_{J^1\xi}\,\mathrm{d}\theta_\lambda = 0$ or $J^2\gamma^* E_\lambda = 0$, or in coordinates $E_\sigma \circ J^2\gamma = 0$, where $E_\sigma = A_\sigma + B_{\sigma\nu}\ddot q^\nu$, where $E_\lambda = E_\sigma\,\omega^\sigma \wedge \mathrm{d}t$ and

$$A_\sigma = \frac{\partial L}{\partial q^\sigma} - \frac{\mathrm{d}'}{\mathrm{d}t}\frac{\partial L}{\partial \dot q},\ \ B_{\sigma\nu} = -\frac{\partial^2 L}{\partial \dot q^\sigma \partial \dot q^\nu},\ \ \frac{\mathrm{d}'}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} - \ddot q^\sigma\frac{\partial}{\partial \dot q^\sigma} = \frac{\partial}{\partial t} + \dot q\frac{\partial}{\partial q^\sigma}. \tag{2.2}$$

A π-projectable vector field ξ on Y is called a *symmetry* of the Lagrange structure $(\pi,\, \lambda)$ if it holds $\partial_{J^1\xi}\lambda = 0$. This condition is the *Noether equation*. The Noether equation in coordinates reads

$$\frac{\partial L}{\partial t}\xi^0 + \frac{\partial L}{\partial q^\sigma}\xi^\sigma + \frac{\partial L}{\partial \dot q^\sigma}\left(\frac{\mathrm{d}\xi^\sigma}{\mathrm{d}t} - \dot q^\sigma\frac{\mathrm{d}\xi^0}{\mathrm{d}t}\right) + L\frac{\mathrm{d}\xi^0}{\mathrm{d}t} = 0. \tag{2.3}$$

Taking into account the first variation formula we can see that if ξ is a symmetry of the Lagrange structure then the quantity

$$i_{J^1\xi}\theta_\lambda = \left(L - \dot q^\sigma\frac{\partial L}{\partial \dot q^\sigma}\right)\xi^0 + \frac{\partial L}{\partial \dot q^\sigma}\xi^\sigma \tag{2.4}$$

(called the *current*) is constant along extremals. This result represents conservation laws and is well known as the *Emmy Noether theorem.*

2.3 Nonholonomic dynamics

Suppose that an unconstrained (first order) Lagrangian mechanical system is subjected to a nonholonomic constraint given by k equations, $1 \le k \le m-1$,

$$f^a(t,\, q^\sigma,\, \dot q^\sigma) = 0, \quad 1 \le a \le k, \quad \text{where} \quad \operatorname{rank}\left(\frac{\partial f^a}{\partial \dot q^\sigma}\right) = k,$$

or in a normal form $\dot q^{m-k+a} = g^a(t,\, q^\sigma,\, \dot q^l)$, $1 \le l \le m-k$. These equations define a *constraint submanifold* $Q \subset J^1Y$ of codimension k fibred over Y. Denote

$$\iota:\ Q \ni (t,\, q^\sigma,\, \dot q^l) \longrightarrow (t,\, q^\sigma,\, \dot q^l,\, g^a(t,\, q^\nu,\, \dot q^s)) \in J^1Y$$

the canonical embedding of Q into J^1Y. On Q there arise the *induced contact ideal* $\tilde{\mathcal{I}}_C$ generated by forms $\bar{\omega}^\sigma = \iota^*\omega^\sigma$ and the *canonical distribution*

$$\mathcal{C} = \{\operatorname{span}\varphi^a \,|\, 1 \le a \le k\}, \quad \varphi^a = \iota^*\omega^{m-k+a} - \frac{\partial g^a}{\partial \dot{q}^l}\iota^*\,\omega^l. \tag{2.5}$$

The vector fields belonging to $\mathcal{C}$ are called *Chetaev vector fields*. They represent *admissible variations* in the nonholonomic variational principle (first introduced in [12]). Let us briefly recall this principle and its consequences. Let $(\pi,\,\lambda)$ be an unconstrained Lagrangian structure and θ_λ the corresponding Poincaré-Cartan form. By the *constraint system* on Q defined by λ we mean the differential form $\iota^*\theta_\lambda$. Denote $\bar{\lambda} = \iota^*\lambda = (L\circ\iota)\,\mathrm{d}t$ and $\theta_{\bar{\lambda}} = \theta_{\iota^*\lambda}$. Calculating $\iota^*\theta_\lambda$ we obtain

$$\iota^*\theta_\lambda = \bar{L}\,\mathrm{d}t + \frac{\partial\bar{L}}{\partial\dot{q}^l}\bar{\omega}^l + \bar{L}_a\varphi^a = \theta_{\iota^*\lambda} + \bar{L}_a\varphi^a, \quad \bar{L} = L\circ\iota, \quad \bar{L}_a = \frac{\partial L}{\partial \dot{q}^{m-k+a}}\circ\iota. \tag{2.6}$$

Let δ be a section of the projection $\bar{\pi}_1 : Q \to X$ defined on an open subset $U \subset X$ containing a compact set $\Omega \subset X$. Let $Z \in \mathcal{C}$ be a $\bar{\pi}_1$-projectable vector field and let $(\phi_u,\,\phi_{0u})$ be its one-parameter group. The constraint variational integral and its variational derivative are

$$S_\Omega[\delta] = \int_\Omega \delta^*\iota^*\theta_\lambda, \qquad \frac{\mathrm{d}S[\delta_u]}{\mathrm{d}u}\Big|_{u=0} = \int_\Omega \delta^*\partial_Z\iota^*\theta_\lambda, \quad \text{or} \quad \int_\Omega J^1\gamma^*\partial_Z\iota^*\theta_\lambda$$

in the case of restriction to holonomic sections. Nonholonomic first variation formula then reads

$$\int_\Omega J^1\gamma^*\,\partial_Z\iota^*\theta_\lambda = \int_\Omega J^1\gamma^*\,i_Z\,\mathrm{d}\iota^*\theta_\lambda + \int_{\partial\Omega} J^1\gamma^* i_Z\theta_{\iota^*\lambda} \tag{2.7}$$

The integrand in the first integral on the right-hand side depends only on components of Z on Y. The requirement of its vanishing for arbitrary Ω gives the equations of motion

$$J^1\gamma^* i_Z\,\mathrm{d}\iota^*\theta_\lambda = 0 \implies \big(\varepsilon_s(\bar{L}) - \bar{L}_a\varepsilon_s(g^a)\big)\circ J^2\gamma = 0, \;\; 1 \le s \le m-k, \tag{2.8}$$

where for a function $f = f(t,\,q^\sigma,\,\dot{q}^l)$ we use the constraint derivative operators:

$$\varepsilon_s(f) = \frac{\partial_c f}{\partial q^s} - \frac{\mathrm{d}_c}{\mathrm{d}t}\frac{\partial f}{\partial \dot{q}^s}, \qquad \frac{\partial_c}{\partial q^s} = \frac{\partial}{\partial q^s} + \frac{\partial g^i}{\partial \dot{q}^s}\frac{\partial}{\partial q^{m-k+i}}.$$

$$\frac{\mathrm{d}_c}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}^l\frac{\partial}{\partial q^l} + g^i\frac{\partial}{\partial q^{m-k+i}} + \ddot{q}^l\frac{\partial}{\partial \dot{q}^l} = \frac{\mathrm{d}'_c}{\mathrm{d}t} + \ddot{q}^l\frac{\partial}{\partial \dot{q}^l}.$$

The equations (2.8) can be written as $\bar{A}_s + \bar{B}_{sr}\ddot{q}^r = 0$, $1 \le l \le m-k$,

$$\bar{A}_s = \frac{\partial_c\bar{L}}{\partial q^s} - \frac{\mathrm{d}'_c}{\mathrm{d}t}\frac{\partial\bar{L}}{\partial\dot{q}^s} - \bar{L}_a\left(\frac{\partial_c g^a}{\partial q^s} - \frac{\mathrm{d}'_c}{\mathrm{d}t}\frac{\partial g^a}{\partial \dot{q}^s}\right), \;\; \bar{B}_{sr} = -\frac{\partial^2\bar{L}}{\partial\dot{q}^s\partial\dot{q}^r} + \bar{L}_a\frac{\partial^2 g^a}{\partial\dot{q}^s\partial\dot{q}^r}. \tag{2.9}$$

We obtained the *reduced equations*, which together with equations of the constraint form a complete set of equations of motion of the nonholonomic system.

2.4 Chetaev equations

The geometrical theory of nonholonomic systems leads also to the well known Chetaev equations of motion. They are obtained by introducing the *Chetaev constraint force* into equations of motion. Suppose that $A_\sigma + B_{\sigma\nu}\ddot{q}^\nu = 0$, $1 \leq \sigma, \nu \leq m$, are equations of motion of an unconstrained system. The Chetaev force is defined as the form $\phi = \mu^a \frac{\partial f^a}{\partial \dot{q}^\sigma}\, \omega^\sigma \wedge \mathrm{d}t$. The coefficients μ^a, $1 \leq a \leq k$ on $J^1 Y$ are *Lagrange multipliers*. The Chetaev equations read

$$\left(A_\sigma + B_{\sigma\nu}\ddot{q}^\nu - \mu^a \frac{\partial f^a}{\partial \dot{q}^\sigma}\right) \circ J^2\gamma = 0. \tag{2.10}$$

Together with the equations of the constraint $f^a = 0$, $1 \leq a \leq k$, we obtain $(m+k)$ equations for trajectories and Lagrange multipliers.

3. Nonholonomic symmetries and conservation laws

We present a definition of a nonholonomic symmetry and derive general equations for symmetries of a constrained mechanical system arising from a first order Lagrangian structure.

3.1 Nonholonomic Noether symmetries

The concept of a nonholonomic Noether symmetry arises form the nonholonomic first variation formula (2.7) (see [19]). Let Z be a Chetaev vector field, i.e. $Z \in \mathcal{C}$. The chart expression of Z is

$$Z = Z^0 \frac{\partial}{\partial t} + Z^l \frac{\partial}{\partial q^l} + \left[Z^0 g^a + (Z^s - \dot{q}^s Z^0) \frac{\partial g^a}{\partial \dot{q}^s} \right] \frac{\partial}{\partial q^{m-k+a}} + \tilde{Z}^l \frac{\partial}{\partial \dot{q}^l}.$$

Z is called a *symmetry* of the nonholonomic mechanical system arising from a primarily unconstrained Lagrangian structure $(\lambda,\, \pi)$ subjected to constraints $\dot{q}^{m-k+a} = g^a(t,\, q^\sigma,\, \dot{q}^l)$ if the constrained system $\iota^*\theta_\lambda$ on Q defined by λ remains invariant under transformations given by the one-parameter group of the vector field Z up to a constraint form. This means that

$$\partial_Z \iota^*\theta_\lambda = i_Z\, \mathrm{d}\iota^*\theta_\lambda + \mathrm{d} i_Z \iota^*\theta_\lambda = F_a \varphi^a, \tag{1.11}$$

where F_a are some functions on Q. The relation (1.11) is the *constraint Noether equation.* From the nonholonomic variation formula (2.7) we can see that if Z is a symmetry of a nonholonomic mechanical system and γ is a solution of the corresponding reduced equations together with constraints, then $\mathrm{d}J^1\gamma^* i_Z \iota^*\theta_\lambda = 0$, i.e. $(i_Z \iota^*\theta_\lambda) \circ J^1\gamma = \mathrm{const}$. This means that the quantities $\Phi = i_Z \iota^*\theta_\lambda$ are constant along solutions. We obtain

$$\Phi = \left(\bar{L} - \dot{q}^l \frac{\partial \bar{L}}{\partial \dot{q}^l}\right) Z^0 + \frac{\partial \bar{L}}{\partial \dot{q}^l} Z^l. \tag{1.12}$$

The quantities Φ are called *currents* and the conditions $\Phi = \mathrm{const.}$ are the corresponding *conservation laws.*

3.2 Equations for nonholonomic symmetries

Using the definition of nonholonomic symmetries and relations (2.9) we obtain after some tedious calculations the following set of partial differential equations for $2(m-k)+1$ components of the symmetries:

$$\begin{aligned}
&\frac{\mathrm{d}'_c\Phi}{\mathrm{d}t} + \bar{A}_l(Z^l - \dot{q}^l Z^0) = 0,\\
&\frac{\partial_c\Phi}{\partial q^l} - \bar{A}_l Z^0 + \left\{\frac{\partial \bar{A}_s}{\partial \dot{q}^l} + \frac{\partial \bar{L}_a}{\partial \dot{q}^l}\varepsilon'_s(g^a)\right\}_{\mathrm{alt}(l,s)} (Z^s - \dot{q}^s Z^0) - \bar{B}_{ls}\tilde{Z}^s = 0,\\
&\frac{\partial\Phi}{\partial \dot{q}^l} + \bar{B}_{ls}(Z^s - \dot{q}^s Z^0) = 0,
\end{aligned} \tag{1.13}$$

where $1 \le l,\ s \le m-k$. These equations enable us to express symmetries of the mechanical system via currents: Denoting $\mathcal{B} = \bar{B}^{-1}$ we obtain

$$Z^0 = \frac{1}{\bar{L}}\left(\Phi + \mathcal{B}^{ls}\frac{\partial \bar{L}}{\partial \dot{q}^l}\frac{\partial \Phi}{\partial \dot{q}^s}\right), \quad Z^l = \dot{q}^l Z^0 - \mathcal{B}^{ls}\frac{\partial\Phi}{\partial \dot{q}^s}, \tag{1.14}$$

$$\tilde{Z}^l = \mathcal{B}^{ls}\left(\frac{\partial_c\Phi}{\partial q^s} - \bar{A}_s Z^0 + \left\{\frac{\partial \bar{A}_r}{\partial \dot{q}^s} + \frac{\partial \bar{L}_a}{\partial \dot{q}^s}\varepsilon'_r(g^a)\right\}_{\mathrm{alt}(l,s)}(Z^r - \dot{q}^r Z^0)\right). \tag{1.15}$$

The computations of symmetries simplify if we know the currents (constants of motion). This might arise during the process of solving the motion equations. We take advantage of this simplification in what follows.

4. Example: Chaplygin sleigh

The setting of the problem can be found in [1]. Here we study the dynamics by the geometric tools exposed above. The main results are an explicit description of the constraint dynamics and analysis of the constraint Noether symmetries and corresponding conservation laws for this classical constraint system.

4.1 Chaplygin sleigh and its motion

The Chaplygin sleigh is a rigid body of mass m sliding on the horizontal plane without friction (figure 1), C is the center of mass. The inertia of the sleigh with respect to the axis going through C perpendicularly to the plane xy is J. The constraint is imposed by a sharp blade placed at a point A, $AC = a$. The blade prevents the sleigh to move in the direction perpendicular to the straight line AC. The constraint in fibred coordinates $(t,\ \varphi,\ x,\ y,\ \dot{\varphi},\ \dot{x},\ \dot{y})$ reads

$$\dot{y} = \dot{x}\,\mathrm{tg}\,\varphi \;\Rightarrow\; \iota : Q \ni (t,\ \varphi,\ x,\ y,\ \dot{\varphi},\ \dot{x}) \to (t,\ \varphi,\ x,\ y,\ \dot{\varphi},\ \dot{x},\ \dot{x}\,\mathrm{tg}\,\varphi) \in J^1Y. \tag{1.16}$$

Unconstrained Lagrange function and constraint Lagrange functions (2.6) are

$$L = \frac{1}{2}m\left[(\dot{x} - a\dot{\varphi}\sin\varphi)^2 + (\dot{y} + a\dot{\varphi}\cos\varphi)^2\right] + \frac{1}{2}J\dot{\varphi}^2,$$

$$\bar{L} = \frac{m}{2}\left(\frac{\dot{x}^2}{\cos^2\varphi} + a^2k^2\dot{\varphi}^2\right), \quad \bar{L}_1 = m(\dot{x}\,\mathrm{tg}\,\varphi + a\dot{\varphi}\cos\varphi), \quad k^2 = 1 + \frac{J}{ma^2}. \tag{1.17}$$

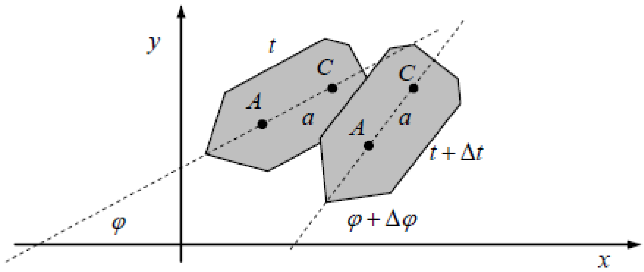

Figure 1: Chaplygin sleigh.

Putting this into (2.9) we obtain the matrices $\bar{A}$, $\bar{B}$ and $\mathcal{B} = \bar{B}^{-1}$, and the equations of motion

$$(1.18)\quad 0 \;=\; -ma^2k^2\ddot{\varphi} - \frac{ma}{\cos\varphi}\dot{\varphi}\dot{x}, \quad -\frac{m}{\cos^2\varphi}\ddot{x} + \frac{ma}{\cos\varphi}\dot{\varphi}^2 - \frac{m\,\mathrm{tg}\,\varphi}{\cos^2\varphi}\dot{\varphi}\dot{x} = 0.$$

Their solutions take the form (with integration constants C_1, C_2, C_3)

$$(1.19)\quad \varphi(t) = k\,\mathrm{arcsin\,tgh}\left(\frac{C_1}{k^2}(t - C_2)\right) + C_3, \quad \varphi = k\psi + C_3,$$

$$x(t) = ak^2\int \cos\,(k\psi + C_3)\mathrm{tg}\,\psi\,\mathrm{d}\psi, \;\; y(t) = ak^2\int \sin\,(k\psi + C_3)\mathrm{tg}\,\psi\,\mathrm{d}\psi,$$

For initial conditions $\varphi(0) = 0$, $\dot{\varphi}(0) = \omega_0 > 0$, $x(0) = 0$, $\dot{x}(0) = y(0) = 0$ we obtain $C_1 = k\omega_0$, $C_2 = 0$, $C_3 = 0$. For illustration, figure 2. presents a graphical output (for $a = 1$, $\frac{\omega_0}{k} = 1$, $m = 2$ and $k = 4$). We note that in [1] equivalent equations of motion

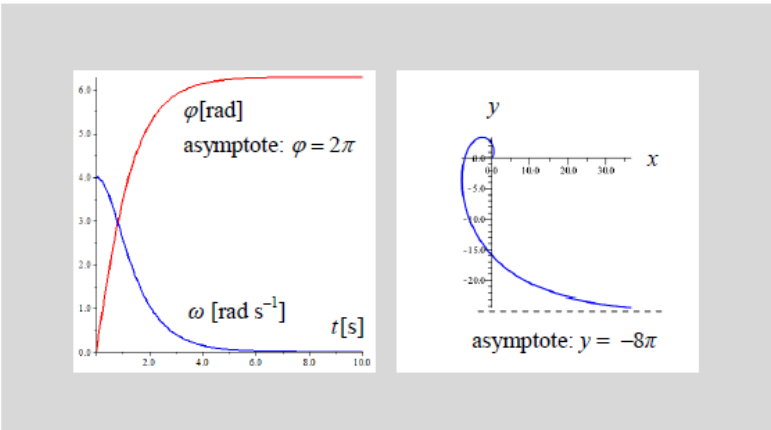

Figure 2: An example of Chaplygin sleigh motion.

are obtained by formulating the second Newton's law in the reference frame connected with the sleigh for components u and v of the sleigh velocity and the angular velocity $\omega = \dot{\varphi}$. The solution is then transformed into the inertial reference frame. Our solution is the same as the last cited one.

4.2 Symmetries and currents

Putting expressions (1.17) into equations (1.13) we obtain

$$\begin{aligned}
&\frac{\mathrm{d}'_c\Phi}{\mathrm{d}t} - \frac{ma\dot{\varphi}\dot{x}}{\cos\varphi}(Z^\varphi - \dot{\varphi}Z^0) + m\left(\frac{a\dot{\varphi}^2}{\cos\varphi} - \frac{\dot{\varphi}\dot{x}\,\mathrm{tg}\,\varphi}{\cos^2\varphi}\right)(Z^x - \dot{x}Z^0) = 0,\\
&\frac{\partial_c\Phi}{\partial\varphi} + \frac{m\dot{x}^2\,\mathrm{tg}\,\varphi}{\cos^2\varphi}Z^0 - m\left(\frac{\dot{x}\,\mathrm{tg}\,\varphi}{\cos^2\varphi} - \frac{a\dot{\varphi}}{\cos\varphi}\right)Z^x + ma^2k^2\tilde{Z}^\varphi = 0,
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial_c\Phi}{\partial x} + m\left(\frac{\dot{x}\,\mathrm{tg}\,\varphi}{\cos^2\varphi} - \frac{a\dot{\varphi}}{\cos\varphi}\right)Z^\varphi + \frac{m}{\cos^2\varphi}\tilde{Z}^x = 0,\\
&\frac{\partial\Phi}{\partial\dot{\varphi}} - ma^2k^2(Z^\varphi - \dot{\varphi}Z^0) = 0, \quad \frac{\partial\Phi}{\partial\dot{x}} - \frac{m}{\cos^2\varphi}(Z^x - \dot{x}Z^0) = 0.
\end{aligned} \tag{1.20}$$

Expressing the components $(Z^\varphi - \dot{\varphi}Z^0)$ and $(Z^x - \dot{x}Z^0)$ from the last two of these equations, putting them into the first equation and substituting $v = \frac{\dot{x}}{\cos\varphi}$ we obtain

$$\left(\frac{\partial}{\partial t} + \dot{\varphi}\frac{\partial}{\partial\varphi} + v\cos\varphi\frac{\partial}{\partial x} + v\sin\varphi\frac{\partial}{\partial y} - \frac{\dot{\varphi}v}{ak^2}\frac{\partial}{\partial\dot{\varphi}} + a\dot{\varphi}^2\frac{\partial}{\partial v}\right)\Phi = 0. \tag{1.21}$$

So, we have the characteristics ODE's

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}\varphi}{\dot{\varphi}} = \frac{\mathrm{d}x}{v\cos\varphi} = \frac{\mathrm{d}y}{v\sin\varphi} = -ak^2\frac{\mathrm{d}\dot{\varphi}}{\dot{\varphi}v} = \frac{\mathrm{d}v}{a\dot{\varphi}^2}.$$

Integrating the last equation we obtain

$$\frac{1}{2}v^2 + \frac{1}{2}a^2k^2\dot{\varphi}^2 = \text{const.}, \quad \text{i.}e. \quad \frac{1}{2}\frac{\dot{x}^2}{\cos^2\varphi} + \frac{1}{2}a^2k^2\dot{\varphi}^2 = \text{const.}$$

This quantity multiplied by mass m is the total mechanical energy E_0 of the sleigh which is the sum of the translational energy $E_T = \frac{1}{2}\frac{m\dot{x}^2}{\cos^2\varphi}$ and the rotational energy $E_R = \frac{1}{2}(J + ma^2)\dot{\varphi}^2$ with respect to the vertical axis going through the point A. $J + ma^2$ is the inertia of the sleigh with respect to this axis. The total mechanical energy of the sleigh expressed via the components of the velocity of the center od mass (x_C, y_C) is

$$E = \frac{m}{2}\left(\dot{x}_C^2 + \dot{y}_C^2\right) + \frac{1}{2}J\dot{\varphi}^2.$$

Taking into account that $x_C = x + a\cos\varphi$, $y_C = y + a\sin\varphi$ and considering the constraint we can see that $E = E_0$. For the solution (1.19) we have

$$E_0 = \frac{1}{2}ma^2k^2\omega_0^2, \quad C_1 = k\omega_0 = \sqrt{\frac{2E_0}{ma^2}}.$$

We obtain the corresponding current taking into account that the constrained Lagrange function $\bar{L}$ does not depend on time explicitly,

$$\Phi_1 = -\frac{m}{2}\left(\frac{\dot{x}^2}{\cos^2\varphi} + a^2k^2\dot{\varphi}^2\right). \tag{1.22}$$

Putting this expression into (1.14) we can verify that the corresponding symmetry is $Z = \frac{\partial}{\partial t}$. Using the solution of equations of motion (1.19) we obtain the expressions for the translational and rotational energy and the angle φ as functions of time:

$$E_T = E_0 \operatorname{tgh}^2\frac{\omega_0 t}{k}, \quad E_R = E_0 \cosh^{-2}\frac{\omega_0 t}{k}, \quad \sin\frac{\varphi}{k} = \operatorname{tgh}\frac{\omega_0 t}{k}. \tag{1.23}$$

It is more correct from the point of view of physics to decompose the total energy

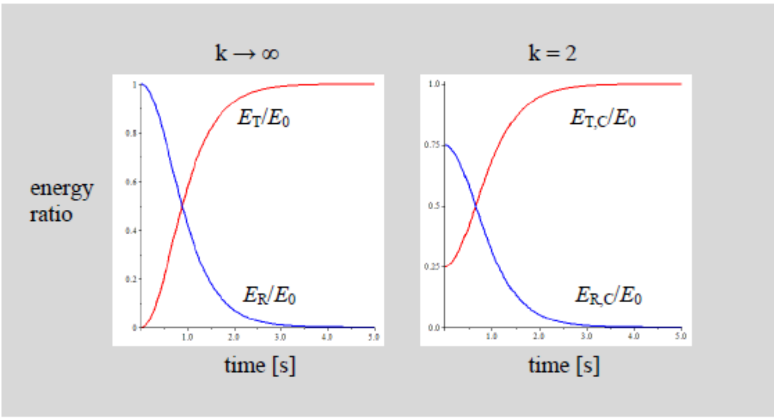

Figure 3: Translational and rotational energy, $\frac{\omega_0}{k} = 1$.

into the translational energy of the center of mass C, $E_{T,C} = \frac{1}{2}m(\dot{x}_C^2 + \dot{y}_C^2)$, and the rotational energy of the sleigh with respect to the center of mass, $E_{R,C} = \frac{1}{2}J\dot{\varphi}^2$. Using the solution of equations of motion we obtain

$$E_{T,C} = E_0\left(\operatorname{tgh}^2\frac{\omega_0 t}{k} + \frac{1}{k^2\cosh^2\frac{\omega_0 t}{k}}\right), \; E_{R,C} = \frac{E_0}{\cosh^2\frac{\omega_0 t}{k}}\left(1 - \frac{1}{k^2}\right). \tag{1.24}$$

Figures 2 and 3 show the asymptotic behaviour of the sleigh motion. Relations (1.23) represent the limit case of (1.24) for $J \gg ma^2$, i.e. $k \to \infty$, as expected. Notice that for $k = 1$ ($J \ll ma^2$) we have $E_{T,C} = E_0$ and $E_{R,C} = 0$. This is not in contradiction with the initial condition for $\dot{\varphi}$: $E_{R,C}$ vanishes because of zero inertia, even though $\omega_0 \neq 0$.

Expressing constants C_2 and C_3 we obtain the following currents

$$\Phi_2 = \frac{m\dot{x}}{\cos\varphi}\sin\left(\frac{\varphi}{k}\right) + mak\dot{\varphi}\cos\left(\frac{\varphi}{k}\right), \;\; \Phi_3 = \frac{1}{2}ma^2k^2\ln\frac{\sqrt{E_0}+\sqrt{E_T}}{\sqrt{E_0}-\sqrt{E_T}} - at\sqrt{2mE_0}$$

For the special case of zero inertia J, i.e. $k = 1$, the current Φ_2 represents the y-component of the impulse of the sleigh, $p_{C,y} = m\dot{y}_C = m(\dot{x}\,\mathrm{tg}\,\varphi + a\dot{\varphi}\cos\varphi)$. (We shall see later that in such a case $p_{C,x}$ must be conserved as well.) The rather complicated formulas for the corresponding symmetries can be found in [22].

4.3 Chetaev equations and constraint forces

Finally, let us express Chetaev equations of motion and the constraint forces using (2.10)). Rewriting the constraint as $f \equiv \dot{y} - \dot{x}\mathrm{tg}\,\varphi = 0$ we obtain

$$\begin{aligned} -mak^2\ddot{\varphi} + m\ddot{x}\sin\varphi - m\ddot{y}\cos\varphi &= \tfrac{\mu}{a}\tfrac{\partial f}{\partial\dot{\varphi}}, \quad \tfrac{\partial f}{\partial\dot{\varphi}} = 0, \\ ma\ddot{\varphi}\sin\varphi - m\ddot{x} + ma\dot{\varphi}^2\cos\varphi &= \mu\tfrac{\partial f}{\partial\dot{x}}, \quad \tfrac{\partial f}{\partial\dot{x}} = -\mathrm{tg}\,\varphi, \\ -ma\ddot{\varphi}\cos\varphi - m\ddot{y} + ma\dot{\varphi}^2\sin\varphi &= \mu\tfrac{\partial f}{\partial\dot{y}}, \quad \tfrac{\partial f}{\partial\dot{y}} = 1, \end{aligned} \tag{1.25}$$

μ being a Lagrange multiplier. The constraint force $\phi = \mu(0,\, -\mathrm{tg}\,\varphi,\, 1)$ has a clear physical meaning in the reference frame connected with the point A and rotating with the sleight: Denote $\vec{r}\,' = (0,\, a\cos\varphi,\, a\sin\varphi)$, $\vec{\omega} = (\dot{\varphi},\, 0,\, 0)$, $\vec{\varepsilon} = (\ddot{\varphi},\, 0,\, 0)$, $\vec{A}(0,\, \ddot{x},\, \ddot{y})$, and $\phi = \vec{F}^*$ as usual in physics, we obtain

$$\vec{F}^* = -m\vec{\varepsilon}\times\vec{r}\,' - m\vec{\omega}\times(\vec{\omega}\times\vec{r}\,') - m\vec{A}. \tag{1.26}$$

This is the sum of Euler, centrifugal and translational force. The Coriolis force is missing because the velocity of C with respect to the reference system connected with A is zero. Using the constraint to write $\ddot{y} = \ddot{x}\,\mathrm{tg}\,\varphi + \frac{\dot{\varphi}\dot{x}}{\cos^2\varphi}$ and putting into (1.25) we obtain the Lagrange multiplier μ and the constraint force ϕ:

$$\mu = -\frac{mJ}{J + ma^2}\dot{\varphi}\dot{x}, \quad \phi = \frac{mJ}{J + ma^2}\left(0,\, \dot{\varphi}\dot{x}\,\mathrm{tg}\,\varphi,\, -\dot{\varphi}\dot{x}\right). \tag{1.27}$$

This force is not variational in the sense of e.g. [21]. For $k = 1$ the constraint force vanishes. This is consistent with the limit case $J \to 0$ in relations (1.24): The motion of the center of mass is uniform and straightforward (both components of the impulse of the center of mass are conserved), while the sleigh rotates around it with the initial angular velocity ω_0 but with zero energy due to $J = 0$.

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