Efficiency for multitime variational problems with geodesic quasiinvex functionals on Riemannian manifolds

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Abstract. We study the connection between a multitime scalar variational problem (SVP), a multitime vector variational problem (VVP) and a multitime vector fractional variational problem (VFP). For (SVP), we establish necessary optimality conditions. For both vector variational problems, we define the notions of Pareto efficient solution and of normal efficient solution and we establish necessary efficiency conditions for (VVP) and (VFP) using both notions. The main purpose of the paper is to establish sufficient efficiency conditions for the vector problems (VVP) and (VFP). Moreover, we obtain sufficient optimality conditions for (SVP). The sufficient conditions are based on our original notion of (ρ, b)-geodesic quasiinvexity.

M.S.C. 2010: 65K10, 90C29, 26B25.
Key words: multitime fractional variational problem; efficient solution; normal efficient solution; (ρ, b)-geodesic quasiinvexity.

1 Introduction and preliminaries

Beginning with Valentine [16] in 1937, during the years, the variational problem with constraints experienced different stages of development. In 2008, Mititelu [6] studied a single-time vector (or multiobjective) fractional variational problem. Mititelu [6] and Mititelu & Stancu-Minasian [9] established for this problem necessary efficiency (Pareto minimum) conditions. Using generalized quasiinvex functions, they developed a duality theory including weak, direct and converse duality theorems.

In 2007 Udrişte and Ţevy [15] gave new results for a multitime variational problem of vector variable. In 2009 Pitea, Udrişte and Mititelu [11], [12] considered the multitime vector variant of the problem (MSP) into geometrical language, using curvilinear integrals, and establishing necessary efficiency conditions and developed a duality theory for this problem. Recently Mititelu and Postolache [8] studied the same subject

BSG Proceedings 22. The International Conference "Differential Geometry - Dynamical Systems" DGDS-2014, September 1-4, 2014, Mangalia-Romania, pp. 38-50.

for multitime vector fractional and nonfractional variational problems on Riemannian manifolds, but using multiple integrals.

The purpose of this work is to deduce necessary and sufficient optimality conditions for the multitime scalar problem (SVP) (sections 2, 4) and of Pareto efficiency for the multitime vector variational problems (VVP) and (VFP) (section 3, 4), in a geometrical framework [11], [12].

Let $(T, h)$and (M, g) be two Riemannian manifolds of dimensions m and n. In addition, M is a complete manifold. Denote $t = (t^1, ..., t^m) = (t^\upsilon)$ the points of a measurable set Ω in T and $x = (x^1, ..., x^n) = (x^i)$ the points of M. Consider the first order jet bundle $J^1(T, M) = \Omega \times R^n \times R^{nm}$ and the functions

$$x : \Omega \subset T \to M,\ X : J^1(T, M) \to \mathrm{R},$$

$$f = (f_r) : J^1(T, M) \to R^p,\ k = (k_r) : J^1(T, M) \to R^p,$$
$$g = (g_\alpha) : J^1(T, M) \to \mathrm{R}^m,\ h = (h_s) : J^1(T, M) \to R^q,$$

where $m, p, q \in N^*$, $r = \overline{1, p}$, $\alpha = \overline{1, m}$ and $s = \overline{1, q}$, all of C^2-class.

The argument of each function X, f, k, g, h is $j^1 x = (t, x, x_\upsilon)$, the first prolongation jet of x. For functionals, based on Lagrangians X, f, k, g, h, we use the pullback $j^1_t x = (j^1 x)(t))$, where $t \in \Omega$ and $x(t) = (x^i(t))$, and $(\partial x/\partial t^\upsilon(t)) = (x_\upsilon(t)) = (x^i_\upsilon(t))$.

The Euler-Ostrogradsky PDEs produced by the Lagrangian X are

$$\frac{\partial X}{\partial x^k} - \frac{\partial}{\partial t^\upsilon}\left(\frac{\partial X}{\partial x^k_\upsilon}\right) = 0,\ k = \overline{1, n};\ \upsilon = \overline{1, m}. \tag{1.1}$$

We shall use a normed vector space of functions $(\mathcal{F}(\Omega, M), || \cdot ||)$, where

$$\mathcal{F}(\Omega, M) = \{x : \Omega \to M\ \mid x \quad \text{is piecewice } C^1\},$$

and

$$\|x\| = \|x\|_\infty + \sum_{\gamma=1}^{m}\sum_{k=1}^{n} ||x^k_\gamma||_\infty.$$

The induced distance is $d(x^0(\cdot), x(\cdot)) = ||x(\cdot) - x^0(\cdot)||$, $x^0(\cdot)$, $x(\cdot) \in \mathcal{F}(\Omega, M)$. In this sense, $(\mathcal{F}(\Omega, M), d)$ is a metric space.

The following partial ordering is used for two n-tuples $v = (v_1, \dots, v_n)$ and $w = (w_1, \dots, w_n)$:

$$v = w\ \Leftrightarrow\ v_i = w_i,\ i = \overline{1, n};\ v < w\ \Leftrightarrow\ v_i < w_i,\ i = \overline{1, n}$$

$$v \preceq w \Leftrightarrow v_i \leq w_i,\ i = \overline{1, n};\ v \leq w \Leftrightarrow v \preceq w \text{ and } v \neq w.$$

Similar partial relations are used also for m-tuples.

Let $dv = \sqrt{\det h}\, dt^1 \wedge dt^2 \ldots \wedge dt^m$ be the volume element on Ω. We use the functionals

$$F_r(x(\cdot)) = \int_a^b f_r(j^1_t x)dv,\ K_r(x(\cdot)) = \int_a^b k_r(j^1_t x)dv,\ r = \overline{1, p}$$

and a vector fractional functional

$$J(x(\cdot)) = \left(\frac{F_1}{K_1}(x(\cdot)), \dots, \frac{F_p}{K_p}(x(\cdot))\right).$$

The general problem of study is the multitime vector fractional problem

$$(VFP) \qquad \begin{cases} \text{Maximize Pareto } J(x(\cdot)) \\ \text{subject to} \quad g(j_t^1x) \preceq 0, \ h(j_t^1x) = 0, \\ \qquad x(t)\,|_{\partial\Omega} = u(t) \text{ (given)}. \end{cases}$$

This problem include the multitime vector variational problem

$$(VVP) \qquad \begin{cases} \text{Minimize Pareto } (F_1(\cdot), \dots, F_p(x(\cdot)) \\ \text{subject to} \quad g(j_t^1x) \preceq 0, \ h(j_t^1x) = 0, \\ \qquad x(t)\,|_{\partial\Omega} = u(t). \end{cases}$$

and the following multitime scalar variational problem

$$\text{(SVP)} \qquad \begin{cases} \text{Minimize } E(x(\cdot)) = \int_\Omega X(j_t^1x)dv \\ \text{subject to} \quad g(j_t^1x) \preceq 0; \ h(j_t^1x) = 0 \\ \qquad x(t)\,|_{\partial\Omega} = u(t). \end{cases}$$

The three variational problems have the same domain

$$\mathcal{D} = \{x \in \mathcal{F}(\Omega, M) \ \big| g(j_t^1x) \preceq 0, \ h(j_t^1x) = 0, \ x(t)\,|_{\partial\Omega} = u(t)\}.$$

Definition 1.1. [1] Let (M, g) be a complete Riemannian manifold. Let $\eta : M \times M \to TM$, $\eta(u, x) \in T_uM$, $u, x \in M$ be a vector function and $S \subset M$ a nonempty set.

(ii) The set S is called η-geodesic invex if, for every $u, x \in S$, there exists exactly one geodesic $\gamma_{u,x} : [0, 1] \to M$ such that

$$\gamma_{u,x}(0) = u, \dot{\gamma}_{u,x}(0) = \eta(u, x), \ \gamma_{u,x}(\tau) \in S, \ \forall \tau \in [0, 1].$$

(ii) Let $S \subset M$ be an open η-geodesic invex set and $f : S \to R$ be a C^1 function. The function f is called η-geodesic invex on S if

$$f(x) - f(u) \geq df_u(\eta(u, x)), \ \forall u, x \in S.$$

Definition 1.2. [16] Let $x^0(\cdot), x(\cdot) \in \mathcal{F}(\Omega, M)$. A function $\varphi(t, \tau)$, $t \in \Omega$, $\tau \in [0, 1]$ is called geodesic deformation of the pair of functions $(x^0(\cdot), x(\cdot))$, if it satisfies the properties: (1) the function $\tau \to \varphi(t, \tau)$ is a geodesic; (2) $\varphi(t, 0) = x^0(t), \varphi(t, 1) = x(t)$.

Definition 1.3. The set $\mathcal{S} = \mathcal{F}(\Omega, S) \subset \mathcal{F}(\Omega, M)$ is called η-geodesic invex if, for every $x^0(\cdot), x(\cdot) \in \mathcal{S}$, there exists exactly one geodesic deformation $\varphi(t, \tau)$, $t \in \Omega$, $\tau \in [0, 1]$ such that the vector function

$$\eta(x^0(t), x(t)) = \frac{\partial \varphi}{\partial \tau}(t, \tau)|_{\tau=0} \in T_{x^0(t)}M \equiv \eta(t) = (\eta^1(t), \dots, \eta^n(t))$$

is of class C^1 and satisfies $\eta(t)\,|_{\partial\Omega} = 0$.

For our sufficient conditions of efficiency and optimality, we shall introduce the notion of (ρ, b)-geodesic quasiinvex functionals. We fix a number $\rho \in R$, a functional $b : \mathcal{F}(\Omega, M) \times \mathcal{F}(\Omega, M) \to [0, \infty)$ and the distance function $d(x(\cdot), y(\cdot))$ on $\mathcal{F}(\Omega, M)$. We consider the functional

$$E : \mathcal{F}(\Omega, M) \to R, \ \ E(x(\cdot)) = \int_\Omega X(j^1_t x) \, \mathrm{dv}.$$

Definition 1.4. Let (M, g) be a complete Riemannian manifold. Let $\mathcal{S}$ be an open η-geodesic invex subset of $\mathcal{F}(\Omega, M)$.

(i) The functional E is called (strictly) (ρ, b)-geodesic quasiinvex at $x^0(\cdot) \in \mathcal{S}$, with respect to $\eta(t)$, if $E(x(\cdot)) \leq E(x^0(\cdot))$ implies

$$b(x, x^0) \int_\Omega \left(\eta^i \frac{\partial X}{\partial x^i}(j^1_t x^0) + \frac{\partial \eta^i}{\partial t^\upsilon} \frac{\partial X}{\partial x^i_\upsilon}(j^1_t x^0) \right) \mathrm{d}v \quad (<) \ \leq \ -\rho b(x, x^0) d^2(x, x^0),$$

for any $x(\cdot) \in \mathcal{S}$.

(ii) The functional E is called monotonic (ρ, b)-geodesic quasiinvex at $x^0(t) \in S$, with respect to $\eta(t)$, if $E(x(\cdot)) \leq E(x^0(\cdot))$ implies

$$b(x, x^0) \int_\Omega \left(\eta^i \frac{\partial X}{\partial x^i}(j^1_t x^0) + \frac{\partial \eta^i}{\partial t^\upsilon} \frac{\partial X}{\partial x^i_\upsilon}(j^1_t x^0) \right) \mathrm{d}v \ = -\rho b(x, x^0) d^2(x, x^0),$$

for any $x(\cdot) \in \mathcal{S}$.

Example 1.5. Let us fix the domain

$$E = \{x : \Omega = [0, 1]^m \subset R^m \to R_+ \ \Big| \ x(\cdot) \text{ continuous}\}$$

and the "negative" Boltzmann-Shannon functional

$$J : E \to R, \ \ J(x(\cdot)) = \int_\Omega x(t) \ln x(t) dv.$$

This functional is geodesic quasiinvex with respect to

$$\eta(t) = \begin{cases} -(\ln x(t) + 1) \, d^2(x, x^0) & if \quad t \in \text{int } \Omega \\ 0 & if \quad t \in \partial\Omega. \end{cases}$$

2 Necessary optimality conditions for scalar problem (SVP)

In all the paper, we simplify supposing $T = R^m$ and hence $\det h = 1$.

We start with variational problem (SVP), recalling a well-known result.

Theorem 2.1. *(Necessary optimality to (SVP) (Mititelu, Postolache [8, Theorem 2.1]). If $x^0(\cdot) \in \mathcal{D}$ is an optimal solution to problem (SVP), then there exist a scalar*

$\tau \in R$ and the piecewise smooth multipliers $\lambda(t) = (\lambda^\alpha(t)) \in R^m$ and $\mu(t) = (\mu^s(t)) \in R^q$ that satisfy the following conditions:

$$
(SFJ) \qquad \begin{cases} \tau \dfrac{\partial X}{\partial x^0} + \lambda^\alpha(t)\dfrac{\partial g_\alpha}{\partial x^0} + \mu^s(t)\dfrac{\partial h_s}{\partial x^0} - \\ -\dfrac{\partial}{\partial t^\upsilon}\left(\tau \dfrac{\partial X}{\partial x^0_\upsilon} + \lambda^\alpha(t)\dfrac{\partial g_\alpha}{\partial x^0_\upsilon} + \mu^s(t)\dfrac{\partial h_s}{\partial x^0_\upsilon}\right) = 0 \\ \lambda^\alpha(t) g_\alpha(j^1_t x^0) = 0, \text{ for each } \alpha = \overline{1,m} \text{ (no summation)} \\ \tau \ \geq 0, \quad (\lambda^\alpha(t)) \ \succeq \ 0, \ t \in \Omega, \end{cases}
$$

where

$$
x^0 = (x^k)^0, \frac{\partial X}{\partial x^0} := \frac{\partial X}{\partial x}(j^1_t x^0), \frac{\partial X}{\partial x^0_\upsilon} := \frac{\partial X}{\partial x_\upsilon}(j^1_t x^0)
$$

etc.

Definition 2.1. A point $x^0(\cdot) \in \mathcal{D}$ is called normal optimal solution to (SVP) if $\tau > 0$.

3 Necessary efficiency conditions for multitime vector variational problems (VVP) and (VFP)

3.1 Efficiency for multitime vector variational problems (VVP)

We consider the vector functional

$$
F(x(\cdot)) = (F_1(x(\cdot)), \ldots, F_p(x(\cdot)))
$$

and the multitime vector variational problem

$$
(VVP) \qquad \begin{cases} \text{Minimize Pareto} \quad F(x(\cdot)) \\ \text{subject to} \quad g(j^1_t x) \ \preceq \ 0, \ h(j^1_t x) \ = 0, \\ \qquad\qquad\quad x(t)\,|_{\partial\Omega} = u(t), \ \forall t \in \Omega\,. \end{cases}
$$

The domain of (VVP) is just $\mathcal{D}$.

In this section we establish necessary Pareto efficiency conditions for the program (VVP).

Definition 3.1. A function $x^0(\cdot) \in \mathcal{D}$ is called an efficiency solution (Pareto minimum) to (VVP) if there exists no $x(\cdot) \in \mathcal{D}$ such that $F(x(\cdot)) \preceq F(x^0(\cdot))$.

Theorem 3.1. *(Necessary efficiency to (VVP) (Mititelu, Postolache [8, Theorem 3.1]). Consider the vector multitime variational problem (VVP) in the framework presented in Section 1.1 and let $x^0(\cdot) \in \mathcal{D}$ be an efficiency solution to (VVP). Then* *there are the vector Lagrange multipliers* $\tau = (\tau^r) \in \mathbf{R}^p$ *and* $\lambda(t) = (\lambda^\alpha(t)) \in R^m$ *and* $\mu(t) = (\mu^s(t)) \in R^q$*, all functions being piecewise smooth, which satisfy the conditions*

$$(VFJ) \qquad \begin{cases} \tau^r \dfrac{\partial f_r}{\partial x^0} + \lambda^\alpha(t) \dfrac{\partial g_\alpha}{\partial x^0} + \mu^s(t) \dfrac{\partial h_s}{\partial x^0} - \\ -\dfrac{\partial}{\partial t^v} \left(\tau^r \dfrac{\partial f_r}{\partial x^0_v} + \lambda^\alpha(t) \dfrac{\partial g_\alpha}{\partial x^0_v} + \mu^s(t) \dfrac{\partial h_s}{\partial x^0_v} \right) = 0 \\ \lambda^\alpha(t) g_\alpha(j^1_t x^0) = 0, \quad \text{for each} \quad \alpha = \overline{1,m} \\ (\tau^r) \succeq 0, \quad (\lambda^\alpha(t)) \succeq 0, \; t \in \Omega, \end{cases}$$

where

$$\frac{\partial f}{\partial x^0} := \frac{\partial f}{\partial x}(t, x^0(t), x^0_v(t)), \quad \frac{\partial f_r}{\partial x^0_v} := \frac{\partial f_r}{\partial x_v}(j^1_t x^0)$$

etc.

Definition 3.2. The function $x^0(\cdot) \in \mathcal{D}$ is called a normal efficient solution to (VVP) if, in the conditions (VFJ), there exists τ with $\tau \succeq 0$, $< e, \tau >= 1$, where $e = (1, \dots 1) \in R^p$.

3.2 Necessary efficiency for multitime vector fractional variational problems (VFP)

In this section we recall some definitions and results that will be needed later in our discussion about Pareto efficiency conditions for the multitime vector fractional variational problem

$$(VFP) \qquad \begin{cases} \text{Maximize } J(x(\cdot)) = \left(\frac{F_1}{K_1}, \dots, \frac{F_p}{K_p} \right) (x(\cdot)) \\ \text{subject to} \quad g(j^1_t x) \preceq 0, \; h(j^1_t x) = 0, \\ \qquad\qquad x(t)\,|_{\partial\Omega} = u(t) \text{ (given)}, \; \forall t \in \Omega. \end{cases}$$

The domain of (VFP) is the same set $\mathcal{D}$.

Definition 3.3. A feasible solution $x^0(\cdot) \in \mathcal{D}$ is called efficient solution of (VFP) if there is no $x(\cdot) \in \mathcal{D}$, $x(\cdot) \neq x^0(\cdot)$ such that $J(x(\cdot)) \preceq J(x^0(\cdot))$.

To present efficiency necessary conditions for (VFP), we need the followings statements. Let $x^0(\cdot)$ be an efficient solution to (FVP). Consider the problem

$$(FP)_r(x^0) \qquad \begin{cases} \underset{x}{\text{Minimize}} \dfrac{F_r(x(\cdot))}{K_r(x(\cdot))} \\ \text{subject to} \quad x(t)\,|_{\partial\Omega} = u(t), \; \forall t \in \Omega \\ \qquad g(j^1_t x) \preceq 0, \quad h(j^1_t x) = 0 \\ \qquad \dfrac{F_j(x(\cdot))}{K_j(x(\cdot))} \leq \dfrac{F_j(x^0(\cdot))}{K_j(x^0(\cdot))}, \; j = \overline{1,p}, \; j \neq r. \end{cases}$$

Denoting

$$R_r^0 = \frac{F_r(x^0(\cdot))}{K_r(x^0(\cdot))} = \min_{x(\cdot)} \frac{F_r(x(\cdot))}{K_r(x(\cdot))},\ r = \overline{1,p},$$

the problem $(\mathrm{FP})_r(x^0)$ can be written as

$$(FPR)_r \qquad \begin{cases} \min\limits_{x} \dfrac{F_r(x(\cdot))}{K_r(x(\cdot))} \\ \text{subject to} \quad x(t)\,|_{\partial\Omega} = u(t),\ \forall t \in \Omega, \\ g(j_t^1 x) \preceq 0, \quad h(j_t^1 x) = 0 \\ F_j(x(\cdot)) - R_j^0 K_j(x(\cdot))\ \leq 0,\ j = \overline{1,p},\ j \neq r. \end{cases}$$

We add now the problem

$$(SPR)_r \qquad \begin{cases} \min\limits_{x}(F_r(x(\cdot)) - R_r^0 K_r(x(\cdot))) \\ \text{subject to} \quad (t)\,|_{\partial\Omega} = u(t),\ \forall t \in \Omega \\ g(j_t^1 x)\ \preceq 0,\ h(j_t^1 x) = 0 \\ F_j(x(\cdot)) - R_j^0 K_j(x(\cdot)) \leq\ 0,\ j = \overline{1,p},\ j \neq r. \end{cases}$$

Lemma 3.2. *(Jaganathan [2]). The function $x^0(\cdot) \in \mathcal{D}$ is optimal to (FRP)$_r$ if and only if it is optimal to (SPR)$_r$.*

Definition 3.4. The efficient solution $x^0(\cdot) \in \mathcal{D}$ is called normal efficient solution to (VFP) if $x^0(\cdot)$ is normal optimal to a least one of scalar problems (SPR)$_r$, $r = \overline{1,p}$.

Theorem 3.3. *(Necessary efficiency in (VFP) (Mititelu, Postolache [8, Theorem 3.2])). Let $x^0(\cdot) \in \mathcal{D}$ be a normal efficient solution to problem (FVP). Then there exist a vector $\tau = (\tau^r) \in \mathbf{R}^p$ and piecewise smooth functions $\lambda = (\lambda^\alpha(t)) \in \mathbf{R}^m$ and $\mu = (\mu^s(t)) \in \mathbf{R}^q$ (Lagrange multipliers) that satisfy the conditions*

$$(MFJ) \qquad \begin{cases} \tau^r \left[\dfrac{\partial f_r}{\partial x^0} - R_r^0 \dfrac{\partial k_r}{\partial x^0}\right] + \lambda^\alpha(t)\dfrac{\partial g_\alpha}{\partial x^0} + \mu^s(t)\dfrac{\partial h_s}{\partial x^0} - \\ -\dfrac{\partial}{\partial t^\upsilon}\left(\tau^r \left[\dfrac{\partial f_r}{\partial x_\upsilon^0} - R_r^0 \dfrac{\partial k_r}{\partial x_\upsilon^0}\right] + \lambda^\alpha(t)\dfrac{\partial g_\alpha}{\partial x_\upsilon^0} + \mu^\beta(t)\dfrac{\partial h_\beta}{\partial x_\upsilon^0}\right) = 0 \\ \lambda^\alpha(t) g_\alpha(j_t^1 x^0) = 0, \quad \textit{for each} \quad \alpha = \overline{1,m} \\ \tau \geq 0, \quad < e, \tau >= 1,\ (\lambda^\alpha(t))\ \succeq\ 0, \quad t \in \Omega, \end{cases}$$

where

$$\frac{\partial f_r}{\partial x^0} = \frac{\partial f_r}{\partial x}(j_t^1 x^0),\ \frac{\partial f_r}{\partial x_\upsilon^0} = \frac{\partial f_r}{\partial x_\upsilon}(j_t^1 x^0)$$

etc.

Obviously, we have $R_r^0 = F_r(x^0(\cdot))/K_r(x^0(\cdot))$, $r = \overline{1,p}$. Taking into account these relations and denoting $\lambda(t) := K_r(x^0(\cdot))\lambda(t)$, $\mu(t) := K_r(x^0(\cdot))\mu(t)$, Theorem 3 becomes

Theorem 3.4. *(Necessary efficiency in (VFP) (Mititelu, Postolache [8, Theorem 3.3])). Let $x^0(\cdot) \in \mathcal{D}$ be a normal efficient solution to problem (VFP). Then there exist a vector $\tau = (\tau^r) \in \mathbf{R}^p$ and piecewise smooth functions $\lambda = (\lambda^\alpha(t)) \in \mathbf{R}^m$ and $\mu = (\mu^\beta(t)) \in \mathbf{R}^q$ (Lagrange multipliers) that satisfy the conditions*

$$(MFJ)_0 \quad \begin{cases} \tau^r \left[K_r(x^0)\dfrac{\partial f_r}{\partial x^0} - F_r(x^0)\dfrac{\partial k_r}{\partial x^0} \right] + \lambda^\alpha(t)\dfrac{\partial g_\alpha}{\partial x^0} + \mu^s(t)\dfrac{\partial h_s}{\partial x^0} - \\ -\dfrac{\partial}{\partial t^\upsilon}\left(\tau^r \left[K_r(x^0)\dfrac{\partial f_r}{\partial x^0_\upsilon} - F_r(x^0)\dfrac{\partial k_r}{\partial x^0_\upsilon} \right] + \lambda(t)\dfrac{\partial g}{\partial x^0_\upsilon} + \mu(t)\dfrac{\partial h}{\partial x^0_\upsilon} \right) = 0 \\ \lambda^\alpha(t) g_\alpha(t, x^0(t), x^0_\upsilon(t)) = 0, \quad \textit{for each} \quad \alpha = \overline{1,m} \\ \tau \ \geq \ 0, \quad < e, \tau >= 1, \ (\lambda^\alpha(t)) \ \succeq \ 0, \ \ t \in \Omega. \end{cases}$$

Definition 3.5. (Equivalent to Definition 8). The efficient solution $x^0(\cdot) \in \mathcal{D}$ is called normal efficient solution to (VFP) if the conditions (MFJ) or $(\text{MFJ})_0$ exist with $\tau \succeq 0$, $\langle e, \tau \rangle = 1$.

4 Sufficient efficiency conditions for problems (VVP) and (VFP)

In this section we shall establish *sufficient efficiency conditions for* (VVP) and (VFP). In all our theory it is used the essential

Main Condition: Suppose that the subset $\mathcal{S} \subset \mathcal{F}(\Omega, M)$ is η-geodesic invex set, where the C^1 vector function $\eta(t)$ is as in Definition 2, and instead of $\mathcal{F}(\Omega, M)$ we use $\mathcal{S}$.

Theorem 4.1. *(Sufficient efficiency for (VVP)). Let $x^0(\cdot) \in \mathcal{S}$, $\tau = (\tau^r)$, $\lambda = (\lambda^\alpha)$ and $\mu = (\mu^s)$ be multipliers satisfying the relations (MFJ) from Theorem 3. For each $x^0(\cdot) \in \mathcal{S}$, let $x(\cdot) \in \mathcal{S}$ be an arbitrary geodesic perturbation. If the following conditions are fulfilled:*

a) Each functional $F_r(x(\cdot))$ is (ρ^1_r, b)-geodesic quasiinvex at $x^0(\cdot)$, with respect to η and d.

b) The functional $\displaystyle\int_\Omega \lambda^\alpha(t) g_\alpha(j^1_t x)\ dv$ is (ρ_2, b)-geodesic quasiinvex at $x^0(\cdot)$, with respect to η and d.

c) The functional $\displaystyle\int_\Omega \mu^s(t) h_s(j^1_t x)\ dv$ is monotonic (ρ_3, b)-geodesic quasiinvex at $x^0(\cdot)$, with respect to η and d.

d) One of the functional of a)-b) is strictly (ρ, b)-geodesic quasiinvex at $x^0(\cdot)$.

e) $\tau^r \rho^1_r + \rho_2 + \rho_3 \geq 0 \quad (\rho^1_r,\ \rho_2,\ \rho_3 \in \mathbf{R})$,

then $x^0(\cdot)$ is an efficient solution to(VVP).

Proof. . Let us suppose toward a contradiction, that $x^0(\cdot)$ is not an efficient solution for (VVP). Then, for each $r = \overline{1,p}$, there exists $x(\cdot) \neq x^0(\cdot)$, a feasible solution to (VVP), such that

$$\int_\Omega f_r(j^1_t x)\ dv \ \leq \ \int_\Omega f_r(j^1_t x^0)\ dv.$$

According to hypothesis a), it follows that

$$b(x,x^0)\int_\Omega [\eta_i(t)\frac{\partial f_r}{\partial x^i}(j_t^1x^0)+\frac{\partial \eta_i}{\partial t^\upsilon}\frac{\partial f_r}{\partial x^i_\upsilon}(j_t^1x^0)]\ dv\ \leq\ -\rho_r^1 b(x,x^0)d^2(x,x^0)$$

(see Definition 2).

Transvecting this inequality by $\tau^r \geq 0$, we obtain

$$\begin{aligned}b(x,x^0)\int_\Omega [\eta_i(t)\ \tau^r\frac{\partial f_r}{\partial x^i}(j_t^1x^0)+\frac{\partial \eta_i}{\partial t^\upsilon}\tau^r\frac{\partial f_r}{\partial x^i_\upsilon}(j_t^1x^0)]\ dv\\ \leq -(\tau^r\rho_r^1)b(x,x^0)d^2(x,x^0).\end{aligned} \tag{4.1}$$

From the continuity of the functions, we choose $x(t)\in S$ such that

$$\int_\Omega \lambda^\alpha(t)g_\alpha(t,x,x_\upsilon)\ dv\ \leq\ \int_\Omega \lambda^\alpha(t)g_\alpha(t,x^0,x^0_\upsilon)\ dv.$$

Then, taking into account the condition b) and Definition 2, it follows

$$\begin{aligned}b(x,x^0)\int_a^b [\eta_i(t)\lambda^\alpha(t)\frac{\partial g_\alpha}{\partial x^i}(j_t^1x^0)+\frac{\partial \eta_i}{\partial t^\upsilon}\lambda^\alpha(t)\frac{\partial g_\alpha}{\partial x^i_\upsilon}(j_t^1x^0)]\ dv\\ \leq -\rho_2 b(x,x^0)d^2(x,x^0).\end{aligned} \tag{4.2}$$

Taking into account the condition c) and Definition 2, the equality

$$\int_\Omega \mu^s(t)h_s(j_t^1x)\ dv\ =\ \int_\Omega \mu^s(t)h_s(j_t^1x^0)\ dv$$

implies

$$\begin{aligned}b(x,x^0)\int_\Omega [\eta_i(t)\mu^s(t)\frac{\partial h_s}{\partial x^i}(j_t^1x^0)\ +\frac{\partial \eta_i}{\partial t^\upsilon}\mu^s(t)\frac{\partial h_s}{\partial x^i_\upsilon}(j_t^1x^0)]\ dv\\ \leq -\rho_3 b(x,x^0)d^2(x,x^0).\end{aligned} \tag{4.3}$$

We sum side by side the relations (4.1), (4.2) and (4.3) and take into account d). Then we obtain

$$\begin{aligned}&b(x,x^0)\int_\Omega \eta_i(t)\left[\tau^r\frac{\partial f_r}{\partial x^i}+\lambda^\alpha(t)\frac{\partial g_\alpha}{\partial x^i}+\mu^s(t)\frac{\partial h_s}{\partial x^i}\right]\ (j_t^1x^0)dv+\\ &+b(x,x^0)\int_\Omega \frac{\partial \eta_i}{\partial t^\upsilon}\left[\tau^r\frac{\partial f_r}{\partial x^i_\upsilon}+\lambda^\alpha(t)\frac{\partial g_\alpha}{\partial x^i_\upsilon}+\mu^s(t)\frac{\partial h_s}{\partial x^i_\upsilon}\right](j_t^1x^0)\ dv\\ &< -(\tau^r\rho_r^1+\rho_2+\rho_3)b(x,x^0)d^2(x,x^0).\end{aligned} \tag{4.4}$$

From (4.4) it follows $b(x,x^0)>0$ and then

$$\begin{aligned}&\int_\Omega \eta_i(t)\left[\tau^r\frac{\partial f_r}{\partial x^i}+\lambda^\alpha(t)\frac{\partial g_\alpha}{\partial x^i}+\mu^s(t)\frac{\partial h_s}{\partial x^i}\right]\ (j_t^1x^0)dv+\\ &+\int_\Omega \frac{\partial \eta_i}{\partial t^\upsilon}\left[\tau^r\frac{\partial f_r}{\partial x^i_\upsilon}+\lambda^\alpha(t)\frac{\partial g_\alpha}{\partial x^i_\upsilon}+\mu^s(t)\frac{\partial h_s}{\partial x^i_\upsilon}\right](j_t^1x^0)\ dv\\ &< -(\tau^r\rho_r^1+\rho_2+\rho_3)d^2(x,x^0).\end{aligned} \tag{4.5}$$

We denote $V = [\tau^r f_r + \lambda^\alpha(t) g_\alpha + \mu^s(t) h_s](j_t^1 x^0)$ and then the relation (4.5) becomes

$$\int_\Omega \eta_i(t) \frac{\partial V}{\partial x^i} dv + \int_\Omega \frac{\partial \eta_i}{\partial t^\upsilon} \frac{\partial V}{\partial x^i_\upsilon} dv < -(\tau^r \rho_r^1 + \rho_2 + \rho_3) d^2(x, x^0), \tag{4.6}$$

where we denoted $\frac{\partial V}{\partial x} = \frac{\partial V}{\partial x}(j_t^1 x^0)$, $\frac{\partial V}{\partial x^i_\upsilon} = \frac{\partial V}{\partial x^i_\upsilon}(j_t^1 x^0)$.

For an integration by parts in the second integral of (4.5), we have

$$\frac{\partial}{\partial t^\upsilon}\left(\eta_i(t) \frac{\partial V}{\partial x^i}\right) = \frac{\partial \eta_i}{\partial t^\upsilon} \frac{\partial V}{\partial x^i} + \eta_i(t) \frac{\partial}{\partial t^\upsilon}\left(\frac{\partial V}{\partial x^i_\upsilon}\right), \upsilon = \overline{1, m}.$$

and

$$\int_\Omega \frac{\partial \eta_i}{\partial t^\upsilon} \frac{\partial V}{\partial x^i} dv = \int_\Omega \frac{\partial}{\partial t^\upsilon}\left(\eta_i(t) \frac{\partial V}{\partial x^i}\right) dv - \int_\Omega \eta_i(t) \frac{\partial}{\partial t^\upsilon}\left(\frac{\partial V}{\partial x^i_\upsilon}\right) dv. \tag{4.7}$$

Using the flow-divergence formula, we find

$$\int_\Omega \frac{\partial}{\partial t^\upsilon}\left(\eta_i(t) \frac{\partial V}{\partial x^i_\gamma}\right) dv = \int_{\partial\Omega} \left(\eta_i(t) \frac{\partial V}{\partial x^i_\upsilon}\right) n_\upsilon d\sigma = 0, \tag{4.8}$$

where $\bar{n} = (n_\upsilon)$ is the normal unit vector of the hypersurface $\partial\Omega$ and $\eta(t)\,|_{\partial\Omega} = 0$. Then, relation (4.7) becomes

$$\int_\Omega \frac{\partial \eta_i}{\partial t^\upsilon} \frac{\partial V}{\partial x^i} dv = -\int_\Omega \eta_i(t) \frac{\partial}{\partial t^\upsilon}\left(\frac{\partial V}{\partial x^i_\upsilon}\right) dv \tag{4.9}$$

and according to (4.9), the relation (4.6) can be written

$$\int_\Omega \eta_i(t) \left[\frac{\partial V}{\partial x^i} - \frac{\partial}{\partial t^\upsilon}\left(\frac{\partial V}{\partial x^i_\upsilon}\right)\right] dv < -(\tau^r \rho_r^1 + \rho_2 + \rho_3) d^2(x, x^0). \tag{4.10}$$

Taking into account the first relation of (VVP), the relation (4.10) becomes $0 < -(\tau^r \rho_r^1 + \rho_2 + \rho_3) d^2(x, x^0)$. Having $d(x, x^0) \geq 0$ and the hypothesis e), we obtain the inequality $0 < 0$ that is a false. Therefore x^0 is an efficient solution to (VVP), because M is a complete Riemannian manifold. $\square$

In what follows we establish *efficiency sufficient conditions for the problem* (VFP).

Theorem 4.2. *(Sufficient efficiency for (VFP)). Let $x^0(\cdot) \in \mathcal{S}$ and $\tau = (\tau^r)$, $\lambda = (\lambda^\alpha)$, $\mu = (\mu^s)$ be multipliers satisfying the relations(MFJ) from Theorem 3. Suppose fulfilled the following conditions:*

a′) Each functional $F_r(x(\cdot)) - R_r^0 K_r(x(\cdot))$ is (ρ_i^1, b)-geodesic quasiinvex at $x^0(\cdot)$, with respect to η and d.

b′) The functional $\int_\Omega \lambda^\alpha(t) g_\alpha(j_t^1 x)\, dv$ is (ρ_2, b)-geodesic quasiinvex at $x^0(\cdot)$, with respect to η and d.

c′) The functional $\int_\Omega \mu^s(t) h_s(j_t^1 x)\, dv$ is (ρ_3, b)-geodesic quasiinvex at $x^0(\cdot)$, with respect to η and d.

d′) One of the functionals from a′), b) and c) is strictly (ρ, b)-geodesic quasiinvex at $x^0(\cdot)$, with respect to η and d ($\rho = \rho_i^1$, ρ_2 or ρ_3, respectively).

e′) $\tau^r \rho_r^1 + \rho_2 + \rho_3 \ \geq\ 0$.

Then $x^0(\cdot)$ is an efficient solution to (VFP).

Proof. It is similar to those of Theorem 5, where, for each $r = \overline{1,p}$, the Lagrangian $f_r(j_t^1 x)$ is replaced by $f_r(j_t^1 x) - R_r^0 k_r(j_t^1 x)$.

Theorem 4.3. *(Sufficient efficiency for (VFP)). Let $x^0(\cdot) \in \mathcal{S}$ and $\tau = (\tau^r)$, $\lambda = (\lambda^\alpha)$, $\mu = (\mu^s)$ be multipliers satisfying the relations (MFV)^0 from Theorem 4. Suppose*

a$''$) Each functional $r = \overline{1,p}$, $\int_\Omega [K_r(x^0) f_r(j_t^1 x) - F_r(x^0) k_r(j_t^1 x)] dv$ is (ρ_r^1, b)-geodesic quasiinvex at $x^0(\cdot)$, with respect to η and d.

b), c) and e) of Theorem 5.

d$''$) One of the functionals from a$''$), b) and c) is strictly (ρ, b)-geodesic quasiinvex at $x^0(\cdot)$, with respect to η and d ($\rho = \rho_r^1, \rho_2$ or ρ_3 respectively).

Then $x^0(\cdot)$ is a geodesic efficient solution to (VFP).

Proof. It is similar to those of Theorem 5, where the hypothesis a) is replaced by hypothesis a$''$) of this theorem.

If, in Theorems 5-7, the functionals from the hypotheses b) and c) are replaced by the functional $\int_\Omega [\lambda^\alpha(t) g_\alpha(j_t^1 x) + \mu^s(t) h_s(j_t^1 x)] dv$, then we have the following results:

Corollary 4.4. *(Sufficient efficiency conditions for (VVP)). Let $x^0(\cdot) \in \mathcal{S}$ and τ, λ, μ be multipliers satisfying the relations (VFJ) from Theorem 3. Suppose*

a) Each functional $r = \overline{1,p}$, $F_r(x(\cdot))$ is (ρ_r^1, b)-geodesic quasiinvex at $x^0(\cdot)$, with respect to η and d.

b$'$) The functional $\int_\Omega [\lambda^\alpha(t) g_\alpha(j_t^1 x) + \mu^s(t) h_s(j_t^1 x)] dv$ is (ρ_2, b)-geodesic quasiinvex at $x^0(\cdot)$, with respect to η and d.

d$'$) The functionals from a) and b$'$)are strictly (ρ, b)-geodesic quasiinvex at $x^0(\cdot)$, with respect to η and d ($\rho = \rho_r^1$ or ρ_2, respectively).

e$'$) $\tau^r \rho_r^1 + \rho_2 \geq 0$.

Then $x^0(\cdot)$ is an efficient solution to (VVP).

Corollary 4.5. *(Sufficient efficiency conditions for (VFP) Let $x^0(\cdot) \in \mathcal{S}$, and τ, λ, μ be multipliers satisfying the relations (MFJ) from Theorem 3. Suppose satisfied the following conditions:*

a$'$) Each functional $r = \overline{1,p}$, $F_r(x(\cdot)) - R_r^0 K_r(x(\cdot))$ is (ρ_r^1, b)-geodesic quasiinvex at $x^0(\cdot)$, with respect to η and d_1.

b$'$) and e$'$) from Corollary 1.

d$''$) One of the functionals from a$'$) and b$'$) is strictly (ρ, b)-geodesic quasiinvex at $x^0(\cdot)$, with respect to η and d.

Then $x^0(\cdot)$ is an efficient solution to (VFP).

Corollary 4.6. *(Sufficient efficiency conditions for (VFP)). Let $x^0(\cdot) \in \mathcal{S}$, and τ ,λ, μ be multipliers satisfying the relations (MFV)$_0$ from Theorem 4. Also, we consider a vector function η as in Definitions* 3. *Suppose the following conditions are satisfied:*

a$''$) Each functional $r = \overline{1,p}$, $\int_\Omega [K_r(x^0) f_r(j_t^1 x) - F_r(x^0) k_r(j_t^1 x)]\ dv$ is (ρ_r^1, b)-geodesic quasiinvex at $x^0(\cdot)$, with respect to η and d.

b$'$) and e$'$) from Corollary 1.

d''') One of the functionals from a'') and b') is strictly (ρ, b)-geodesic quasiinvex at $x^0(\cdot)$, with respect to η and d, as in Corollary 1.

Then $x^0(\cdot)$ is an efficient solution to (VFP).

Corollary 4.7. *(Sufficient optimality conditions for (SVP)). Let $x^0(\cdot) \in \mathcal{S}$, and τ, λ, μ be multipliers satisfying the relations (SFJ) from Theorem 1. If the following conditions*

a) $\int_\Omega X(j_t^1 x)dv$ is (ρ, b)-geodesic quasiinvex at $x^0(\cdot)$, with respect to η and d.

b') From Corollary 1, c), we have $\tau\rho + \rho_2 \geq 0$.

$\overline{d}$) One of the functionals from $\overline{a}$ and b') is strictly (ρ, b)-geodesic quasiinvex at $x^0(\cdot)$, with respect to η and d,

are satisfied, then $x^0(\cdot)$ is an efficient solution to (SVP).

Acknowledgements. Partially supported by University Politehnica of Bucharest and by Academy of Romanian Scientists. Part of this paper was presented at The VIII-th International Conference "Differential Geometry and Dynamical Systems" (DGDS-2014), 1 - 4 September 2014, Mangalia, Romania.

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