Geometric and numerical analysis of mathematical models for multi-species interactions

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Abstract. In this paper, by using appropriate numerical methods, we perform a computational analysis of different types of invariants and main sizes for some mathematical models of the multi-species interactions given by Volterra-Lotka type equations.

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Key words: conservation laws; Volterra-Lotka equations; Runge-Kutta methods.

1 Introduction

The study of Lotka-Volterra type equations has been the subject of thousands of scientific papers and books in the last 90 years, immediately after the introduction of these mathematical models of evolution of prey-predator populations by Alfred James Lotka in 1925 ([9]) and Vito Volterra in 1931 ([17]), 1937 ([18]). More precisely, the $2D$ Lotka–Volterra model was initially proposed by A. J. Lotka in the theory of autocatalytic chemical reactions in 1910. This system of equations is a first-order, non-linear system of differential equations and it is used to describe the dynamics of biological systems in which two or more species interact, one as a predator and the others as prey.

In this paper we will make a numerical study of some conservation laws and main sizes for 2D and 3D Volterra-Lotka systems, using the Hamilton-Poisson realisations of these dynamical systems. The viewpoint is geometric and we also compute and characterize objects of dynamical significance, in order to understanding the mathematical properties observed in numerical computation for dynamical models arising in many important theoretical and practical situations from mathematics, science and engineering. Our study is a interplay between dynamical systems geometrical theory and computational calculus.

We will discuss two very important examples. First example represent so called variational dynamical systems, i.e. these dynamical systems are described by a system

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of ordinary differential equations (SODE) which can be written as the Euler-Lagrange equations associated to a Lagrangian L,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right) - \frac{\partial L}{\partial x^i} = 0\,.$$

This example is the prey-predator $2D$ Lotka-Volterra system ([9], [16], [17], [18]). This dynamical system is included in the presymplectic case because the 2-form ω_L associated to the corresponding Lagrangian is degenerate ([13], [14]). Finally, we present a version of the well-known prey-predator $3D$ Lotka-Volterra system. This system isn't a variational dynamical system ([5], [11], [12]). However, we can give more Hamilton-Poisson realizations of this bi-Hamiltonian system like in the $2D$ case. Important studies related with this subject was also made in [1], [5], [8], [15].

Between dynamical systems theory and computational analysis of dynamical systems there is a strong interplay. The theory provides a framework for interpreting numerical observations and foundations for efficient numerical algorithms. Numerical integration is a important and actual subject, but due to the today's high computer efficiency (speed and memory) it is still very active, being analyzed by extensive theory and a vast range of software, platforms or libraries, [2], [3], [7], [10]. Taking into account that even for the simplest 2D Lotka-Volterra system, the analytical solution is useless: root of a polynomial with an integral plus the special function Lambert, we must resort to numerical methods in order to have information about the trajectories.

Thus, for $2D$ and $3D$ Volterra-Lotka systems, by constructing a Matlab-based numerical code, we are looking to approximate and characterize different types of invariants and also to extract information on the dynamical behavior and perform comparisons for both different initial conditions associated to the considered problem and for different values of the parameters involved in the analysed problems. Firstly, we focus on the numerical solving of the initial value problems by appropriate numerical methods, such as Runge-Kutta methods (for the 2D case we use a fourth order Runge-Kutta method, [4], and for the 3D case we used a fifth order Runge-Kutta method, [19]). Secondly, using this approach we perform a numerical analysis of the conservation laws and main sizes, like the Lagrangian and the Hamiltonian function.

2 The prey-predator $2D$ Lotka-Volterra system

Let us consider the well-known system of ordinary differential equations:

$$\left\{\begin{array}{lcl}\dot{x} & = & ax - bxy\\ \dot{y} & = & -cy + dxy\end{array}\right., \ a,\, b,\, c,\, d > 0. \tag{2.1}$$

This system is called *Lotka-Volterra system* and represent a complex biological system model, in which two species x and y live in a limited area, so that individuals of the species y (predator) feed only individuals of species x (prey) and they feed only resources of the area in which they live.

The evolution system (2.1) can be written in the form of Euler-Lagrange equations:

$$\begin{array}{lcl}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} & = & 0\\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} & = & 0,\end{array}$$

where the Lagrangian L is

$$L = \frac{1}{2}\left(\frac{\ln y}{x}\dot{x} - \frac{\ln x}{y}\dot{y}\right) + c\ln x - a\ln y - dx + by.$$

Let us remark that the function

$$H = -c\ln x + a\ln y + dx - by$$

is a *a conservation law* for prey-predator system (2.1), according also with Figure 4 and Figure 5. The Lotka-Volterra equations (2.1) has the *Hamilton-Poisson realization*

$$\dot{X} = J\nabla H$$

where $X^t = (x,\, y)$, $H = c\ln x + a\ln y - dx - by$ is the Hamiltonian function and $J = \begin{pmatrix} 0 & xy \\ -xy & 0 \end{pmatrix}$ is the Poisson structure.

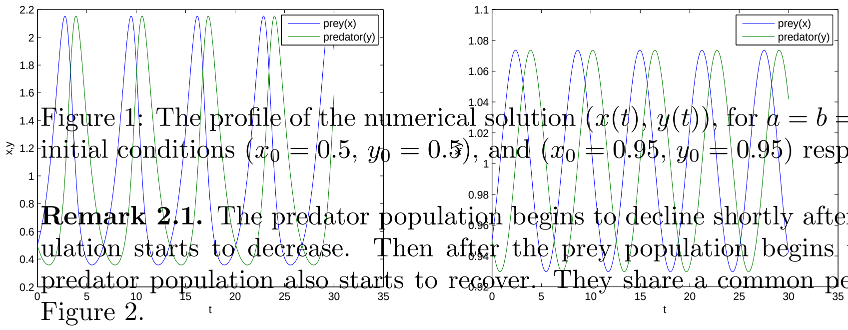

Figure 1: The profile of the numerical solution $(x(t),\, y(t))$, for $a = b = c = d = 1$ and initial conditions $(x_0 = 0.5,\, y_0 = 0.5)$, and $(x_0 = 0.95,\, y_0 = 0.95)$ respectively.

Remark 2.1. The predator population begins to decline shortly after the prey population starts to decrease. Then after the prey population begins to recover, the predator population also starts to recover. They share a common period, Figure 1, Figure 2.
H takes constant values for different values of parameters a, b, c, d and different initial conditions, as we presented in Figure 4. Thus we are in agreement with the property to be a conservation law.

3 The $3D$ Lotka-Volterra system

In [6] was discussed the next three-dimensional Lotka-Volterra system which models the evolution of competition between three species:

$$\text{(3.1)} \qquad \begin{cases} \dot{x} &= x(cy + z + \lambda) \\ \dot{y} &= y(x + az + \mu) \\ \dot{z} &= z(bx + y + \nu) \end{cases}, \quad a, b, c \in \mathbf{R},\ \lambda, \mu, \nu > 0.$$

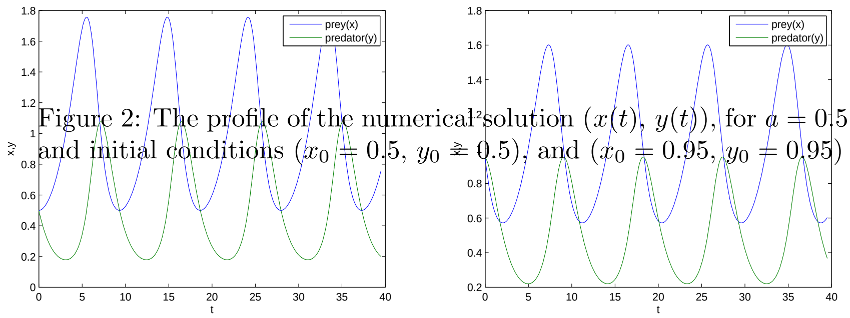

Figure 2: The profile of the numerical solution $(x(t),\, y(t))$, for $a = 0.5$, $b = c = d = 1$ and initial conditions $(x_0 = 0.5,\, y_0 = 0.5)$, and $(x_0 = 0.95,\, y_0 = 0.95)$ respectively.

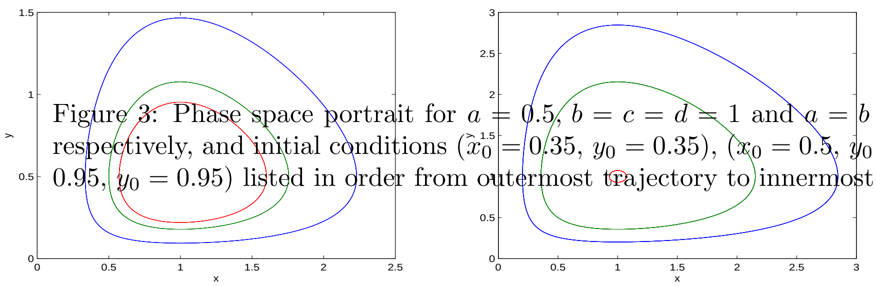

Figure 3: Phase space portrait for $a = 0.5$, $b = c = d = 1$ and $a = b = c = d = 1$, respectively, and initial conditions $(x_0 = 0.35,\, y_0 = 0.35)$, $(x_0 = 0.5,\, y_0 = 0.5)$, $(x_0 = 0.95,\, y_0 = 0.95)$ listed in order from outermost trajectory to innermost trajectory

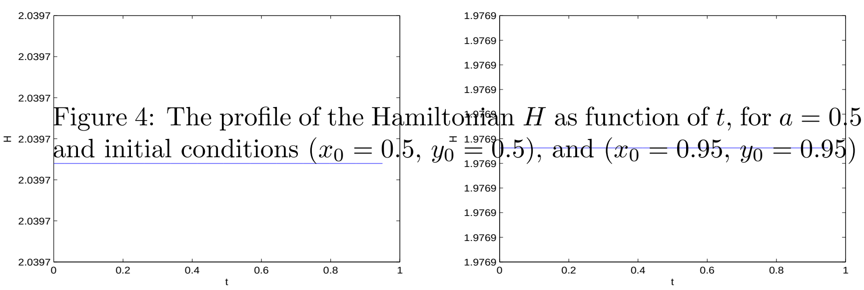

Figure 4: The profile of the Hamiltonian H as function of t, for $a = 0.5$, $b = c = d = 1$ and initial conditions $(x_0 = 0.5,\, y_0 = 0.5)$, and $(x_0 = 0.95,\, y_0 = 0.95)$ respectively.

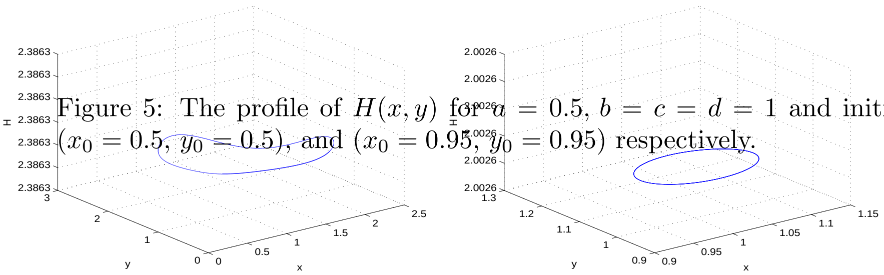

Figure 5: The profile of $H(x,y)$ for $a = 0.5$, $b = c = d = 1$ and initial conditions ($x_0 = 0.5$, $y_0 = 0.5$), and ($x_0 = 0.95$, $y_0 = 0.95$) respectively.

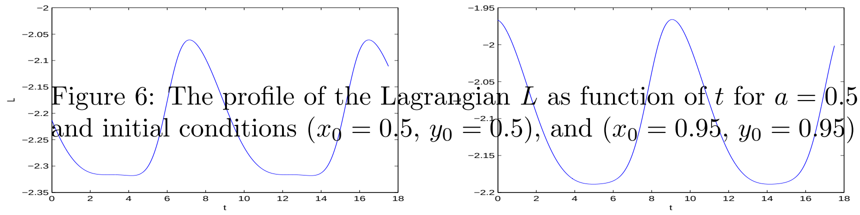

Figure 6: The profile of the Lagrangian L as function of t for $a = 0.5$, $b = c = d = 1$ and initial conditions ($x_0 = 0.5$, $y_0 = 0.5$), and ($x_0 = 0.95$, $y_0 = 0.95$) respectively.

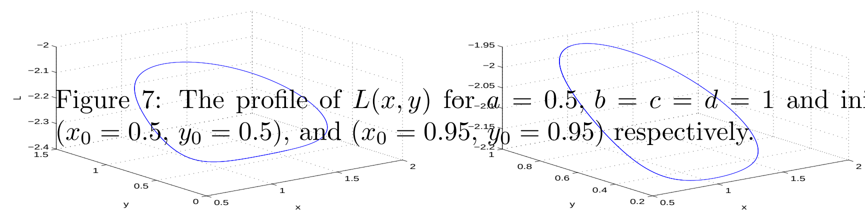

Figure 7: The profile of $L(x,y)$ for $a = 0.5$, $b = c = d = 1$ and initial conditions ($x_0 = 0.5$, $y_0 = 0.5$), and ($x_0 = 0.95$, $y_0 = 0.95$) respectively.

Following [6], if $abc = -1$ and $\nu = \mu b - \lambda ab$, then the $3D$ Lotka-Volterra system (3.1) admit two conservation laws

$$H_1 = ab \ln x - b \ln y + \ln z \,, \;\; H_2 = abx + y - az + \nu \ln y - \mu \ln z \,,$$

because (3.1) is a particular case of a *bi-Hamiltonian system.*

In this case, the dynamics of (3.1) has two distinct Hamilton-Poisson realizations $\dot{X} = J_1 \nabla H_2$ and $\dot{X} = J_2 \nabla H_1$, where $X^t = (x, y)$, and

$$J_1 = \begin{pmatrix} 0 & cxy & bcxz \\ -cxy & 0 & -yz \\ -bcxz & yz & 0 \end{pmatrix}, \; J_2 = \begin{pmatrix} 0 & cxy(az+\mu) & cxz(y+\nu) \\ -cxy(az+\mu) & 0 & xyz \\ -cxz(y+\nu) & -xyz & 0 \end{pmatrix}.$$

From $J_1 \nabla H_1 = 0$ and $J_2 \nabla H_2 = 0$ we have that H_1 and H_2 are Casimir functions of J_1 and J_2, respectively ([15]).

For the system (3.1) we consider the case $a = b = c = -1$ and $\mu = 1$, $\lambda = 0$, $\nu = -1$.

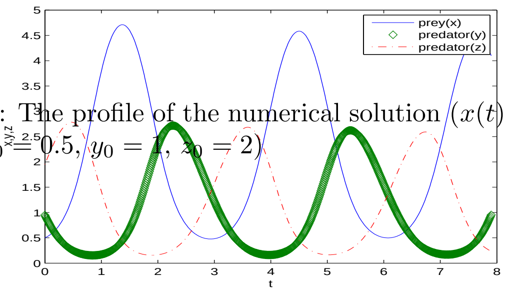

Figure 8: The profile of the numerical solution $(x(t),\, y(t),\, z(t))$, for the initial conditions $(x_0 = 0.5,\, y_0 = 1,\, z_0 = 2)$

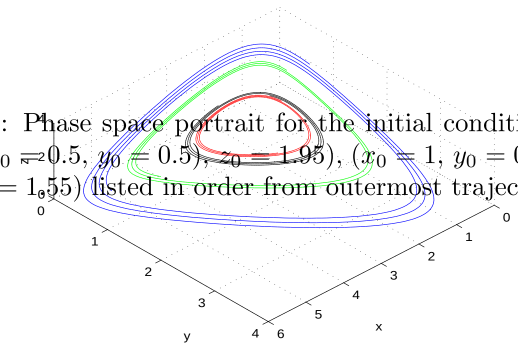

Figure 9: Phase space portrait for the initial conditions ($x_0 = 0.5$, $y_0 = 0.95$, $z_0 = 2.95$), ($x_0 = 0.5$, $y_0 = 0.5$), $z_0 = 1.95$), ($x_0 = 1$, $y_0 = 0.75$, $z_0 = 1.25$), ($x_0 = 2.1$, $y_0 = 0.35$, $z_0 = 1.55$) listed in order from outermost trajectory to innermost trajectory

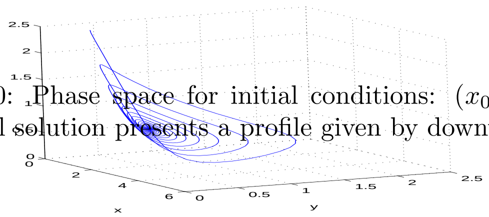

Figure 10: Phase space for initial conditions: ($x_0 = 1$, $y_0 = 0.25$, $z_0 = 2.5$) The numerical solution presents a profile given by downward spirals

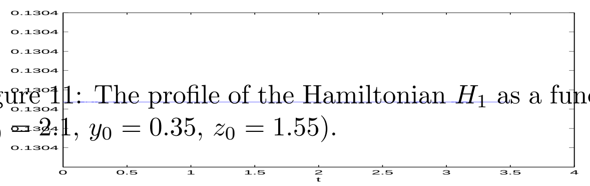

Figure 11: The profile of the Hamiltonian H_1 as a function of t, for initial conditions: ($x_0 = 2.1$, $y_0 = 0.35$, $z_0 = 1.55$).

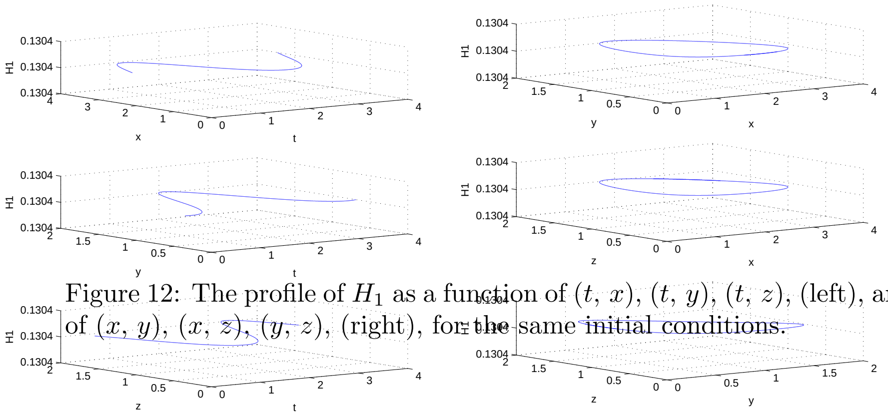

Figure 12: The profile of H_1 as a function of $(t,\, x)$, $(t,\, y)$, $(t,\, z)$, (left), and as function of $(x,\, y)$, $(x,\, z)$, $(y,\, z)$, (right), for the same initial conditions.

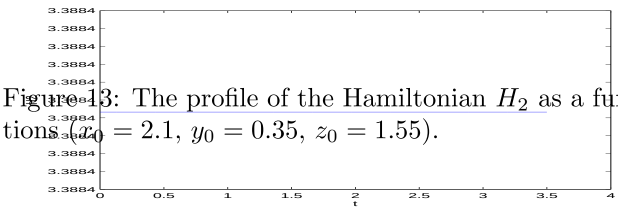

Figure 13: The profile of the Hamiltonian H_2 as a function of t, for the initial conditions $(x_0 = 2.1,\, y_0 = 0.35,\, z_0 = 1.55)$.

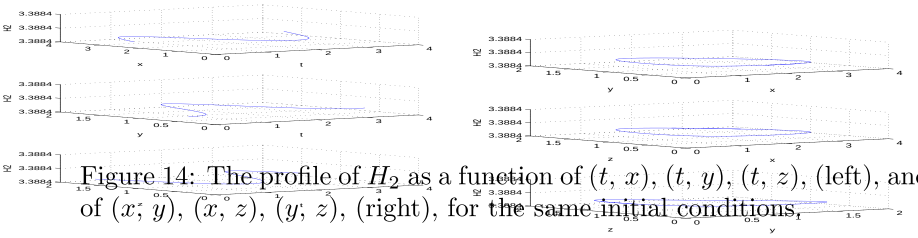

Figure 14: The profile of H_2 as a function of $(t,\, x)$, $(t,\, y)$, $(t,\, z)$, (left), and as function of $(x,\, y)$, $(x,\, z)$, $(y,\, z)$, (right), for the same initial conditions.

Remark 3.1. In the 3D case displaying the graph of x, y and z across time t, one observes the periodic behavior of the system. Each predator population also peaks and then begins to decrease shortly after its respective prey population peaks and begins to decrease, Figure 8.
The two Hamiltonians H_1 and H_2 associated to the 3D case of Lotka-Volterra system are characterized thorough our numerical study by constant values, for different initial conditions, as we presented in Figure 11-14.

4 Conclusions

We perform a computational analysis of these mathematical models, in order to approximate different types of invariants and main sizes, through numerical codes based on appropriate numerical calculus techniques for numerical integration of these type problems. Thus, starting from certain initial value problems associated to our models, we obtain the numerical solution and we develop the numerical characterization of the main sizes previously analysed from the geometrical point of view. Thus we are able to make different comparisons between these studied quantities for different values of parameters, for different initial conditions etc. We emphasize that the present study can be useful also to make certain adjustments for the parameters involved in the differential equations in order to calibrate more precisely the models to better predict some realistic situations.

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