On hypersurfaces in a real space form with the Ricci tensor satisfying certain conditions

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Abstract. The purpose of the present paper is to classify hypersurfaces M^n in a real space form $\tilde{M}^{n+1}(c)$ which satisfy the harmonic curvature or the harmonic conformal curvature. We show that such hypersurfaces have the parallel second fundamental form satisfying certain conditions.

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1 Introduction

Let $\tilde{M}^{n+1}(c)$ be an $(n+1)$-dimensional space form of constant curvature c (i.e. complete, simply connected Riemannian manifold with constant sectional curvature, say c). For each real number c there is (up to isometry) exactly one space form in every dimension of sectional curvature c. The space forms of sectional curvature c are denoted by $S^{n+1}(c)$, R^{n+1} and H^{n+1} depending on whether c is positive, zero or negative, respectively. $S^{n+1}(c)$ is a Euclidean sphere of constant curvature c. R^{n+1} is a Euclidean space. H^{n+1} is a hyperbolic space of constant curvature c

Let M^n be a hypersurface in a space form $\tilde{M}^{n+1}(c)$. Let ∇ and S be the covariant differentiation on M^n and the Ricci tensor of M^n, respectively. P. J. Ryan [7] classified these hypersurfaces with regards to the parallel Ricci tensor, i.e., $\nabla S = 0$. He proved that if the Ricci tensor S of M^n is parallel and the mean curvature is constant, then M^n has the parallel second fundamental tensor, that is, M^n is locally symmetric and M^n is either a space form or the product manifold of two space forms.

The Ricci tensor S is called the *harmonic curvature* if M^n satisfies the condition of vanishing the codifferential of the curvature tensor, i.e., $(\nabla_X S)Y = (\nabla_Y S)X$ for any X, Y tangent to M^n. The Ricci tensor S is also called the *harmonic conformal curvature* if M^n satisfies the condition of vanishing the codifferential of the conformal curvature tensor, i.e., $(\nabla_X S)Y - (\nabla_Y S)X = \frac{1}{2(n-1)}\{(\nabla_X s)Y - (\nabla_Y s)X\}$ for any X, Y tangent to M^n, where s is the scalar curvature of M^n.

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The purpose of this paper is to classify hypersurfaces with the harmonic curvature and with the harmonic conformal curvature in a space form. We note that these conditions are weaker than $\nabla S = 0$. We prove the following theorems:

Theorem 1. *Let M^n be a hypersurface of dimension $n \geq 3$ without constant curvature c and with harmonic curvature in a space form $\tilde{M}^{n+1}(c)$. Then M^n has the parallel second fundamental form or $c = 0$ and* $\mathrm{rank} A = 2$.

Remark The notion of the harmonic curvature was introduced in [5] for the first time and Omachi showed such hypersurfaces have pararell second fundamental tensor under the assumptions of complete, constant mean curvature and trace $A^4 = $ constant. In [8] Umehara showed by using the fact that M^n has the parallel second fundamental form if M^n has a symmetric tensor which satisfies ϕ and ϕ^2 are the Codazzi type: If M^n is a hypersurface of dimension $n \geq 3$ with constant mean curvature and harmonic curvature, then M^n has the parallel second fundamental form. As a corollary we will also give a simple proof of his result.

Theorem 2. *Let M^n be a hypersurface of dimension $n \geq 3$ with constant mean curvature and harmonic conformal curvature in a space form $\tilde{M}^{n+1}(c)$. Then M^n has the parallel second fundametal form.*

In [3] the second author applied the condition of Ricci semi-symmetric the affine hypersurfaces in an affine space. In the future we want to apply the harmonci curvature or harmonic conformal curvature the affine hypersurfaces in an affine space.

2 Preliminaries

Let M^n be a hypersurface of dimension n in a space form $\tilde{M}^{n+1}(c)$. For each point $x_0 \in M^n$, we choose an unit normal vector field ξ defined in a neighborhood $U(x_0)$ of x_0. Let $\tilde{\nabla}$ (resp. ∇) be the covariant differentiation on $\tilde{M}^{n+1}(c)$ (resp. M^n). Then for any vector fields X, Y tangent to M^n on $U(x_0)$, we have

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)\xi, \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -AX, \tag{2.2}$$

where g and A are the induced metric on M^n and the (0, 2)-type symmetric tensor field called the *second fundamental form*, respectively ([1], [2]).

Let R be the curvature tensor of M^n. Then, for any vector fields X, Y and Z on $U(x_0)$, we have the following:

$$R(X, Y)Z = \tilde{R}(X, Y)Z + g(AY, Z)AX - g(AX, Z)AY, \tag{2.3}$$

–Gauss equation,

$$(\nabla_X A)Y = (\nabla_Y A)X, \tag{2.4}$$

–Codazzi equation,

where $\tilde{R}$ is the curvature tensor of $\tilde{M}^{n+1}(c)$. Since $\tilde{M}^{n+1}(c)$ is of constant curvature c, $\tilde{R}(X, Y)Z$ can be written as

$$\tilde{R}(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y\}, \tag{2.5}$$

In particular, if the secnod funamental tensor A satisfies

$$(\nabla_X A)Y = 0 \tag{2.6}$$

on a neighborhood of every point in M^n, then we say that *parallel second fundamental tensor.*

Next, we denote the (1, 1)-type Ricci tensor of M^n by S. For any point x of $U(x_0)$, S is defined by

$$SX = \sum_{i=1}^{n} R(X, e_i)e_i, \tag{2.7}$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of the tangent space T_xM^n. Using the Gauss equation (2.3) and the equation (2.5) , we obtain

$$SX = (n-1)cX + (\mathrm{trace}A)AX - A^2X \tag{2.8}$$

for any X tangent to M^n on $U(x_0)$.

Moreover, we prepare the following lemmas to prove the Theorems ([4]). We shall assume that M^n is oriented (so that a unit normal vector field ξ is defined on the whole M^n). Suppose that M^n has $k(x)$-distinct eigenvalues $\lambda_i(x), 1 \leq i \leq k(x)$ at x. Then we may assume that M^n has $k(x)$-distinct eigenvalues on a neighborhood U of x. Hence we may speak of the differentiable function $\lambda_i(x)(1 \leq i \leq k(x))$ which assign to each $x \in M^n$ eigenvalues of A at x. Thus, at each $x \in M^n$, $\lambda_i(x)$'s are an eigenvalues of A with multiplicities i_ℓ. We define distributions on M^n as follows:

$$T_{\lambda_i}(x) = \{X \in T_x(M); AX = \lambda_i(x)X\}.$$

Lemma 1. *T_{λ_i}'s are differentiable.*

Proof. Let $\{X_1, \ldots, X_{i_\ell}\}$ be a basis of $T_{\lambda_i}(x)$. We extend $\{X_1, \ldots, X_{i_\ell}\}$ to vector fields on U and define vector fields

$$Y_t = (A - \lambda_1 I)\cdots(A - \overset{\vee}{\lambda_i} I)\cdots(A - \lambda_{k(x)}I)X_t \qquad \text{for} \quad 1 \leq t \leq i_\ell, \tag{2.9}$$

where I denotes the identity transformation and $\vee$ means to neglect $A - \lambda_i I$. At x, we have $Y_t = (\lambda_i - \lambda_1)\cdots(\lambda_i - \lambda_{k(x)})X_t$ for $1 \leq t \leq i_\ell$. Thus $Y_t, 1 \leq t \leq i_\ell$ are linearly independent at x and hence in a neighborhood U of x. At each point of U, we have

$$(A - \lambda_i I)Y_t = 0 \quad \text{for} \quad 1 \leq t \leq i_\ell.$$

Hence $Y_t, 1 \leq t \leq i_\ell$ forms a basis of T_{λ_i}. Therefore $T'_{\lambda_i}s, 1 \leq t \leq k(x)$ are differentiable.

Lemma 2. *T_{λ_i}'s are involutive.*

Proof. We recall the Codazzi equation

$$(\nabla_X A)(Y) = (\nabla_Y A)(X).$$

Suppose that X and Y are vector fields belonging to T_{λ_i}. Then we obtain

$$(\nabla_X A)(Y) = \nabla_X(AY) - A(\nabla_X Y) = (X\lambda_i)Y - (\lambda_i I - A)(\nabla_X Y)$$

and

$$(\nabla_Y A)(X) = \nabla_Y(AX) - A(\nabla_Y X) = (Y\lambda_i)X - (\lambda_i I - A)(\nabla_Y X).$$

Thus we get $(X\lambda_i)Y - (Y\lambda_i)X - (\lambda_i I - A)[X,Y] = 0$. Since $(X\lambda_i)Y - (Y\lambda_i)X \in T_{\lambda_i}$ and $(\lambda_i I - A)[X,Y] \notin T_{\lambda_i}$, we get $(X\lambda_i)Y - (Y\lambda_i)X = 0$ and $A[X,Y] = \lambda_i[X,Y]$. The second identity shows that $[X,Y] \in T_{\lambda_i}$, proving that T_{λ_i} is involutive. The first identity will establish the following:

Lemma 3. *If X belongs to $T_{\lambda_i}(x)$ and* $\dim T_{\lambda_i}(x) \geq 2$, *then* $X\lambda_i = 0$.

Proof. Since $\dim T_{\lambda_i}(x) \geq 2$, we may choose X and $Y \in T_{\lambda_i}(x)$ such that X and Y are linearly independent. Extending X and Y to vector fields belonging to T_{λ_i}, we have $(X\lambda_i)Y - (Y\lambda_i)X = 0$ at x. Thus $X\lambda_i = Y\lambda_i = 0$ at x.

Now, we prepare the following results without proof:

Theorem A [7]. *Let $M^n (n > 2)$ in a real space form $\tilde{M}^{n+1}(c)$. If M^n is not of constant curvature c and if $\nabla S = 0$ on M^n, then M^n is either the product manifold of two space forms or $c = 0$ and rank $A = 2$ on M^n.*

Theorem B [7]. *Let M^n be a hypersurface with constant mean curvature of the dimension $n > 2$ in a space $M^{n+1}(c)$. If the Ricci tensor of M^n is parallel, then M^n is locally symmetric and M^n is either a space form or the product manifold of two space forms.* (See Introduction)

3 Proof of Theorem 1

For each point x of $U(x_0)$, we choose an orthonormal basis $\{e_1, \ldots, e_n\}$ of $T_x M^n$ which A is diagonalized.

Since M^n has the harmonic curvature, it follows that

$$(\nabla_X S)Y = (\nabla_Y S)X$$

for any X, Y tangent to M^n on $U(x_0)$. From the equation (2.8) , we have

$$S = (n-1)cI + (\mathrm{trace}A)A - A^2.$$

Thus we obtain

$$\begin{aligned} (3.1) \qquad & (X\mathrm{trace}A)AY + (\mathrm{trace}A)(\nabla_X A)Y - (\nabla_X A)AY - A(\nabla_X A)Y \\ = \quad & (Y\mathrm{trace}A)AX + (\mathrm{trace}A)(\nabla_Y A)X - (\nabla_Y A)AX - A(\nabla_Y A)X. \end{aligned}$$

The following Codazzi equation (2.4) will be useful.

$$(\nabla_X A)Y = (\nabla_Y A)X.$$

Combining (3.1) with (2.4) , we get

$$(3.2) \qquad (\nabla_X A)AY - (\nabla_Y A)AX = (X\mathrm{trace}A)AY - (Y\mathrm{trace}A)AX.$$

We consider the distribution on $U(x_0)$ defined by

$$T_\lambda(x) = \{X \in T_x M^n | AX = \lambda X\}.$$

From the assumption of Theorem 1 we may assume that rank$A \geq 2$. At first, suppose that rank$A > 2$. Let $X \in T_\lambda, Y \in T_\mu, \lambda \neq \mu$ and $\mu \neq 0$. From (3.2) , we have

$$(\mu - \lambda)(\nabla_X A)Y = (\mu X \mathrm{trace} A)Y - (\lambda Y \mathrm{trace} A)X.$$

Let x be an arbitrary point of M^n. We may choose the tangent vector Y on a neighborhood U_x of x such that $\nabla_Y Y = 0$ at x. Hence we obtain

$$\mu X \mathrm{trace} A = 0,$$

since $(\nabla_Y A)X = (Y\lambda)X + (\lambda I - A)\nabla_Y X$ and Y are orthogonal. Assume that dim$T_\mu \geq 2$. From (2.4) and (3.2) we get

$$0 = (\mu Y_1 \mathrm{trace} A)Y_2 - (\mu Y_2 \mathrm{trace} A)Y_1$$

for $Y_1, Y_2 \in T_\mu$. Since dim$T_\mu \geq 2$, we obtain

$$\mu Y \mathrm{trace} A = 0.$$

Thus we have $(\nabla_X A)Y = 0$. When μ is simple, we choose $Z \in T_\nu, \nu \neq \mu, \nu \neq \lambda$ and $\nu \neq 0$ and we may choose the tangent vector Z on a neighborhood U_x of x such that $\nabla_Z Z = 0$ at x. Hence by the equation (3.2) we get

$$(\nu - \mu)(\nabla_Y A)Z = (\nu Y \mathrm{trace} A)Z - (\mu Z \mathrm{trace} A)Y.$$

Thus we obtain

$$\nu Y \mathrm{trace} A = 0.$$

Therefore we have $(\nabla_X A)Y = 0$. Next, if we assume that rank$A = 2$, then it holds $R(X,Y)\cdot R = 0$ holds. Then we see that in the case of $c \neq 0$ rank$A \leq 1$ or rank$A = n$ (See [6]). Hence it remains the case $c = 0$ and rank$A = 2$.

Corollary *Let M^n be a hypersurface of dimension $n > 2$ with constant mean curvature in a space form $M^{n+1}(c)$. If M^n has the harmonic curvature, then M^n has the parallel second fundamental form.*

Proof. From the proof of Theorem 1, we see that

$$(\nabla_X A)Y = 0$$

for $X \in T_\lambda, Y \in T_\mu, \lambda \neq \mu$. Then we have

$$(\nabla_X A)Y = (X\mu)Y + (\lambda I - A)\nabla_X Y.$$

Hence we get $X\mu = 0$ for any $\mu \neq \lambda$. Since the mean curvature is constant, we obtain $X\lambda = 0$. Therefore

$$(\nabla_{X_1} A)X_2 = (\lambda I - A)\nabla_{X_1} X_2$$

for $X_1, X_2 \in T_\lambda$. Hence we see that M^n has the parallel second fundamenta form.

4 Proof of Theorem 2

For each point x of $U(x_0)$, we choose an orthonormal basis $\{e_1, \ldots, e_n$ of T_xM^n which A is diagonalized.

Since M^n has harmonic conformal curvature, it follows that

$$(\nabla_X S)Y - (\nabla_Y S)X = \frac{1}{2(n-1)}\{(\nabla_X s)Y - (\nabla_Y S)X\}$$

for any X, Y tangent to M^n on $U(x_0)$. From the equation (2.8) we obtain

$$S = (n-1)cI + (\mathrm{trace}A)A - A^2$$

as above. Since $s = \mathrm{trace}S$, we have

$$s = n(n-1)c + (\mathrm{trace}A)^2 - \mathrm{trace}A^2.$$

Hence we get

$$(\nabla_X s)Y = X((\mathrm{trace}A)^2 - \mathrm{trace}A^2))Y$$

from the assumption of Theorem 2. Therefore we have

$$\begin{aligned} -(\nabla_X A)AY + (\nabla_Y A)AX \quad &= \quad -(X\mathrm{trace}A)AY + (Y\mathrm{trace}A)AX \\ &\quad + \frac{1}{2(n-1)}\{(X((\mathrm{trace}A)^2 - \mathrm{trace}A^2))Y \\ &\quad -(Y((\mathrm{traceA})^2 - \mathrm{trace}A^2))X\}. \end{aligned} \tag{4.1}$$

Let $X \in T_\lambda, Y \in T_\mu, Z \in T_\nu$ and $\lambda \neq \mu, \mu \neq \nu, \nu \neq \lambda$. From (4.1) we have

$$\begin{aligned} -(\mu - \lambda)(\nabla_X A)Y \quad &= \quad -(\mu X\mathrm{trace}A)Y + (\lambda Y\mathrm{trace}A)X \\ &\quad + \frac{1}{2(n-1)}\{(X((\mathrm{trace}A)^2 - \mathrm{trace}A^2))Y \\ &\quad -(Y((\mathrm{traceA})^2 - \mathrm{trace}A^2))X\}. \end{aligned}$$

Therefore we see that

$$(\nabla_X A)Y \in T_\lambda \oplus T_\mu. \tag{4.2}$$

Let x be an arbitrary point of M^n. We may choose the tangent vector Y (resp. Z) on a neighborhood U_x of x such that $\nabla_Y Y = 0$ (resp. $\nabla_Z Z = 0$) at x. Then we have

$$(\nabla_Y A)X = (Y\lambda)X + (\lambda I - A)\nabla_Y X = (Y\lambda)X. \tag{4.3}$$

Thus we see that

$$(\nabla_X A)Y \in T_\lambda. \tag{4.4}$$

On the other hand, it holds

$$(\nabla_X A)Y = (X\mu)Y + (\mu I - A)\nabla_X Y.$$

Since $Y \in T_\mu$ and $(\mu I - A)\nabla_X Y \notin T_\mu$, we get $X\mu = 0$ for $\lambda \neq \mu$. Hence $X\lambda = 0$, since $X(\text{trace}A) = 0$. Thus we have

$$(\nabla_{X_1} A)X_2 = (\lambda I - A)\nabla_{X_1} X_2. \tag{4.5}$$

Combining (4.1) with (4.4) , we get

$$X((\text{trace}A)^2 - \text{trace}A^2)) = 0.$$

Therefore we have

$$-(\mu - \lambda)(\nabla_X A)Y = (\lambda Y \text{trace}A)X + \frac{1}{2(n-1)}\{(Y((\text{trace}A)^2 - \text{trace}A^2))X\}. \tag{4.6}$$

On the other hand, for $Z \in T_\nu$ we get

$$\begin{aligned} -(\nu - \mu)(\nabla_Y A)Z &= -(\nu Y \text{trace}A)Z + (\mu Z \text{trace}A)Y \\ &+ \frac{1}{2(n-1)}\{(Y((\text{trace}A)^2 - \text{trace}A^2))Z \\ &-(Z((\text{traceA})^2 - \text{trace}A^2))Y\}. \end{aligned}$$

Since we have $(\nabla_Y A)Y = (Y\mu)Y$ and $(\nabla_Z A)Z = (Z\nu)Z$, we obtain

$$g((\nabla_Y A)Z, Y) = g((\nabla_Y A)Y, Z) = 0.$$

$$g((\nabla_Y A)Z, Z) = g((\nabla_Z A)Z, Y) = 0.$$

Hence we have

$$-(\nu Y \text{trace}A) + \frac{1}{2(n-1)}(Y((\text{trace}A)^2 - \text{trace}A^2)) = 0.$$

Combining this equation with constant mean curvature, we obtain

$$-(\lambda Y \text{trace}A) + \frac{1}{2(n-1)}(Y((\text{trace}A)^2 - \text{trace}A^2)) = 0.$$

Hence we get $(\nabla_X A)Y = 0$ for any X, Y.
This completes the proof of the theorem.

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