

# On discrete geometry of non-holonomic spaces

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**Abstract.** The goal of the paper is to present a discrete version of the geometry of nonholonomic spaces, using nonlinear constraints corresponding to a bundle or a foliation. A concrete numeric examples is considered by comparing the coordinates computed according to the method proposed in the paper, that are very closed to the accurate ones.

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## 1 Introduction

The problem to express the equations of motion of a nonholonomic system, in presence of nonlinear constraints, is studied in a recent paper [11]. A discrete analogue of this problem is presented in this paper. The discrete approach is based on Veselov principle, used to discretize the Euler-Lagrange equation (see, for example [9]). We use variations in discrete time, giving rise to a discrete analogue of the Euler-Lagrange equation. A new discretization method of nonholonomic constraints is given. We consider an analogue of Lagrange-d'Alembert-Chaplygin equations (2.2), as their discrete version (DLACE) by formula (3.2). The main result is Theorem 3.2, where DLACE equations (3.2) are giving in a form involving the Lagrange-d'Alembert form of the induced Lagrangian. This result is a discrete analogue of the Lagrange-d'Alembert-Chaplygin equations in [11, Proposition 4.1]. An example considered in [11] is given in the discrete case, comparing the coordinates computed according to the method proposed in the paper, that very closed to the accurate ones.

## 2 Differentiable and discrete cases

Let  $L : TM \rightarrow \mathbb{R}$  be a Lagrangian on the total space of a fibered manifold  $\pi : M \rightarrow \bar{M}$ . A *nonlinear constraint* is given by a left splitting  $\mathcal{C}$  of  $I'_0$ , or, equivalently, by a right

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splitting  $\mathcal{D}$  of the projection  $\Pi'_0$  in the exact sequence

$$(2.1) \quad 0 \rightarrow \pi_{NM}^* VM \xrightarrow{I''_0} \pi_{NM}^* TM \xrightarrow{\Pi''_0} \pi_{NM}^* NM \rightarrow 0,$$

where  $NM = \pi^* T\bar{M}$ ,  $VM = \ker \pi_*$  (vertical bundle).

As in the case of linear or affine constraints (as in [1, Sect. 5.2]), the equations of motions governed by a Lagrangian and some nonlinear constraints, can be deduced imposing the following principle: first one apply the variation, then the projection of the Lagrange equations according to the constraints. It is a d'Alembert's type principle, that we call *Lagrange-d'Alembert-Chaplygin*:

$$(2.2) \quad \left( \frac{d}{dt} \frac{\partial L}{\partial y^{\bar{u}}} - \frac{\partial L}{\partial x^{\bar{u}}} \right) - C^u_{\bar{u}} \left( \frac{d}{dt} \frac{\partial L}{\partial y^u} - \frac{\partial L}{\partial x^u} \right) = 0,$$

where  $C^u_{\bar{u}} = \frac{\partial C^u}{\partial y^{\bar{u}}}$ . A nonlinear constraint  $C \in \Gamma(\pi_{NM}^* TM)$  can be viewed as a map  $C : NM \rightarrow TM$ , thus any Lagrangian  $L : TM \rightarrow \mathbb{R}$  induces by composition  $NM \xrightarrow{C} TM \xrightarrow{L} \mathbb{R}$  a new Lagrangian  $L_c = L \circ C$  on  $N\mathcal{F}$ , called the *constrained Lagrangian*:

$$(2.3) \quad L_c(x^u, x^{\bar{u}}, y^{\bar{u}}) = L(x^u, x^{\bar{u}}, C^u, y^{\bar{u}}).$$

Notice that along the integral curves we have

$$(2.4) \quad \frac{dx^u}{dt} = C^u(x^u, x^{\bar{u}}, y^{\bar{u}}).$$

A form of Lagrange-d'Alembert-Chaplygin equations is given in [11, Proposition 4.1]. In this paper we obtain some discrete analogues of these equations.

We give below a short presentation of the discretization method, using discrete variations that give the Euler-Lagrange equations. We briefly give the principal ideas of Veselov's discretization, studied in many papers (see, for example [9]).

A *discrete Lagrangian* (of first order) on a differentiable manifold  $M$  is a differentiable map  $L : M^2 \rightarrow \mathbb{R}$ ,  $(z_0, z_1) \rightarrow L(z_0, z_1)$ . The corresponding *action* is  $S : M^n \rightarrow \mathbb{R}$ ,  $S(x_0, \dots, x_n) = \sum_{i=0}^{n-1} L(x_i, x_{i+1})$ . Thus,  $TM$  is replaced, in the discrete case, by  $M \times M$ . Therefore vectors are locally replaced by points and the image of the null section of  $TM \rightarrow M$  is replaced by the diagonal inclusion  $M \subset M \times M$ ,  $x \rightarrow (x, x)$ .

By considering a variation as in [9, Subsect. 1.1.2.]:

$$(2.5) \quad \begin{aligned} dS(x_0, \dots, x_n) \cdot (\delta x_0, \dots, \delta x_n) &= \\ &= \sum_{i=0}^{n-1} \left( \frac{\partial L_d}{\partial z_0^\alpha}(x_i, x_{i+1}) \delta x_i^\alpha + \frac{\partial L_d}{\partial z_1^\alpha}(x_i, x_{i+1}) \delta x_{i+1}^\alpha \right) \\ &= \sum_{i=0}^{n-1} \frac{\partial L_d}{\partial z_0^\alpha}(x_i, x_{i+1}) \delta x_i^\alpha + \sum_{i=1}^n \frac{\partial L_d}{\partial z_1^\alpha}(x_{i-1}, x_i) \delta x_i^\alpha = \frac{\partial L_d}{\partial z_0^\alpha}(x_0, x_1) \delta x_0^\alpha \\ &\quad + \sum_{i=0}^{n-1} \left( \frac{\partial L_d}{\partial z_0^\alpha}(x_i, x_{i+1}) + \frac{\partial L_d}{\partial z_1^\alpha}(x_{i-1}, x_i) \right) \delta x_i^\alpha + \frac{\partial L}{\partial z_1^\alpha}(x_{n-1}, x_n) \delta x_n^\alpha, \end{aligned}$$

taken such that the points  $\{x_0, x_1, x_{n-1}, x_n\}$  are fixed, thus  $\delta x_\alpha = 0$ , for each  $\alpha \in \{0, n\}$ . Thus a condition that the variation vanishes is

$$(2.6) \quad \mathcal{E}\mathcal{L}_{i,\alpha} = \frac{\partial L_d}{\partial z_0^\alpha}(x_i, x_{i+1}) + \frac{\partial L_d}{\partial z_1^\alpha}(x_{i-1}, x_i) = 0,$$

$i = \overline{1, n-1}, \alpha = \overline{1, m}$ , called the *discrete Euler-Lagrange* equation (DEL equation for short). One say that  $L_d$  is *regular* if the matrix  $\left(\frac{\partial^2 L_d}{\partial z_0 \partial z_1}(z_0, z_1)\right)$  is non-degenerate for every  $(z_0, z_1) \in M^2$ . This condition enables to consider the discrete Euler-Lagrange equation and to solve it locally with respect to  $x_{i-1}$  or to  $x_{i+1}$  provided that it has a solution in a given point (using the implicit function theorem). We say that  $L_d$  is *left (right) regular* if the equation  $\frac{\partial L_d}{\partial z_1}(z', z_1)dz_0 = pdz_0$  (respectively  $\frac{\partial L_d}{\partial z_0}(z_0, z'')dz_0 = pdz_0$ ) can be solved with respect to  $z'$  and  $z''$ , for every  $z_0, z_1 \in M$  and  $pdz_0 \in \mathcal{X}^*(M)$ ; if it is the case of regularity, then  $z' = \varphi(p, z_1, z_2)$  (respectively  $z'' = \varphi(z_1, z_2, p)$ ). It is easy to see that if  $L_d$  is regular, then it is left and right regular. We say that  $L_d$  is *hyperregular* if the solutions  $z'$  and  $z''$  are both unique.

### 3 A new discretization method of nonholonomic constraints

The new fact in our approach is that the constraints are involved directly from the differentiable case in the Euler-Lagrange formulas.

Let  $M$  be a fibered manifold  $\pi : M \rightarrow \bar{M}$ . Consider now the equivalence relation on  $M$ :  $x \sim_N y$  iff  $\pi(x) = \pi(y)$  and the quotient manifold  $NM = (M \times M) / \sim_N$ . Consider now some local coordinates  $(x^{\bar{u}})$  on  $\bar{M}$ ,  $(x^\alpha) = (x^u, x^{\bar{u}})$  on  $M$ , adapted to the fibered manifold structure. A *constraint* is a map

$$C : NM \rightarrow M, \quad (x^u, x^{\bar{u}}, y^{\bar{u}}) \xrightarrow{C} (C^u(x^u, x^{\bar{u}}, y^{\bar{u}})).$$

We can suppose that  $M \subset \mathbb{R}^k$  (i.e. a vector space), for example using local coordinates, then gluing the resulting curves. The differential constraints can be taken  $\tilde{C}^{\bar{u}}(x^u, x^{\bar{u}}, y^{\bar{u}}) = \frac{C^u(x^u, x^{\bar{u}}, y^{\bar{u}}) - x^u}{h}$ , since the manifold  $M$  is embedded in a vector space. If it is the case and  $\tilde{C} : N\mathcal{F} \rightarrow \pi_{N\mathcal{F}}^* TM$  are some differentiable constraints, then we can take  $\tilde{C}(x^u, x^{\bar{u}}, y^{\bar{u}}) = \tilde{C}^u(x^u, x^{\bar{u}}, \frac{y^{\bar{u}} - x^{\bar{u}}}{h})$ , thus the effective constraints reads

$$(3.1) \quad C^u(x^u, x^{\bar{u}}, y^{\bar{u}}) = x^u + h\tilde{C}^u(x^u, x^{\bar{u}}, \frac{y^{\bar{u}} - x^{\bar{u}}}{h}).$$

Denoting  $C_{\bar{u}}^u = \frac{\partial C^u}{\partial y^{\bar{u}}}$ , then considering the virtual displacements induced by constraints and the discrete analogous of equation (2.4) as  $x_{i+1}^u = C^u(x^u, x^{\bar{u}}, y^{\bar{u}})$ , it follows that

$$(3.2) \quad \mathcal{E}\mathcal{L}_{i,\bar{u}} - C_{\bar{u}}^u \cdot \mathcal{E}\mathcal{L}_{i,u} = 0.$$

We call (3.2) the *discrete Lagrange-d'Alembert-Chaplygin equations* (DLACE), where the induced objects look as follows:

$$\begin{aligned}
\mathcal{E}\mathcal{L}_{i,\bar{u}} &= \frac{\partial L_d}{\partial z_0^{\bar{u}}}(x_i^v, x_i^{\bar{v}}, C^v(x_i^v, x_i^{\bar{v}}, x_{i+1}^{\bar{v}}), x_{i+1}^{\bar{v}}) + \\
&\quad \frac{\partial L_d}{\partial z_1^{\bar{u}}}(x_{i-1}^v, x_{i-1}^{\bar{v}}, C^v(x_{i-1}^v, x_{i-1}^{\bar{v}}, x_i^{\bar{v}}), x_i^{\bar{v}}), \\
(3.3) \quad \mathcal{E}\mathcal{L}_{i,u} &= \frac{\partial L_d}{\partial z_0^u}(x_i^v, x_i^{\bar{v}}, C^v(x_i^v, x_i^{\bar{v}}, x_{i+1}^{\bar{v}}), x_{i+1}^{\bar{v}}) + \\
&\quad \frac{\partial L_d}{\partial z_1^u}(x_{i-1}^v, x_{i-1}^{\bar{v}}, C^v(x_{i-1}^v, x_{i-1}^{\bar{v}}, x_i^{\bar{v}}), x_i^{\bar{v}}), \\
C_{\bar{u}}^u &= \frac{\partial C^u}{\partial x_1^{\bar{u}}}(x_{i-1}^v, x_{i-1}^{\bar{v}}, x_i^{\bar{v}}).
\end{aligned}$$

One can consider two DLACE's; the second one can be  $C_{\bar{u}}^u = \frac{\partial C^u}{\partial x_1^{\bar{u}}}(x_i^v, x_i^{\bar{v}}, x_{i+1}^{\bar{v}})$ , but in the sequel we take  $C_{\bar{u}}^u$  as above, in (3.3).

**Proposition 3.1.** For  $C_{\bar{u}}^u = \frac{\partial C^u}{\partial x_1^{\bar{u}}}(x_{i-1}^v, x_{i-1}^{\bar{v}}, x_i^{\bar{v}})$ , we have

$$\begin{aligned}
\frac{\partial}{\partial x_{i+1}^{\bar{v}}}(\mathcal{E}\mathcal{L}_{i,\bar{u}} - C_{\bar{u}}^u \cdot \mathcal{E}\mathcal{L}_{i,u}) &= \left( \frac{\partial C^v}{\partial z_1^{\bar{v}}}(x_i^v, x_i^{\bar{v}}, x_{i+1}^{\bar{v}}) \quad \delta_{\bar{v}}^{\bar{v}} \right) \cdot \\
&\cdot \left( \begin{array}{cc} \frac{\partial^2 L_d}{\partial z_1^v \partial z_0^u} & \frac{\partial^2 L_d}{\partial z_1^v \partial z_0^{\bar{v}}} \\ \frac{\partial^2 L_d}{\partial z_1^{\bar{v}} \partial z_0^u} & \frac{\partial^2 L_d}{\partial z_1^{\bar{v}} \partial z_0^{\bar{v}}} \end{array} \right) (x_i^v, x_i^{\bar{v}}, C^v(x_i^v, x_i^{\bar{v}}, x_{i+1}^{\bar{v}}), x_{i+1}^{\bar{v}}) \cdot \left( \begin{array}{c} -C_{\bar{u}}^u \\ \delta_{\bar{u}}^{\bar{u}} \end{array} \right).
\end{aligned}$$

Let us consider the induced Lagrangian

$$LL_d(x_0^u, x_0^{\bar{u}}, x_1^{\bar{u}}) = L_d(x_0^u, x_0^{\bar{u}}, C^u(x_0^u, x_0^{\bar{u}}, x_1^{\bar{u}}), x_1^{\bar{u}})$$

and its Euler-Lagrange forms

$$\begin{aligned}
\mathcal{E}\mathcal{L}\mathcal{L}_{i,\bar{u}} &= \frac{\partial LL_d}{\partial z_0^{\bar{u}}}(x_i^v, x_i^{\bar{v}}, x_{i+1}^{\bar{v}}) + \frac{\partial LL_d}{\partial z_1^{\bar{u}}}(x_{i-1}^v, x_{i-1}^{\bar{v}}, x_i^{\bar{v}}), \\
\mathcal{E}\mathcal{L}\mathcal{L}_{i,u} &= \frac{\partial LL_d}{\partial z_0^u}(x_i^v, x_i^{\bar{v}}, x_{i+1}^{\bar{v}}), \\
C_{\bar{u}}^u &= \frac{\partial C^u}{\partial x_1^{\bar{u}}}(x_i^v, x_i^{\bar{v}}, x_{i+1}^{\bar{v}}) \quad (\text{or} : (x_{i-1}^v, x_{i-1}^{\bar{v}}, x_i^{\bar{v}})).
\end{aligned}$$

Its discrete Lagrange-d'Alembert form is

$$E\mathcal{L}\mathcal{L}_{i,\bar{u}} = \mathcal{E}\mathcal{L}\mathcal{L}_{i,\bar{u}} - \tilde{C}_{\bar{u}}^u \cdot \mathcal{E}\mathcal{L}\mathcal{L}_{i,u}.$$

The following result is a discrete analogous of the Lagrange-d'Alembert-Chaplygin equations, as in [11, Proposition 4.1].

**Theorem 3.2.** *The discrete Lagrange-d'Alembert-Chaplygin equations (3.2) have the following form, involving the Lagrange-d'Alembert form  $E\mathcal{L}\mathcal{L}$  of the induced Lagrangian  $LL_d$ :*

$$\begin{aligned} E\mathcal{L}\mathcal{L}_{i,\bar{u}} &= \mathcal{E}\mathcal{L}\mathcal{L}_{i,\bar{u}} - \tilde{C}_{\bar{u}}^u \cdot \mathcal{E}\mathcal{L}\mathcal{L}_{i,u} = \\ &= \mathcal{E}\mathcal{L}_{i,\bar{u}} - \tilde{C}_{\bar{u}}^u \cdot \mathcal{E}\mathcal{L}_{i,u} + \frac{\partial L_d}{\partial z_1^u}(x_i^u, x_i^{\bar{u}}, C^u(x_i^u, x_i^{\bar{u}}, x_{i+1}^{\bar{u}}, x_{i+1}^{\bar{u}})) \cdot \\ &\quad \cdot \left( \frac{\partial C^u}{\partial z_0^{\bar{u}}}(x_i^u, x_i^{\bar{u}}, x_{i+1}^{\bar{u}}) - \frac{\partial C^v}{\partial z_1^{\bar{u}}}(x_i^u, x_i^{\bar{u}}, x_{i+1}^{\bar{u}}) \frac{\partial C^u}{\partial z_0^v}(x_i^u, x_i^{\bar{u}}, x_{i+1}^{\bar{u}}) \right) + \\ &\quad + \frac{\partial L_d}{\partial z_1^u}(x_{i-1}^u, x_{i-1}^{\bar{u}}, C^u(x_{i-1}^u, x_{i-1}^{\bar{u}}, x_i^{\bar{u}}, x_i^{\bar{u}})) \cdot \frac{\partial C^u}{\partial z_1^{\bar{u}}}(x_{i-1}^u, x_{i-1}^{\bar{u}}, x_i^{\bar{u}}). \end{aligned}$$

We say that the constraint  $C$  is *linear* if  $\frac{\partial^2 C^u}{\partial z_1^{\bar{u}} \partial z_1^{\bar{v}}} = 0$ , i.e.  $\frac{\partial C^u}{\partial z_1^{\bar{u}}}(z_0^u, z_0^{\bar{u}}, z_1^{\bar{u}}) = \frac{\partial C^u}{\partial z_1^{\bar{u}}}(z_0^u, z_0^{\bar{u}})$  does not depend on  $z_1^{\bar{u}}$ . For example, if the constraint  $C$  comes from a linear differentiable constraint  $\bar{C}$  according to formula (3.1), then  $C$  is linear or affine, since  $C^u(z_0^u, z_0^{\bar{u}}, z_1^{\bar{u}}) = z_0^u + b(z_0^u, z_0^{\bar{u}}) + C_{\bar{u}}^u(z_0^u, z_0^{\bar{u}})(z_1^{\bar{u}} - z_0^{\bar{u}})$ , provided that  $\bar{C}(z_0^u, z_0^{\bar{u}}, w_1^{\bar{u}}) = w_1^{\bar{u}} C_{\bar{u}}^u(z_0^u, z_0^{\bar{u}}) + b(z_0^u, z_0^{\bar{u}})$  is affine or linear (when  $b = 0$ ).

**Example.** We consider the Lagrangian

$$L : T\mathbb{R}^3 = \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}, \quad L(x^i, y^i) = \frac{1}{2} \left( (y^{\bar{1}})^2 + (y^{\bar{2}})^2 + (y^1)^2 \right) + f(x^i),$$

the fibered manifold  $pr_3 : M = \mathbb{R}^3 \rightarrow \mathbb{R}$ ,  $pr_3(x^{\bar{1}}, x^{\bar{2}}, x^1) = x^1$  and the constraint  $y^{\bar{1}}x^{\bar{2}} - y^1 = 0$ ; more specifically,  $C^1(x^1, x^{\bar{1}}, x^{\bar{2}}, y^{\bar{1}}, y^{\bar{2}}) = y^1 = y^{\bar{1}}x^{\bar{2}}$ . We have

$$\frac{\partial C^1}{\partial y^{\bar{1}}} = x^{\bar{2}}, \quad \frac{\partial C^1}{\partial y^{\bar{2}}} = 0 \quad \text{and} \quad \frac{d}{dt}C^1 = \frac{dy^{\bar{1}}}{dt}x^{\bar{2}} + y^{\bar{1}}y^{\bar{2}}.$$

The constrained EL equations are  $E_{\bar{u}} - C_{\bar{u}}^u E_u = 0$ , where  $E_i = \frac{d}{dt} \frac{\partial L}{\partial y^i} - \frac{\partial L}{\partial x^i}$ :

$$\begin{aligned} \frac{dy^{\bar{1}}}{dt} - \frac{\partial f}{\partial x^{\bar{1}}} - x^{\bar{2}} \left( \frac{dy^{\bar{1}}}{dt}x^{\bar{2}} + y^{\bar{1}}y^{\bar{2}} - \frac{\partial f}{\partial x^{\bar{1}}} \right) &= 0, \\ \frac{dy^{\bar{2}}}{dt} - \frac{\partial f}{\partial x^{\bar{2}}} - 0 \cdot \left( \frac{dy^{\bar{1}}}{dt}x^{\bar{2}} + y^{\bar{1}}y^{\bar{2}} - \frac{\partial f}{\partial x^{\bar{2}}} \right) &= 0. \end{aligned}$$

Take  $f = -\left((x^{\bar{1}})^2 + (x^{\bar{2}})^2\right)$ ; then  $\frac{dy^{\bar{2}}}{dt} + 2x^{\bar{2}} = 0$ , or  $\frac{d^2x^{\bar{2}}}{dt^2} + 2x^{\bar{2}} = 0$ , thus  $x^{\bar{2}}(t) = C_1 \cos(t\sqrt{2}) + C_2 \sin(t\sqrt{2})$ . In the discrete case, we consider the curve that starts from  $(0, 0, 0)$  and has the initial velocities  $\frac{dx^{\bar{1}}}{dt}(0) = \frac{dx^{\bar{2}}}{dt}(0) = 0$ ; the curve contains the point  $(0.00097, 0.001999, -0.00002)$ , for  $h = 0.001$ . In the next table, the values obtained using our formulas are  $(x_d, y_d, z_d)$  and the exact values are  $(x_e, y_e, z_e)$ , using  $h = 0.001$ . Below, 51 steps are performed.

h	$x_d$	$y_d$	$z_d$	$x_e$	$y_e$	$z_e$
0.	0.	0.	0.	0.	0.	0.
.001	0.000050	0.010000	0.010000	0.000050	0.010000	0.010000
.002	0.000150	0.019998	0.019997	0.000200	0.019999	0.019997
.003	0.000350	0.029992	0.029991	0.000450	0.029996	0.029991
.004	0.000649	0.039983	0.039979	0.000799	0.039989	0.039979
.005	0.001049	0.049969	0.049958	0.001249	0.049979	0.049958
...	...	...	...	...	...	...
h	$x_d$	$y_d$	$z_d$	$x_e$	$y_e$	$z_e$
...	...	...	...	...	...	...
0.046	0.093560	0.440241	0.428234	0.095813	0.441111	0.428234
0.047	0.097235	0.448823	0.436148	0.099535	0.449723	0.436149
0.048	0.100945	0.457330	0.443976	0.103292	0.458261	0.443976
0.049	0.104688	0.465760	0.451714	0.107081	0.466721	0.451715
0.050	0.108460	0.474110	0.459362	0.110898	0.475100	0.459363
0.051	0.112257	0.482376	0.466918	0.114740	0.483396	0.466919

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