Eigenvalues and eigenvectors of Laplacian on a parallelepiped

I. Ţevy, C. Udrişte, F. Munteanu, D. Zugrăvescu, L. Petrescu

Abstract. The purpose of this paper is threefold: (i) to prove new properties of eigenvalues and eigenvectors of the Laplacian on a parallelepiped; (ii) to interchange the parameters with the variables, in order to find qualitative properties of Laplacian eigenvalues and eigenvectors; (iii) to show that time waves are born on a bed of standing waves.

M.S.C. 2010: 35J91, 35L05.
Key words: Laplacian; eigenvalues; eigenvectors; time-independent waves.

1 Eigenvalues and eigenvectors of the Laplacian on a parallelepiped

In functional analysis one defines the *spectrum* of a linear operator T as the set of all scalars λ for which the operator $T - \lambda I$ has no bounded inverse. Thus the spectrum of an operator always contains all its eigenvalues, but is not limited to them. When the eigenvectors are functions, the corresponding eigenvalues are interpreted as energies of eigenfunctions.

In R^2, the Laplacian (Dirichlet) eigenvalues are the fundamental modes of vibration of an idealized drum with a given shape. The problem of whether one can hear the shape of a drum is: given the Laplacian eigenvalues, what features of the shape of the drum can one deduce. Here a "drum" is thought of as an elastic membrane, which is represented as a planar domain whose boundary is fixed.

Let us explain what we understand by Laplacian eigenvalues and eigenvectors in a parallelepiped ([6]). In other words, we analyze the PDE

$$\Delta u(x, y, z) + \lambda u(x, y, z) = 0$$

on the parallelepiped

$$\Omega : 0 \leq x \leq a_1,\ 0 \leq y \leq a_2,\ 0 \leq z \leq a_3.$$

BSG Proceedings 22. The International Conference "Differential Geometry - Dynamical Systems" DGDS-2014, September 1-4, 2014, Mangalia-Romania, pp. 75-83.

To understand this problem, we remark that for each natural number n, the function
$$u_n(x) = \sin \frac{n\pi x}{a}$$
satisfies the second order ODE
$$u_n''(x) + \frac{n^2\pi^2}{a^2}\, u_n(x) = 0$$
with the boundary condition $u_n(0) = u_n(a) = 0$. It follows that the functions
$$u_{n_1n_2n_3}(x, y, z) = A \sin \frac{n_1\pi x}{a_1} \sin \frac{n_2\pi y}{a_2} \sin \frac{n_3\pi z}{a_3},\ A = \text{const}$$
satisfy the PDE
$$\Delta u(x, y, z) + \lambda u(x, y, z) = 0,\ \lambda = \pi^2 \left(\frac{n_1^2}{a_1^2} + \frac{n_2^2}{a_2^2} + \frac{n_3^2}{a_3^2} \right)$$
and vanish on the boundary of the parallelepiped Ω. In this way, the triple sequence of functions $u_{n_1n_2n_3}(x, y, z)$ are eigenvectors, and the numbers
$$\lambda_{n_1n_2n_3} = \pi^2 \left(\frac{n_1^2}{a_1^2} + \frac{n_2^2}{a_2^2} + \frac{n_3^2}{a_3^2} \right)$$
are eigenvalues for the Dirichlet problem attached to the Laplace operator on the parallelepiped Ω. The system $u_{n_1n_2n_3}(x, y, z)$ is orthogonal in $L_2(\Omega)$; it is also orthonormal if we select
$$A = 2^{\frac{3}{2}} \frac{1}{\sqrt{a_1a_2a_3}}.$$
Moreover, our system is complete in $L_2(\Omega)$.

Laplacian eigenfunctions appear as vibration modes in acoustics, as electron wave functions in quantum waveguides, as natural basis for constructing heat kernels in the theory of diffusion, etc [4].

The dependence of eigenvalues on parameters proves to be full of surprises, in the context of differential geometry and optimization problems.

Proposition 1.1. *The eigenvalue $\lambda_{n_1n_2n_3}$ as function of a_1, a_2, a_3 is a convex one.*

Proof. The first order differential is
$$d\lambda_{n_1n_2n_3} = -2\pi^2 \left(\frac{n_1^2}{a_1^3}\, da_1 + \frac{n_2^2}{a_2^3}\, da_2 + \frac{n_3^2}{a_3^3}\, da_3 \right).$$
Then the second order differential
$$d^2\lambda_{n_1n_2n_3} = 6\pi^2 \left(\frac{n_1^2}{a_1^4}\, da_1^2 + \frac{n_2^2}{a_2^4}\, da_2^2 + \frac{n_3^2}{a_3^4}\, da_3^2 \right)$$
is positive definite. □

Proposition 1.2. *The Gauss curvature K of the constant level surface*

$$M : \frac{n_1^2}{a_1^2} + \frac{n_2^2}{a_2^2} + \frac{n_3^2}{a_3^2} = 1$$

verifies the condition $\frac{K}{d^4}(vol(\Omega))^4 = -9\; n_1^2 n_2^2 n_3^2$, where d is the distance from the origin to the tangent plane to surface at a generic point.

Proof. Let us compute the curvature of the surface M. We introduce the function $g = \sum_{i=1}^3 \frac{n_i^2}{a_i^2}$ and we use the (nonvanishing) normal vector field

$$Z = \frac{1}{2}\nabla g = -\sum_{i=1}^{3} \frac{n_i^2}{a_i^3} U_i.$$

If $V = \sum_{i=1}^3 v_i U_i$ is a tangent vector field on M, then $\nabla_V Z = 3\sum_{i=1}^3 \frac{n_i^2 v_i}{a_i^4} U_i$. Similar computations for another tangent vector field W, and using the position vector field $X = \sum_{i=1}^3 a_i U_i$, yield

$$Z\cdot(\nabla_V Z \times \nabla_W Z) = \begin{vmatrix} -\frac{n_1^2}{a_1^3} & -\frac{n_2^2}{a_2^3} & -\frac{n_3^2}{a_3^3} \\ 3\frac{n_1^2 v_1}{a_1^4} & 3\frac{n_2^2 v_2}{a_2^4} & 3\frac{n_3^2 v_3}{a_3^4} \\ 3\frac{n_1^2 w_1}{a_1^4} & 3\frac{n_2^2 w_2}{a_2^4} & 3\frac{n_3^2 w_3}{a_3^4} \end{vmatrix}$$

$$= -\frac{n_1^2 n_2^2 n_3^2}{a_1^4 a_2^4 a_3^4} \begin{vmatrix} a_1 & a_2 & a_3 \\ 3v_1 & 3v_2 & 3v_3 \\ 3w_1 & 3w_2 & 3w_3 \end{vmatrix} = -9\;\frac{n_1^2 n_2^2 n_3^2}{a_1^4 a_2^4 a_3^4}\; X\cdot(V\times W).$$

Now we select V and W so that $V \times W = Z$. Then

$$X\cdot(V\times W) = X\cdot Z = -\sum \frac{n_i}{a_i^2} = -1.$$

Applying the formula

$$K = \frac{Z\cdot(\nabla_V Z \times \nabla_W Z)}{||Z||^4},$$

we find

$$K = -9\;\frac{n_1^2 n_2^2 n_3^2}{a_1^4 a_2^4 a_3^4}\;\frac{1}{||Z||^4}$$

(surface with negative Gauss curvature). On the other hand, we have

$$d = |X\cdot N| = \left| X \cdot \frac{Z}{||Z||}\right| = \frac{1}{||Z||}.$$

□

Constant proper value surfaces are really creations of nature, of the natural world of mathematical objects, which mathematicians just endeavor to discover. Let us plot a constant proper value surface (algebraic surface, sextic, Figure 1)

> with(plots);

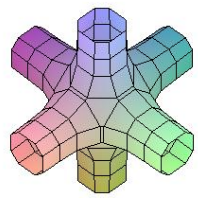

Figure 1: Constant eigenvalue surface

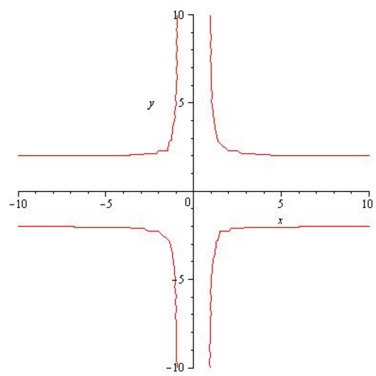

Figure 2: Constant eigenvalue curve

> implicitplot3$d(y^2z^2 + 4x^2z^2 + 9x^2y^2 = x^2y^2z^2, x = -10..10, y = -20..20,$
$z = -30..30)$

In projection on R^2, now let us plot some curves that appear in kaleidoscopic variations of nodal lines on thin vibration plates used in the construction of musical instruments (Chladni figures, Figure 2).

> with(plots);

> implicitplot3$d(y^2 + 4x^2 = x^2y^2, x = -10..10, y = -10..10)$

Theorem 1.3. *: Let n_1, n_2, n_3 be fixed integers. The minimum value of the eigenvalue*

$$\frac{1}{\pi^2}\lambda_{n_1n_2n_3} = \frac{n_1^2}{a_1^2} + \frac{n_2^2}{a_2^2} + \frac{n_3^2}{a_3^2},$$

at constant volume (Tzitzeica surface) $a_1a_2a_3 = \frac{2^3}{A^2}$, is $\frac{3}{4}(n_1n_2n_3)^{\frac{2}{3}}A^{\frac{4}{3}}$. This value is attained for the edge lengths

$$a_i = \frac{2n_i}{(n_1n_2n_3)^{\frac{1}{3}}A^{\frac{2}{3}}},\ i = 1, 2, 3.$$

Hint. AM-GM inequality.

Corollary 1.4. *: All eigenvalues* $\lambda_{kn_1;kn_2;kn_3}$, $k \in N$ *are minimized via the previous theorem.*

Consider two different parallelepipeds

$$\Omega_a : 0 \leq x \leq a_1,\ 0 \leq y \leq a_2,\ 0 \leq z \leq a_3,\ \ a = (a_1, a_2, a_3).$$

and

$$\Omega_b : 0 \leq x \leq b_1,\ 0 \leq y \leq b_2,\ 0 \leq z \leq b_3,\ \ b = (b_1, b_2, b_3).$$

Let $\lambda_{n_1n_2n_3}$ be the eigenvalues of the Laplacian on Ω_a and $\lambda'_{n_1n_2n_3}$ be the eigenvalues of the Laplacian on Ω_b.

Proposition 1.5. *(i) If* $a = (a_1, a_2, a_3) < b = (b_1, b_2, b_3)$*, then* $\lambda_{n_1n_2n_3} > \lambda'_{n_1n_2n_3}$.

(ii) If $\lambda_{n_1n_2n_3} = \lambda'_{n_1n_2n_3}, \forall (n_1, n_2, n_3) \in N^3$ *(i.e., both domains are isospectral), then* $\Omega_a = \Omega_b$.

(iii) For each triplet (n_1, n_2, n_3) *and* $\lambda_{n_1n_2n_3} \geq \lambda'_{n_1n_2n_3}$*, there exists a rotation* R *such that*

$$\left(\frac{n_1}{a_1}, \frac{n_2}{a_2}, \frac{n_3}{a_3}\right) \geq \left(\frac{n_1}{b_1}, \frac{n_2}{b_2}, \frac{n_3}{b_3}\right) R^T.$$

For a fixed point (x, y, z), the proper vector $u_{n_1n_2n_3}$ may be consider as function of (a_1, a_2, a_3).

2 Parameters are variables and variables are parameters

In the foregoing theory (x, y, z) are variables and (a_1, a_2, a_3) are parameters. Our aim is to interchange their role, in order to find qualitative properties of Laplacian eigenvalues and eigenvectors.

Proposition 2.1. *Let us consider the transformation (change of variables)*

$$\mathcal{T} : x = \frac{1}{a_1}, y = \frac{1}{a_2}, z = \frac{1}{a_3}.$$

(i) The surface $\mathcal{T}(M)$ *is an ellipsoid (Tzitzeica surface).*

(ii) In the space of C^2 *functions, the eigenvectors* $u_{n_1n_2n_3}$ *are fixed points of the transformation* $\mathcal{T}$.

(iii) The functions

$$(a_1, a_2, a_3) \to u_{n_1n_2n_3}(x, y, z) = A \sin \frac{n_1\pi x}{a_1} \sin \frac{n_2\pi y}{a_2} \sin \frac{n_3\pi z}{a_3},\ A = ct$$

are orthogonal, with weight $\frac{1}{a_1^2a_2^2a_3^2}$*, on the octant* $[x, \infty) \times [y, \infty) \times [z, \infty)$.

(iv) The transformation $\mathcal{T}$ *changes the Laplacian* $\Delta_{(x,y,z)}$ *into the Laplacian*

$$\Delta_{(a_1,a_2,a_3)} = \sum_{i=1}^{3} a_i^4 \frac{\partial^2 u}{\partial a_i^2} + 2\sum_{i=1}^{3} a_i^3 \frac{\partial u}{\partial a_i}.$$

Proof. (ii) The ratios $\frac{x}{a_1}, \frac{y}{a_2}, \frac{z}{a_3}$ are invariants of the transformation $\mathcal{T}$.

(iii) The transformation $\mathcal{T}$ changes the parallelepiped $\Omega = [0, a_1] \times [0, a_2] \times [0, a_3]$ into $[x, \infty) \times [y, \infty) \times [z, \infty)$. If J is the Jacobian matrix of the transformation $\mathcal{T}$, then $|\det J| = \frac{1}{a_1^2 a_2^2 a_3^2}$. On the other hand the correspondence $(x, y, z) \to u_{n_1 n_2 n_3}(x, y, z)$ is changed into $(a_1, a_2, a_3) \to u_{n_1 n_2 n_3}(x, y, z)$.

(iv) The Euclidean metric $\delta = I$ is changed into $g = J^t I J$, where J is the Jacobian matrix of the transformation $\mathcal{T}$. The nonzero components of the new metric are $g_{ii} = \frac{1}{a_i^4}, i = 1, 2, 3$. We find $g^{ii} = a_i^4$ and the nonzero connection components $\Gamma_{ii}^i = -\frac{2}{a_i}, i = 1, 2, 3$. On the other hand, the Laplacian is defined by

$$\Delta u = g^{ij} \left(\frac{\partial^2 u}{\partial a_i \partial a_j} - \Gamma_{ij}^k \frac{\partial u}{\partial a_k} \right).$$

□

3 Three-temporal optimal control problem; least square technique

Let us formulate a three-temporal optimal control problem with pointwise state constraints (see also [3]-[10]):

$$\min_{f(\cdot)} \frac{1}{2} \int_{\Omega_a} (u(x, y, z) - \varphi(x, y, z))^2 \, dxdydz + \frac{\alpha}{2} \int_{\Omega_a} f^2(x, y, z) \, dxdydz$$

subject to

$$\Delta u(x, y, z) + \lambda u(x, y, z) = f(x, y, z)$$

$$(x, y, z) \in \Omega_a; \;\; u(x, y, z) = 0 \;\; \text{for} \, (x, y, z) \in \partial\Omega_a.$$

Hint To solve this problem we can apply the three-time maximum principle via second order constraints, with the control f and evolution variables (x, y, z). For that, we introduce the Lagrangian

$$\mathcal{L} = -\frac{1}{2}(u(x, y, z) - \varphi(x, y, z))^2 - \frac{\alpha}{2} f^2(x, y, z)$$

$$+ p(x, y, z)(\Delta u(x, y, z) + \lambda u(x, y, z) - f(x, y, z)).$$

The adjoint PDE is

$$u(x, y, z) - \varphi(x, y, z) + \Delta p(x, y, z) = 0, \;\; p|_{\partial\Omega_a} = 0.$$

On the other hand, the critical point condition gives $p(x, y, z) = -\alpha f(x, y, z)$. Suppose $\alpha > 0$.

4 Time-independent wave PDE

Waves are achieved by imposing a special form of solutions of the system of partial differential equations (PDE) describing electromagnetic waves (see also [1], [2], [6]). One obtains a frequency dependent EDP system. This system can be solved using two methods: (i) eigenvalues eigenvectors, (ii) three-dimensional Fourier transform.

A clever idea embraced by authors is that time-waves are generated by standing waves and by the geometry of the space. In this way we confirm the hopes of some physicists which claim such theory.

Proposition 4.1. *The time waves are born on a bed of standing waves.*

Proof. **Proof via spatial solitons**

($\Longrightarrow$) **From time waves to steady-state waves**

The propagation of the electromagnetic waves in a medium free of charges and currents is described by the wave equation

$$(W) \qquad \left(\nabla^2 - \hat{\mu}\,\hat{\epsilon}\,\frac{\partial^2}{\partial t^2}\right) F(t,\vec{r}) = 0$$

on the Lorenz manifold $(R^4, g_{11} = g_{22} = g_{33} = 1, g_{44} = -\frac{1}{\hat{\mu}\,\hat{\epsilon}})$. Here, $\hat{\mu}$ is the permeability and $\hat{\epsilon}$ is the permittivity of the medium and $F(t,\vec{r})$ stands for the electromagnetic fields, $E(t,\vec{r})$ or $B(t,\vec{r})$. The relative permeability μ and the permittivity ϵ are defined by

$$\mu = \frac{\hat{\mu}}{\mu_0},\ \epsilon = \frac{\hat{\epsilon}}{\epsilon_0},$$

where $\mu_0\epsilon_0 = 1/c^2$ and c is the speed of light in free space.

The parameters μ and ϵ are constants for a linear, homogeneous, isotropic and non-dispersive medium. Generally, they are complex, space-dependent and frequency-dependent corresponding to absorbing or active, non-uniform and dispersive medium.

Suppose that the parameters μ and ϵ are only frequency-dependent. Imposing oscillatory electromagnetic fields as *spatial solitons* (SS)

$$F_\omega(t,\vec{r}) = \varphi_\omega(\vec{r})e^{-i\omega t},\ ||\varphi_\omega(\vec{r})|| \to 0, \text{when } ||\vec{r}|| \to \infty,$$

as solutions of (W), we obtain a time independent wave PDE, but a frequency dependent PDE,

$$(SS) \qquad \left(\nabla^2 + \mu(\omega)\,\epsilon(\omega)\,\frac{\omega^2}{c^2}\right) \varphi_\omega(\vec{r}) = 0.$$

It is known, and all calculations confirm this, that there is a discrepancy between the time-dependent and the frequency-dependent solutions of the wave equation in an active medium.

Adding boundary conditions, the frequency cannot be anything, being joined to the eigenvalues of the Laplacian. Indeed, we can assimilate the PDE (SS) to the PDE of eigenvalues-eigenfunctions

$$(EV) \qquad \left(\nabla^2 + \lambda\right) \varphi(\vec{r}) = 0.$$

□

Theorem 4.2. *With boundary conditions, the set of all frequencies is discrete.*

Proof. By identification, it appears the relation $\mu(\omega)\,\epsilon(\omega)\,\frac{\omega^2}{c^2} = \lambda$. On the other hand, the spectrum of the Laplace operator is known to be discrete, the eigenvalues λ_m are nonnegative and ordered in an ascending order by the index $m = 1, 2, 3, ...,$

$$(0 \leq)\,\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \nearrow \infty.$$

Let φ_m be the eigenfunction associated to the eigenvalue λ_m. The correspondence

$$\begin{array}{ccccc} & & \varphi_m & & \\ & \nearrow & & \searrow & \\ \lambda_m & & & & \varphi_{\omega_m} \\ & \searrow & & \nearrow & \\ & & \omega_m & & \end{array}$$

closed the discussion.

Particularly, for

$$\mu(\omega)\,\epsilon(\omega)\,\frac{\omega^2}{c^2} = 0,$$

i.e., $\omega = 0$, we obtain the PDE (on the Riemannian manifold $(R^3, g_{11} = g_{22} = g_{33} = 1)$)

$$(H) \qquad \nabla^2\varphi_0(\vec{r}) = 0,$$

whose solutions are called *steady-state solutions* (*harmonic functions*). These solutions represent the equilibrium wave distribution (they describes standing waves that do not move at all).

($\Longleftarrow$) **From harmonic functions to time waves (genesis of waves)**

We start from the PDE (H) and we embed the Riemannian manifold $(R^3, g_{11} = g_{22} = g_{33} = 1)$ into the Lorenz manifold $(R^4, g_{11} = g_{22} = g_{33} = 1, g_{44} = -\frac{1}{\hat{\mu}\,\hat{\epsilon}})$. The Riemannian Laplacian ∇^2 is lifted to the Lorenz Laplacian (D'Alembertian) $\nabla^2 - \hat{\mu}\,\hat{\epsilon}\,\frac{\partial^2}{\partial t^2}$. The function $\varphi_0(\vec{r})$ on R^3 is extended to a function $\varphi_\omega(\vec{r})$, then to $F_\omega(t, \vec{r}) = \varphi_\omega(\vec{r})e^{-i\omega t}$, where $F_\omega(t, \vec{r})$ on R^4 is solution of the PDE (W). Consequently, starting from harmonic functions (steady-state waves) and adding ingredients from differential geometry, we create time waves. □

5 Equivalence between PDE (SS) and frequency independent PDE (W), via Fourier transform

Suppose that the coefficients of the PDE (W) depends only on $\vec{r}$.

Let $F(t, \vec{r})$ be a solution of the equation (W), which satisfies the conditions: (i) the improper integral $\int_{-\infty}^{\infty} |F(t, \vec{r})|dt$ is (uniformly) convergent and (ii) $\lim_{|t|\to\infty} F(t, \vec{r}) = 0$. We use the time Fourier transform

$$\tilde{F}(\omega, \vec{r}) = \int_{-\infty}^{\infty} F(t, \vec{r})e^{-i\omega t}dt, \ \lim_{|\omega|\to\infty} \tilde{F}(\omega, \vec{r}) = 0$$

and its inverse

$$F(t,\vec{r}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{F}(\omega,\vec{r})e^{i\omega t}d\omega.$$

The image function $\tilde{F}(\omega,\vec{r})$ verifies the PDE

$$\left(\nabla^2 + \hat{\mu}\,\hat{\epsilon}\;\omega^2\right)\tilde{F}(\omega,\vec{r}) = 0.$$

Conversely, for any solution $\tilde{F}(\omega,\vec{r})$ of this PDE, the inverse Fourier transform $F(t,\vec{r})$ is solution of PDE (W).

References

[1] H. Bahlouli, A. D. Alhaidari, A. Al Zahrani, E. N. Economou, *Study of electromagnetic wave propagation in active medium and the equivalent Schrödinger equation with energy-dependent complex potential*, INTERNET, 2013.

[2] T. E. Bearden, *Extracting and using electromagnetic energy from the active vacuum*, CEO, CTEC, Inc.

[3] L.C. Evans, *Partial Differential Equations*, Graduate Studies in Math. 19, American Mathematical Society, Providence, RI, 1998.

[4] D. S. Grebenkov, B.-T. Nguyen, *Geometrical structure of Laplacian eigenfunctions*, arXiv:1206.1278v2 [math.AP] 8 Feb 2013.

[5] A. Henrot, E. Oudet, *Minimizing the second eigenvalue of the Laplace operator with Dirichlet boundary conditions*, Internet 2015.

[6] C. Udriste, L. Petrescu, L. Matei, *Multitime reaction-diffusion solitons*, Balkan J. Geom. Appl. 17, 2 (2012), 115-128.

[7] C. Udriste, I. Tevy, *Sturm-Liouville operator controlled by sectional curvature on Riemannian manifolds*, Balkan J. Geom. Appl. 17, 2 (2012), 129-140.

[8] C. Udriste, S. Dinu, I. Tevy, *Multitime optimal control for linear PDEs with curvilinear cost functional*, Balkan J. Geom. Appl. 18, 1 (2013), 87-100.

[9] C. Udriste, *Minimal submanifolds and harmonic maps through multitime maximum principle*, Balkan J. Geom. Appl. 18, 2 (2013), 69-82.

[10] C. Udriste, *Multitime optimal control with second order PDEs constraints*, Atti Accad. Pelorit. Pericol. Cl. Sci. Fis. Mat. Nat. 91, 1, A2 (2013), 10 pages.

Authors' addresses:

Ionel Ţevy, Constantin Udrişte, Laura Petrescu
University Politehnica of Bucharest, Faculty of Applied Sciences
Department of Mathematics and Informatics, Splaiul Independentei 313,
Bucharest 060042, Romania.

E-mails: vascatevy@yahoo.com , udriste@mathem.pub.ro , lavinialaurapetrescu@yahoo.com

Dorel Zugrăvescu, Florin Munteanu
Institute of Geodynamics "Sabba S. Stefanescu"
Dr. Gerota 19-21, Bucharest, 020032, Romania.

E-mails: dorezugr@geodin.ro , florin@geodin.ro