Homothetical and translation hypersurfaces with constant curvature in the isotropic space

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Abstract. In this paper, we study homothetical and translation hypersurfaces in the isotropic $(n+1)-$space $\mathbb{I}^{n+1}$. We classify such hypersurfaces of constant curvature in $\mathbb{I}^{n+1}$.

M.S.C. 2010: 53A35, 53A40, 53B25.
Key words: Homothetical hypersurface; translation hypersurface; isotropic space; isotropic mean curvature; relative curvature.

1 Preliminaries

Let $\mathbb{R}^n$ be the Euclidean n-space, i.e., the Cartesian n-space endowed with the Euclidean metric. We will denote the Euclidean scalar product and the induced norm on $\mathbb{R}^n$ by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively.

The isotropic n-space $\mathbb{I}^n$ introduced by H. Sachs ([23]) is the product of $\mathbb{R}^{n-1}$ and the isotropic line equipped with a degenerate parabolic distance metric. It is derived from $\mathbb{R}^n$ by substituting the usual Euclidean distance with the isotropic distance.

The group of motions of $\mathbb{I}^n$ is given by the matrix $\begin{pmatrix} A & 0 \\ B & 1 \end{pmatrix}$, where A is an orthogonal $(n-1, n-1)-$matrix, $\det A = 1$, B a real $(1, n-1)-$matrix (see [28]).

Consider the points $\mathbf{p} = (p, p_n)$ and $\mathbf{q} = (q, q_n)$ in $\mathbb{I}^n$ with $p = (p_1, ..., p_{n-1})$, $q = (q_1, ..., q_{n-1})$. Thus the *isotropic distance* (*i-distance*) of two points $\mathbf{p} = (p, p_n)$ and $\mathbf{q} = (q, q_n)$ is defined as

$$\|\mathbf{p} - \mathbf{q}\|_i = \|p - q\| = \sqrt{\sum_{j=1}^{n-1} (q_j - p_j)^2}. \tag{1.1}$$

The i-metric (1.1) is degenerate along the lines in x_n-direction, and these lines are called *isotropic lines.* The k-planes containing an isotropic line are called *isotropic $k-$planes* [6, 8].

BSG Proceedings 23. The International Conference "Differential Geometry - Dynamical Systems" DGDS-2015, October 8-11, 2015, Bucharest-Romania, pp. 1-10.

Isotropic scalar product (*i-scalar product*) "$\cdot$" of the vectors $\mathbf{u} = (u, u_n)$ and $\mathbf{v} = (v, v_n)$ in $\mathbb{I}^n$ for $u = (u_1, ..., u_{n-1})$ and $v = (v_1, ..., v_{n-1})$ is given by ([10, 28])

$$\mathbf{u} \cdot \mathbf{v} = \begin{cases} \langle u, v \rangle & \text{, if at least one of } u_i \text{ or } v_i \text{ is nonzero, } i = \overline{1, n-1}, \\ u_n v_n & \text{, if } u_i = 0 = v_i \text{ for all } i = \overline{1, n-1}. \end{cases} \tag{1.2}$$

We call the vectors of the form $\mathbf{u} = (0, u_n)$ in $\mathbb{I}^n$, $0 = (\underbrace{0, ..., 0}_{(n-1)-tuple})$, $u_n \neq 0$, *isotropic vectors* and ones of the form $\mathbf{u} = (u \neq 0, u_n)$ *non-isotropic vectors.* With respect to the i-scalar product (1.2), all the isotropic vectors are orthogonal to the non-isotropic ones. Moreover, two non-isotropic vectors $\mathbf{u}, \mathbf{v}$ from $\mathbb{I}^n$ are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$.

Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a smooth function. Consider the graph hypersurface M^n of F in $\mathbb{I}^{n+1}$ parametrized by

$$X : \mathbb{R}^n \longrightarrow \mathbb{I}^{n+1}, \ \mathbf{x} \longmapsto X(\mathbf{x}) = (\mathbf{x}, F(\mathbf{x})), \ \mathbf{x} \in \mathbb{R}^n.$$

The metric on M^n induced from $\mathbb{I}^{n+1}$ is given by $g_* = dx_1^2 + ... + dx_n^2$. This implies that M^n is always a flat space with respect to the induced metric g_*. Thus its Laplacian is given by

$$\triangle = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}.$$

The Hessian matrix D^2F is a square matrix of the second-order partial derivatives of F. Denote $H^{i_1,...,i_s}$ the determinant of the quadratic submatrix of D^2F obtained by taking in D^2F only rows and columns with indices $i_1, ..., i_s$. Then the n *fundamental curvature functions* on M^n are defined as

$$K_j = \frac{1}{\binom{n}{j}} \left(H^{1,...,j} + H^{1,...,j-1,j+1} + ... + H^{n-j+1,...,n} \right). \tag{1.3}$$

In the particular cases $j = 1$ and $j = n$, the fundamental curvature functions K_1 and K_n are respectively called the *isotropic mean curvature* and the *relative curvature*,

$$K_1 = \frac{1}{n} \text{trace} \left(D^2F \right) = \frac{1}{n} \triangle F \quad \text{and} \quad K_n = \det \left(D^2F \right).$$

A hypersurface in $\mathbb{I}^{n+1}$ is called *isotropic minimal* (resp. *isotropic flat*) if it has null isotropic mean curvature K_1 (resp. relative curvature K_n).

More details on $\mathbb{I}^{n+1}$ can be found in [13], [21]-[23].

On the other hand, the isotropic geometry naturally appears when properties of functions shall be geometrically visualized and interpreted via their graph surfaces and has remarkable application areas such as Image Processing, architectural design, economics, etc. [2, 6, 8, 14], [18]-[20].

Z. M. Sipus [27] derived several classifications for the translation surfaces generated by two planar curves in the isotropic 3-space $\mathbb{I}^3$ with constant curvature. Then this classification was generalized to ones generated by a space curve and a planar curve in $\mathbb{I}^3$ [1].

Most recently, the first author and I. Mihai established a method to calculate the second fundamental form of the surfaces in $\mathbb{I}^4$ and classified some surfaces in $\mathbb{I}^4$ with vanishing curvatures.

In the present paper, we classify translation and homothetical hypersurfaces in $\mathbb{I}^{n+1}$ with constant fundamental curvature.

2 Homothetical and translation hypersurfaces in Euclidean spaces

Let $F:\mathbb{R}^n \longrightarrow \mathbb{R}$ be a smooth function and $\mathbf{x}=(x_1,...,x_n)\in\mathbb{R}^n$. Then the function F can be identified with the non-parametric hypersurface M^n of $\mathbb{R}^{n+1}$ given by

$$X:\mathbb{R}^n \longrightarrow \mathbb{R}^{n+1},\ \mathbf{x}\longmapsto(\mathbf{x},F(\mathbf{x})).$$

A hypersurface M^n of $\mathbb{R}^{n+1}$ is called a *translation hypersurface* if it is the graph of a function of the form:

$$F(\mathbf{x})=F(x_1,...,x_n)=f_1(x_1)+...+f_n(x_n),$$

where $f_1,...,f_n$ are smooth functions of one variable, see [4, 5, 9, 16, 17, 30].

A surface of $\mathbb{R}^3$ is called *Scherk's minimal translation surface*, which is one of the famous minimal surfaces of $\mathbb{R}^3$, if it satisfies the equation

$$x_3=\frac{1}{c}\log\left|\frac{\cos\ cx_2}{\cos\ cx_1}\right|,$$

for a nonzero constant c. In 1991, F. Dillen et al. [9] generalized this result to higher-dimensional Euclidean spaces.

The translation hypersurfaces in $\mathbb{R}^{n+1}$ with constant curvature are completely classified by K. Seo as following:

Theorem 2.1. *[26] Let M^n be a translation hypersurface in $\mathbb{R}^{n+1}$ with constant mean curvature H. Then M^n is congruent to a cylinder $\Sigma\times\mathbb{R}^{n-2}$, where Σ is a surface in $\mathbb{R}^3$ with constant mean curvature. In particular, if $H\equiv 0$, then M^n is either a hyperplane or $\Sigma\times\mathbb{R}^{n-2}$, where Σ is a Scherk's minimal translation surface in $\mathbb{R}^3$.*

Theorem 2.2. *[26] Let M^n be a translation hypersurface with constant Gauss–Kronecker curvature K in $\mathbb{R}^{n+1}$. Then M^n is congruent to a cylinder, and hence $K\equiv 0$.*

A hypersurface M^n of $\mathbb{R}^{n+1}$ is called a *homothetical hypersurface* if it can be locally written as the product of the functions of one variable, i.e., it is the graph of a function of the form:

$$F(\mathbf{x})=F(x_1,\dots,x_n)=f_1(x_1)\cdot\dots\cdot f_n(x_n),\quad \mathbf{x}\in\mathbb{R}^n,$$

where $f_1,\dots,f_n$ are smooth functions of one variable.

The homothetical hypersurfaces have been studied by many authors based on minimality property in (semi-) Euclidean spaces [11, 12, 15, 24, 25, 29].

L. Jiu and H. Sun gave an exact classification for minimal homothetical hypersurfaces in $\mathbb{R}^{n+1}$ and proved

Theorem 2.3. *[12] Let M^n $(n\geq 3)$ be an $n-$dimensional minimal homothetical hypersurface in $\mathbb{R}^{n+1}$. Then the hypersurface is a hyperplane, a quadratic cone, a cylinder on a quadratic cone or a cylinder on a helicoid.*

Morever, the first author and M. Ergut classified the homothetical hypersurfaces in $\mathbb{R}^{n+1}$ having null Gauss-Kronecker curvature by proving next result

Theorem 2.4. *[3] Let M^n be a homothetical hypersurface in $\mathbb{R}^{n+1}$. M^n has null Gauss-Kronecker curvature if and only if it has one of the following forms*

(i) $X(\mathbf{x}) = \left(\mathbf{x}, \gamma \exp(\alpha_1 x_1 + \alpha_2 x_2) \prod_{j=3}^{n} h_j(x_j)\right)$ *for nonzero constants* $\gamma, \alpha_1, \alpha_2$, $\mathbf{x} \in \mathbb{R}^n$;

(ii) $X(\mathbf{x}) = \left(\mathbf{x}, \gamma \prod_{j=1}^{n} (x_j + \beta_j)^{\alpha_j}\right)$, *where* $\beta_1, ..., \beta_n$ *are some constants and* $\gamma, \alpha_1, ..., \alpha_n$ *nonzero constants such that* $\sum_{i=1}^{n} \alpha_i = 1$, $\mathbf{x} \in \mathbb{R}^n$.

3 Homothetical hypersurfaces in $\mathbb{I}^{n+1}$

A hypersurface M^n of the isotropic $(n+1)-$space $\mathbb{I}^{n+1}$ is called a *homothetical hypersurface* if it is the graph of a function given by

$$H(x_1, ..., x_n) := h_1(x_1) \cdot ... \cdot h_n(x_n),$$

where $h_1, ..., h_n$ are non-constant smooth functions of one real variable. We denote the homothetical hypersurface M^n by a pair (M^n, H).

Next results classify homothetical hypersurfaces in $\mathbb{I}^{n+1}$ with constant isotropic mean and relative curvature.

Theorem 3.1. *Let (M^n, H) be a homothetical hypersurface in $\mathbb{I}^{n+1}$ with constant isotropic mean curvature K_1. Then it is isotropic minimal, i.e. $K_1 = 0$ and has one of the following forms*

(i)

$$X(\mathbf{x}) = \left(\mathbf{x}, \prod_{j=1}^{n} \{\gamma_j x_j + \varepsilon_j\}\right), \tag{3.1}$$

where $\mathbf{x} \in \mathbb{R}^n$ *and* γ_j, ε_j *are some constants,* $j \in \{1, ..., n\}$;

(ii)

$$X(\mathbf{x}) = \left(\mathbf{x}, \prod_{j=1}^{n} \left\{\gamma_j \exp\left(\sqrt{\alpha_j} x_j\right) + \beta_j \exp\left(-\sqrt{\alpha_j} x_j\right)\right\}\right), \tag{3.2}$$

for $\mathbf{x} \in \mathbb{R}^n$ *and nonzero constants* $\alpha_j, \beta_j, \gamma_j$, $j \in \{1, ..., n\}$ *such that* $\sum_{j=1}^{n} \alpha_j = 0$.

Proof. Assume that (M^n, H) is a homothetical hypersurface with constant isotropic mean curvature K_1 in $\mathbb{I}^{n+1}$. Then it follows

$$\triangle H = \sum_{j=1}^{n} h_1 ... h_{j-1} \ddot{h}_j h_{j+1} ... h_n = n K_1. \tag{3.3}$$

By dividing (3.3) with the product $h_1 h_2 ... h_n$, we write

$$\frac{\ddot{h}_1}{h_1} + \frac{\ddot{h}_2}{h_2} + ... + \frac{\ddot{h}_n}{h_n} = \frac{n K_1}{h_1 h_2 ... h_n}. \tag{3.4}$$

After taking the partial derivative of (3.4) with respect to x_j, we deduce

$$\frac{\dddot{h}_j h_j - \ddot{h}_j \dot{h}_j}{(h_j)^2} = -nK_1 \frac{\dot{h}_j / (h_j)^2}{h_1 h_2 ... \overset{j}{\frown} ... h_n}, \tag{3.5}$$

where $\overset{j}{\frown}$ denotes absence of $j-$th element. We can rewrite (3.5) as

$$\left(\frac{\dddot{h}_j h_j - \ddot{h}_j \dot{h}_j}{(h_j)^2} \right) \frac{(h_j)^2}{\dot{h}_j} = -\frac{nK_1}{h_1 h_2 ... \overset{j}{\frown} ... h_n}. \tag{3.6}$$

The left-hand side of the equation (3.6) is either a function of variable x_j or a constant. However the right-hand side is a non-constant function of variables $x_1, x_2, ..., \overset{j}{\frown}, ..., x_n$. This implies that the constant K_1 must be zero which means that (M^n, H) is isotropic minimal and

$$\frac{\dddot{h}_j h_j - \ddot{h}_j \dot{h}_j}{(h_j)^2} = 0. \tag{3.7}$$

It is easy to see that when h_j is a linear function, it is a solution of (3.7). This gives the proof of the statement (i) of the theorem.

Now assume that $h_1, ..., h_n$ are non-linear functions. Then (3.7) can be rewritten as

$$\ddot{h}_j - \alpha_j h_j = 0, \tag{3.8}$$

for nonzero constants α_j, $j \in \{1, ..., n\}$. Thus by solving (3.8), we derive

$$h_j(x_j) = \gamma_j \exp\left(\sqrt{\alpha_j} x_j\right) + \beta_j \exp\left(-\sqrt{\alpha_j} x_j\right),$$

for nonzero constants β_j, γ_j, $j \in \{1, ..., n\}$ such that $\sum_{j=1}^{n} \alpha_j = 0$. This completes the proof. □

In the Euclidean spaces, B.-Y. Chen [7] proved that a graph hypersurface in $\mathbb{R}^{n+1}$ of a given real-valued function F has null Gauss-Kronoceker curvature if and only if the Hessian matrix $D^2 F$ is singular. This result also works when a graph hypersurface in $\mathbb{I}^{n+1}$ has null relative curvature. Explicitly, the graph hypersurface of F in $\mathbb{I}^{n+1}$ has null relative curvature if and only if the Hessian matrix $D^2 F$ is singular.

Thus the following result can be given by using Theorem 2.4 without proof.

Theorem 3.2. *Let (M^n, H) be an isotropic flat homothetical hypersurface in $\mathbb{I}^{n+1}$. Then it has one of the following forms:*

(ii) $X(\mathbf{x}) = \left(\mathbf{x}, \gamma \exp(\alpha_1 x_1 + \alpha_2 x_2) \prod_{j=3}^{n} h_j(x_j) \right)$ *for nonzero constants* $\gamma, \alpha_1, \alpha_2$;

(ii) $X(\mathbf{x}) = \left(\mathbf{x}, \gamma \prod_{j=1}^{n} (x_j + \beta_j)^{\alpha_j} \right)$, *where* $\mathbf{x} \in \mathbb{R}^n$, $\beta_1, ..., \beta_n$ *are some constants and* $\gamma, \alpha_1, ..., \alpha_n$ *nonzero constants such that* $\sum_{i=1}^{n} \alpha_i = 1$.

4 Translation hypersurfaces in $\mathbb{I}^{n+1}$

Let $\gamma_1, ..., \gamma_n$ be planar curves lying in mutually orthogonal isotropic planes of $\mathbb{I}^{n+1}$. Then we may assume that the planar curves γ_i are parametrized by

$$\gamma_i(x_i) = (\delta_{i1}x_1, \delta_{i2}x_2, ..., \delta_{in}x_n, f_i(x_i)),$$

where δ_{ij} is the Kronocker delta and f_i is a smooth function of one variable for all $i \in \{1, ..., n\}$.

Now, we consider the translation hypersurfaces obtained by translating the planar curves γ_i, $i \in \{1, ..., n\}$, with respect to i-motions in $\mathbb{I}^{n+1}$. In this sense, a *translation hypersurface* M^n in $\mathbb{I}^{n+1}$ is parametrized by

$$X : \mathbb{R}^n \longrightarrow \mathbb{I}^{n+1},\ \mathbf{x} \longmapsto (\mathbf{x}, F(\mathbf{x})),\ F(\mathbf{x}) := \sum_{j=1}^{n} f_j(x_j),\ \mathbf{x} \in \mathbb{R}^n,$$

where f_j is a smooth function of one variable for all $j \in \{1, ..., n\}$. We henceforth denote a translation hypersurface in $\mathbb{I}^{n+1}$ by a pair (M^n, F).

The following results classify translation hypersurfaces in $\mathbb{I}^{n+1}$ with constant curvature.

Theorem 4.1. *Let (M^n, F) be a translation hypersurface in $\mathbb{I}^{n+1}$ with constant isotropic mean curvature K_1. Then it has the form*

$$X(\mathbf{x}) = \left(\mathbf{x}, \sum_{j=1}^{n} \alpha_j x_j^2 + \beta_j x_j + \varepsilon\right), \tag{4.1}$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\alpha_j, \beta_j, \varepsilon$ are some constants for all $j \in \{1, ..., n\}$ such that $\sum_{j=n}^{n} \alpha_j = \frac{n}{2}K_1$.

Proof. Assume that (M^n, F) is a translation hypersurface in $\mathbb{I}^{n+1}$ with nonzero constant isotropic mean curvature K_1. Then we have

$$\ddot{f}_1 + \ddot{f}_2 + ... + \ddot{f}_n = nK_1, \tag{4.2}$$

where $\ddot{f}_j = \frac{d^2 f_j}{dx_j^2}$ for all $j \in \{1, ..., n\}$. Taking partial derivative of (4.2) with respect to x_j, we get $\dddot{f}_j = 0$, which implies

$$f_j(x_j) = \alpha_j x_j^2 + \beta_j x_j + \varepsilon_j, \tag{4.3}$$

for some constants $\alpha_j, \beta_j, \varepsilon_j$, $j \in \{1, ..., n\}$ and $\sum_{j=n}^{n} \alpha_j = \frac{n}{2}K_1$. It gives the proof. □

Remark 4.1. Isotropic minimal translation hypersurfaces in $\mathbb{I}^{n+1}$ are also classified by Theorem 4.1 by taking $K_1 = 0$.

Theorem 4.2. *Let (M^n, F) be a translation hypersurface in $\mathbb{I}^{n+1}$ with nonzero constant relative curvature K_n. Then it is of the form*

$$X(\mathbf{x}) = \left(\mathbf{x}, \sum_{j=1}^{n} \alpha_j x_j^2 + \beta_j x_j + \varepsilon\right),$$

where $\mathbf{x} \in \mathbb{R}^n$, α_j *are nonzero constants and* β_j, ε *some constants for all* $j \in \{1, ..., n\}$ *such that* $\prod_{j=1}^{n} \alpha_j = \frac{1}{2^n} K_1$. *In particular, if* (M^n, F) *is isotropic flat in* $\mathbb{I}^{n+1}$, *then it is congruent to a cylinder from the Euclidean perspective.*

Proof. Let us assume that (M^n, F) is a translation hypersurface in $\mathbb{I}^{n+1}$ with nonzero constant relative curvature K_n. It follows $\det\left(D^2 F\right) = K_n$. Thus we get

$$\ddot{f}_1 \cdot \ddot{f}_2 \cdot ... \cdot \ddot{f}_n = K_n. \tag{4.4}$$

Taking partial derivative of (4.4) with respect to x_j, we derive

$$\ddot{f}_1 \cdot \ddot{f}_2 \cdot ... \cdot \ddot{f}_{j-1} \cdot \dddot{f}_{j-1} \cdot \ddot{f}_{j+1} \cdot ... \cdot \ddot{f}_n = 0. \tag{4.5}$$

Equality (4.5) yields that $f_j(x_j) = \alpha_j x_j^2 + \beta_j x_j + \varepsilon_j$ for nonzero constants α_j and some constants β_j, ε, $j \in \{1, ..., n\}$ such that $\frac{1}{2^n} \prod_{j=1}^{n} \alpha_j = K_1$.

Now let (M^n, F) be an isotropic flat translation hypersurface in $\mathbb{I}^{n+1}$, i.e. $K_1 = 0$. Then from (4.4) it is easy to see that at least one the $f_1, ..., f_n$ is a linear function. Without loss of generality, we may assume that $f_1(x_1) = \beta_1 x_1 + \varepsilon_1$. Thus we have

$$\begin{aligned} X(\mathbf{x}) &= \left(\mathbf{x}, \beta_1 x_1 + \varepsilon_1 + \sum_{j=2}^{n} f_j(x_j)\right) \\ &= x_1 (1, 0..., \beta_1) + \left(0, x_2, ..., x_n, \varepsilon_1 + \sum_{j=2}^{n} f_j(x_j)\right), \end{aligned}$$

which implies that (M^n, F) is a cylinder. This completes the proof. □

In the particular four dimensional case, next provides a classification for the translation hypersurfaces in $\mathbb{I}^4$ with constant second fundamental curvature K_2.

Theorem 4.3. *Let* $\left(M^3, F\right)$ *be a translation hypersurface in* $\mathbb{I}^4$ *with constant second fundamental curvature* K_2. *Then it has the following form*

$$X(x_1, x_2, x_3) = \left(x_1, x_2, x_3, \sum_{j=1}^{3} \alpha_j x_j^2 + \beta_j x_j + \varepsilon\right), \tag{4.6}$$

where $\alpha_j, \beta_j, \varepsilon$ *are some constants and* $\sum_{i<j \in I} \alpha_i \alpha_j = 3K_2$, $I = \{1, 2, 3\}$.

In particular, if $\left(M^3, F\right)$ *has null second fundamental curvature, i.e.* $K_2 = 0$, *then it is a product* $\Gamma \times \mathbb{I}^2$, *where* Γ *is a curve lying in the isotropic* $x_3 x_4-$*plane in* $\mathbb{I}^4$.

Proof. By using (1.3), the second fundamental curvature K_2 of $\left(M^3, F\right)$ in $\mathbb{I}^4$ becomes

$$3K_2 = \left(\ddot{f}_1 \cdot \ddot{f}_2 + \ddot{f}_1 \cdot \ddot{f}_3 + \ddot{f}_2 \cdot \ddot{f}_3\right). \tag{4.7}$$

We divide the proof into two separate cases.

Case (a) K_2 is a nonzero constant. After taking partial derivative of (4.7) with respect to x_i, $i \in \{1,2,3\}$, we derive $0 = \dddot{f}_i \cdot \left(\ddot{f}_j + \ddot{f}_k\right)$, where $j,k \in \{1,2,3\} \setminus \{i\}$. Hence we get

$$f_i = \alpha_i x_i^2 + \beta_i x_i + \varepsilon_i,$$

where $\alpha_i, \beta_i, \varepsilon_i$ are some constants and $\sum_{i<j\in\{1,2,3\}} \alpha_i \alpha_j = 3K_2$.

Case (b) $K_2 = 0$. We divide the proof of case (b) into two cases based on the linearity of the functions f_1, f_2, f_3.

Case (b.1) f_1, f_2, f_3 are non-linear functions. Dividing (4.7) with the product $\ddot{f}_1 \cdot \ddot{f}_2 \cdot \ddot{f}_3$ gives

$$\frac{1}{\ddot{f}_1} + \frac{1}{\ddot{f}_2} + \frac{1}{\ddot{f}_3} = 0.$$

This yields that $\ddot{f}_i = \alpha_i$, $i = 1,2,3$, where α_i are nonzero constants and $\sum_{i<j\in\{1,2,3\}} \alpha_i \alpha_j = 0$. Thus, we obtain

$$f_i = \alpha_i x_i^2 + \beta_i + \varepsilon_i$$

for some constants β_i, ε_i, $i = 1,2,3$. This is a special case of Case (a).

Case (b.2) At least one of f_1, f_2, f_3 is a linear function. Without loss of generality, we may assume f_1 is a linear function. Thus it follows from (4.7) that

$$\ddot{f}_2 \cdot \ddot{f}_3 = 0.$$

This leads that either f_2 or f_3 is a linear function. Without loss of generality, we may assume f_2 is a linear function. This means that the translation hypersurface $\left(M^3, F\right)$ is parametrized by

$$X(x_1, x_2, x_3) = (0, 0, x_3, f_3(x_3)) + (x_1, x_2, 0, \alpha_1 x_1 + \alpha_2 x_2),$$

where α_1, α_2 are some constants, and the proof is complete. □

Acknowledgements. The results of this paper were presented at the IX-th International Conference of Differential Geometry and Dynamical Systems (DGDS-2015), Bucharest, Romania. We thank the organizers for inviting us to present this work there.

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