Jet prolongations of the projectable characteristic vector fields of a commutative and characteristic deformation algebra

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Abstract. Our purpose is to develop the jet prolongations of the projectable characteristic vector fields of a commutative and characteristic deformation algebra.

M.S.C. 2010: 58A20.
Key words: jets; jet prolongations of projectable characteristic vector fields; fibered manifolds; commutative and characteristic deformation algebra.

1 Fibered manifolds

A *fibered manifold* is a triple (Y, π, X), where Y and X are finite dimensional differentiable manifolds and $\pi : Y \to X$ is a surjective submersion with $\dim X = n$ and $\dim Y = m+n$. At every point $y \in Y$, the following two equivalent conditions defining submersions are satisfied:

(1) the tangent mapping $T_y\pi : T_yY \to T_{\pi(y)}X$ is surjective,

(2) there exist a chart (V, ψ), $\psi = (u^i, y^\sigma)$, at y, where $1 \leq i \leq n$, $1 \leq \sigma \leq m$ and a chart (U, φ), $\varphi = (x^i)$, at $x = \pi(y)$ such that $U = \pi(V)$ and $x^i\pi = u^i$.

We write x^i instead of u^i, and call (V, ψ), $\psi = (x^i, y^\sigma)$, a *fibered chart.* The chart (U, φ), $\varphi = (x^i)$ on X is unique, and is said to be *associated with* (V, ψ), $\psi = (x^i, y^\sigma)$.

A *section* of a fibered manifold (Y, π, X), is a mapping $\gamma : U \to Y$, where $U \subset X$, is an open set, such that

$$\pi \circ \gamma = id_U.$$

A vector field Ξ on Y is said to be π*-projectable*, or simply *projectable*, if there exists a vector field ξ on X such that

$$T\pi \cdot \Xi = \xi \circ \pi.$$

If ξ exists, it is unique, and is called the π*-projection* of Ξ.

BSG Proceedings 23. The International Conference "Differential Geometry - Dynamical Systems" DGDS-2015, October 8-11, 2015, Bucharest-Romania, pp. 24-28.

In a fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, a π-projectable vector field Ξ is expressed as

$$\Xi = \xi^i \frac{\partial}{\partial x^i} + \Xi^\sigma \frac{\partial}{\partial y^\sigma}$$

where $\xi^i = xi^i(x^j)$ and $\Xi^\sigma = \Xi^\sigma(x^j, y^\sigma)$.

2 Jet prolongations of a fibered manifold

Let $y \in Y$, $x = \pi(y)$, and let $\Gamma^r_{x,y}$ be the set of smooth sections γ of Y defined at x such that $\gamma(x) = y$. Let $r > 0$ be an integer. We define the binary relation $\gamma_1 \sim \gamma_2$ if there exists a fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, at y such that

$$D_{i_1} D_{i_2} \dots D_{i_k} (y^\sigma \gamma_1 \varphi^{-1})(\varphi(x)) = D_{i_1} D_{i_2} \dots D_{i_k} (y^\sigma \gamma_2 \varphi^{-1})(\varphi(x)),$$

for all $k = 1, 2, \dots, r$ and all $i_1, i_2, \dots, i_k$ such that $1 < i_1 < i_2 < \dots < i_k < n$ is an equivalence on the set $\Gamma^r_{x,y}$. The equivalence class containing a section γ, is called an r-jet with *source* x and *target* y or the r-jet of y at x, and is denoted by $J^r_x \gamma$. We denote by $J^r Y$ the set of r-jets with source in X and target in Y. The *canonical jet projections* are the mappings $\pi^{r,s}$ (respectively π^r) of $J^r Y$ onto $J^s Y$, where $0 < s < r$ (respectively on X), defined by $\pi^{r,s}(J^r_x \gamma) = (J^s_x \gamma)$ (respectively $\pi^r(J^r_x \gamma) = x$).

The *smooth structure* of $J^r Y$ associated with the smooth structure of Y is defined as follows. Let (V, ψ), $\psi = (x^i, y^\sigma)$, where $1 \leq i \leq n$, $1 \leq \sigma \leq m$, be a fibered chart on Y, (U, φ), $\varphi = (x^i)$, the associated chart on X. Then the *associated fibered chart* (V^r, ψ^r), $\psi^r = (x^i, y^\sigma, y^{\sigma j_1}, \dots, y^{\sigma j_1 j_2 \cdots j_r})$ on $J^r Y$ is defined by the following two conditions:

$$V^r = (\pi^{r,0})^{-1}(V),$$

and if $J^r_x \gamma \in V^r$, then

$$y^{\sigma j_1 j_2 \cdots j_k}(J^r_x \gamma) = D_{j_1} D_{j_2} \dots D_{j_k} (y^\sigma \gamma \varphi^{-1})(\varphi(x)),$$

where $k = 1, 2, \dots, r$ and $1 < j_1 < j_2 < \dots < j_k < n$. If (V', ψ'), $\psi' = (x'^i, y'\sigma)$, is another fibered chart such that $V \cap V' \neq \varnothing$, then writing

$$y'^\sigma \gamma \varphi'^{-1} = y' \psi^{-1} \circ \psi \gamma \varphi^{-1} \circ \varphi \varphi'^{-1}$$

we get, using the chain rule, the *transformation formula* in a recurrent form

$$y'^{\sigma j_1 j_2 \cdots j_k} = D_{j_1} D_{j_2} \dots D_{j_k} (y'^\sigma \psi^{-1} \circ \psi \gamma \varphi^{-1} \circ \varphi \varphi'^{-1})(\varphi'(x)).$$

The dimension of $J^r Y$ is given by $\dim J^r Y = n + m \begin{pmatrix} n + r \\ n \end{pmatrix}$.

3 The horizontalization of tangent vectors

A vector bundle morphism acts on tangent spaces to the jet prolongations of a fibered manifold. Similarly as in the case of differential forms, this vector bundle morphism is induced by the structure of the jet prolongations.

Let $r > 0$ be an integer. One can assign to every tangent vector $\xi \in TJ^{r+1}Y$ at a point $J^{r+1}_x\gamma \in J^{r+1}Y$ a tangent vector $h\xi \in TJ^rY$ at $J^r_x\gamma = \pi^{r+1,r}(J^{r+1}_x\gamma) \in J^rY$ by

$$h\xi = T_xJ^r\gamma \circ T\pi^{r+1} \cdot \xi.$$

The mapping $h : TJ^{r+1}Y \to TJ^rY$ defined by this formula is a vector bundle morphism over the jet projection $\pi^{r+1,r}$; we call h the π-*horizontalization*, or simply the *horizontalization* and

$$h\xi = \xi^j \left(\frac{\partial}{\partial x^i} + \sum_{k=0}^{r} \sum_{j_1 \le j_2 \le \ldots \le j_k} y^\sigma_{j_1 j_2 \ldots j_k j} \frac{\partial}{\partial y^\sigma}_{j_1 j_2 \ldots j_k} \right).$$

Lemma 1 [5]. *Let Ξ be a π-projectable vector field on Y, (V,ψ), $\psi = (x^i, y^\sigma)$, a fibered chart on Y, and let Ξ be expressed by $\Xi = \xi^i \frac{\partial}{\partial x^i} + \Xi^\sigma \frac{\partial}{\partial y^\sigma}$. Then $J^r\Xi$ is expressed with respect to the associated chart (V^r, ψ^r)*

$$J^r\Xi = \xi^i \frac{\partial}{\partial x^i} + \left(\sum_{k=0}^{r} \sum_{j_1 \le j_2 \le \ldots \le j_k} \Xi^\sigma_{j_1 j_2 \ldots j_k} \frac{\partial}{\partial y^\alpha_{j_1 j_2 \ldots j_k}} \right),$$

where the components $\Xi^\sigma_{j_1 j_2 \ldots j_k}$ are determined by the recurrent formula

$$\Xi^\sigma_{j_1 j_2 \ldots j_k} = d_{j_k} \Xi^\sigma_{j_1 j_2 \ldots j_{k-1}} - y^\sigma_{j_1 j_2 \ldots j_{k-1} i} \frac{\partial \xi^i}{\partial x^{j_k}}.$$

Lemma 2 [5]. *Let Ξ_1 and Ξ_2 be two π-projectable vector fields on Y. Then the Lie bracket $[\Xi_1, \Xi_2]$ is also π-projectable vector fields on Y, and $J^r[\Xi_1, \Xi_2] = [J^r\Xi_1, J^r\Xi_2]$.*

Let $A(\Xi_1, \Xi_2) = \nabla_{\Xi_1}\Xi_2 - \nabla'_{\Xi_1}\Xi_2$ be a tensor field of type $(1, 2)$ which defines the deformation algebra associated to the pair of linear connections (∇, ∇') on TJ^rY, noted by $U(J^rY, A)$. These exist from [4 Prop. 3 p. 226].

4 Commutative and characteristic deformation algebra

Definition 1 [7]. A tangent vector v of $T_{J^r_x\gamma}J^rY$ on $J^r_x\gamma \in J^rY$ is called characteristic vector of the deformation algebra $U(J^r_x\gamma, A)$, if the vector subspace $\langle v \rangle$ generated by v is a subalgebra of the deformation algebra $(J^r_x\gamma, A)$.

Example 1. Let ω be an 1-form on J^rY and A be the $(1, 2)$ tensor field given by $A = \omega \otimes J + J \otimes \omega$ where J is a $(1, 1)$ tensor field on J^rY. The characteristic vectors of the deformation algebra A given by $v * w = A(v, w)$ for all $v, w \in T_{J^r_x\gamma}J^rY$, are the vectors that verify the equality $\omega_{J^r_x\gamma}(v) = 0$ or the proper vectors of $J_{J^r_x\gamma}$.

Example 2. If $J = I$ then all the elements of $U(J^r_x\gamma, A)$ are characteristic and the proper value of v is given by $\lambda = 2\omega_{J^r_x\gamma}(v)$.

Remark 1 [7]. The set of the characteristic vectors is a cone.

Definition 2 [7]. A deformation algebra $U(J^rY, A)$ is called projective if there exists a 1-form ω on J^rY such that

$$A = \omega \otimes I + I \otimes \omega.$$

Definition 3 [7]. Let U be a open subset of J^rY and $\omega \in \mathcal{T}_s^0(J^rY)$. The $(0, s)$ tensor field ω is called irreducible relative at U if for all $k > 0$, the only tensor fields $\Omega \in \mathcal{T}_{k+s}^0(J^rY)$ that satisfy the property

$$\Gamma_{J_x^r\gamma}(\omega) \subseteq \Gamma_{J_x^r\gamma}(\Omega),$$

for all $J_x^r\gamma \in J^rU$ are of the form $S\Omega_U = S(\omega \otimes \Omega')_U$, where

$$S\omega(X_1, \dots, X_n) = \frac{1}{s!}\sum_\sigma \omega(X_{\sigma(1)}, \dots, X_{\sigma(n)}),$$

is the $(0, s)$ symmetric tensor field of ω, $\Omega' \in \mathcal{T}_k^0(J^rY)$, σ a permutation with s elements and the index U means the restriction at U.

Corollary 1 [7]. *Let $\omega \in \mathcal{T}_1^0(J^rY)$ with the property $S = \{J_x^r\gamma \in J^rY/\omega_{J_x^r\gamma} \neq 0\} = J^rY$. Then ω is an irreducible tensor field.*

Theorem 1 [3]. *Let v be a tangent vector of $T_{J_x^r\gamma}J^rY$ on $J_x^r\gamma \in J^rY$. Then the following statements are equivalent:*

(1) *v is a characteristic vector;*

(2) *There exists a real number λ such that: $v * v = \lambda v$;*

(3) *In the tensor algebra $T(T_{J_x^r\gamma}J^rY)$, there is the equality*

$$(v * v) \otimes v = v \otimes (v * v),$$

for all $v \in T(T_{J_x^r\gamma}J^rY)$.

Theorem 2. *Any commutative and characteristic algebra $U(J^rY, A)$ is projective.*

Proof. Let $\{U_\alpha\}_\alpha$ be an open covering of the manifold J^rY such that on open set U_α there exists a frame $\delta_\alpha = (E_1^\alpha, \dots, E_n^\alpha)$ and the associated co-frame $\delta_\alpha^* = (\omega_1^\alpha, \dots, \omega_n^\alpha)$. Using the fact that $U(J^rY, A)$ is commutative, it results that

$$A = \sum_\alpha \Omega_i^\alpha \otimes E_i^\alpha, \tag{4.1}$$

where $\Omega_i^\alpha \in \mathcal{T}_2^0(U_\alpha)$ are symmetric tensor fields. Let v be a tangent vector of $T_{J_x^r\gamma}J^rY$ and using that $U(J^rY, A)$ is characteristic, we get that there exists a real number λ such that $A(v, v) = \lambda v$, or

$$\Omega_i^\alpha(v, v) = \lambda\omega_i^\alpha(v). \tag{4.2}$$

Using $\Gamma_{J_x^r\gamma}(\omega_i^\alpha) \subseteq \Gamma_{J_x^r\gamma}(\Omega_i^\alpha)$ and Corollary 1. For U_α, there exists $\Omega_i'^\alpha \in \mathcal{T}_1^0(U_\alpha)$ such that:

$$\Omega_i^\alpha = \Omega_i'^\alpha \otimes \omega_i^\alpha + \omega_i^\alpha \otimes \Omega_i'^\alpha. \tag{4.3}$$

Let us consider $v \in T_{J^r_x\gamma} J^r Y - \{\cup_i \Gamma_{J^r_x\gamma} \omega_i^\alpha)\}$. From (4.2) and (4.3), we get $\Omega_i'^\alpha(v) = \frac{\lambda}{2}$, for all $i = 1, \ldots, n + m \begin{pmatrix} n+r \\ n \end{pmatrix}$, thus the 1-forms $\Omega_i'^\alpha$ coincide on the dense set. Then they coincide on all points, or there exists the 1-form $\omega^\alpha \in \mathcal{T}_1^0(U_\alpha)$ such that

$$\Omega_i'^\alpha = \omega^\alpha, \tag{4.4}$$

for all $i = 1, \ldots, n + m \begin{pmatrix} n+r \\ n \end{pmatrix}$. Using (4.1), (4.3) and (4.4), we obtain

$$A = \omega^\alpha \otimes I + I \otimes \omega^\alpha.$$

Using the unity partition $\lambda_{\alpha\alpha}$ associated to the open covering $U_{\alpha\alpha}$ and considering the 1-form $\omega \in \mathcal{T}_1^0(J^r Y)$ given by $\omega = \sum_\alpha \lambda_\alpha \Omega^\alpha$, we get

$$A = \omega \otimes I + I \otimes \omega.$$

It follows that the algebra $U(J^r Y, A)$ is projective. This ends the proof. □

Corollary 2 [7]. *The algebra $U(J^r Y, A)$ is characteristic if and only if the symmetric associated algebra $U(J^r Y, SA)$ is projective.*

Corollary 3 [7]. *Let ∇, ∇' two linear connections on the manifold $J^r Y$. They admit the same autoparallel curves if and only if the symmetric algebra $U(J^r Y, SA)$ associated to its deformation algebra is projective.*
Proof. One can prove the Corollary 3 using the Remark 5.2 from [7, p.76]. □

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