Contact structures on the indicatrix of a complex Finsler space

Elena Popovici

Abstract. Continuing the study of the complex indicatrix I_zM, approached as an embedded CR hypersurface on the punctual holomorphic tangent bundle of a complex Finsler space, we study in this paper the almost contact structures that can be introduced on I_zM. The Levi form and characteristic direction of the complex indicatrix are given and the CR distributions integrability is studied. Using these we construct a natural contact structure subordonated to the CR-structure of the complex indicatrix, which is also normal. Moreover, with respect to the natural contact structure, the associated connections on I_zM, such as Tanaka and Tanaka Webster connections, are determined and the Bochner type tensor field of the complex indicatrix is introduced.

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1 Introduction

The study of unit tangent sphere, or indicatrix, in real Finsler spaces is one of interest ([8, 13, 16, 18], etc.), mainly because it is a compact and strictly convex set surrounding the origin. In the present paper, considering the indicatrix of a complex Finsler space as a CR hypersurface of the holomorphic tangent space in a fixed point, we introduce and analyze almost contact structures on the complex indicatrix and several of them properties are obtained.

Firstly, in Section 1, we recall some basic notions regarding complex Finsler geometry. By taking $z \in M$ an arbitrary point, the punctured holomorphic tangent bundle T'_zM can be locally viewed as a Kähler manifold and the complex indicatrix is a real hypersurface, i.e. a CR hypersurface of T'_zM. Thus, the characteristic direction, Levi distribution and integrability of the CR distributions are studied in Section 2. By using these, we introduce in Section 3 the natural contact structure associated to the

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CR-structure of the complex indicatrix and we prove that any almost contact structure subordonated to the CR structure is normal. Hence, the complex indicatrix is a Sasakian manifold. However, the natural connection is not the only almost contact structure on the complex indicatrix and using basic transformations we can determine other almost contact structures subordonated to the indicatrix CR structure. In the last Section we describe the existence of some fundamental connections related to the natural contact structure subordonated to the complex indicatrix CR structure. In this sense we will present the action of Tanaka and Tanaka Webster connections on the tangent vectors of the complex indicatrix and with their help we are able to analyze geometric invariants of the complex indicatrix under certain transformations. Most of them are obtained from curvature tensor fields of linear connection and for a pseudo-convex CR-structure, the complex indicatrix in particular, it is defined the Bochner type curvature tensor field which is invariant under gauge transformation of almost contact structure associated to the CR-structure.

Now, we make a short overview of the concepts and terminology used in complex Finsler geometry, as in [1, 15]. Let M be an n - dimensional complex manifold, with $z := (z^k),\ k = 1,..,n$, the complex coordinates on a local chart (U, φ). The complexified of the real tangent bundle $T_{\mathbb{C}}M$ splits into the sum of holomorphic tangent bundle $T'M$ and its conjugate $T''M$, i.e. $T_{\mathbb{C}}M = T'M \oplus T''M$. The holomorphic tangent bundle $T'M$ is in its turn a $2n$-dimensional complex manifold and the local coordinates in a local chart in $u \in T'M$ are $u := (z^k, \eta^k),\ k = 1,..,n$.

Definition 1.1. A *complex Finsler space* is a pair (M, F), with $F : T'M \to \mathbb{R}^+,\ F = F(z, \eta)$ a continuous function that satisfies the following conditions:

(i) F is a smooth function on $\widetilde{T'M} := T'M \setminus \{0\}$;
(ii) $F(z, \eta) \geq 0$, the equality holds if and only if $\eta = 0$;
(iii) $F(z, \lambda\eta) = |\lambda| F(z, \eta)$, $\forall \lambda \in \mathbb{C}$;
(iv) the Hermitian matrix $\big(g_{i\bar{j}}(z, \eta)\big)$ is positive definite, where $g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}$ is the fundamental metric tensor, with $L := F^2$ the complex Lagrangian associated to the complex Finsler function F.

The positivity of the fundamental tensor assures the convexity of the Lagrangian L and the strongly pseudoconvex property of the complex indicatrix in a fixed point $I_z M = \{\eta \mid g_{i\bar{j}}(z, \eta)\eta^i \bar{\eta}^j = 1\}$, for any $z \in M$.

Moreover, from iii. it takes that L is homogeneous with respect to the complex norm, $L(z, \lambda\eta) = \lambda\bar{\lambda} L(z, \eta)$, $\forall \lambda \in \mathbb{C}$, and by applying Euler's formula we get that:

$$\frac{\partial L}{\partial \eta^k}\eta^k = \frac{\partial L}{\partial \bar{\eta}^k}\bar{\eta}^k = L; \quad \frac{\partial g_{i\bar{j}}}{\partial \eta^k}\eta^k = \frac{\partial g_{i\bar{j}}}{\partial \bar{\eta}^k}\bar{\eta}^k = 0 \quad \text{and} \quad L = g_{i\bar{j}}\eta^i \bar{\eta}^j. \tag{1.1}$$

An immediate consequence concerns the following Cartan complex tensors:

$$C_{i\bar{j}k} := \frac{\partial g_{i\bar{j}}}{\partial \eta^k} \qquad \text{and} \qquad C_{i\bar{j}\bar{k}} := \frac{\partial g_{i\bar{j}}}{\partial \bar{\eta}^k}.$$

which have the following properties:

$$\begin{aligned} & C_{i\bar{j}k} = C_{k\bar{j}i} \quad ; \quad C_{i\bar{j}\bar{k}} = C_{i\bar{k}\bar{j}} \quad ; \quad C_{i\bar{j}k} = \overline{C_{j\bar{i}\bar{k}}}; \\ & C_{i\bar{j}k}\eta^k = C_{i\bar{j}\bar{k}}\bar{\eta}^j = C_{i\bar{j}k}\eta^i = C_{i\bar{j}\bar{k}}\bar{\eta}^k = 0. \end{aligned} \tag{1.2}$$

The geometry of complex Finsler spaces consists of the study of geometric objects on the complex manifold $T'M$ endowed with a Hermitian metric structure defined by $g_{i\bar{j}}$. We start by analyzing the sections of the complexified tangent bundle $T_{\mathbb{C}}(T'M) = T'(T'M) \oplus T''(T'M)$, where $T''_u(T'M) = \overline{T'_u(T'M)}$. Let $V(T'M) = span\{\frac{\partial}{\partial \eta^k}\} \subset T'(T'M)$ be the vertical bundle and we introduce the complex non-linear connection, denoted by (c.n.c.), as the supplementary complex subbundle to $V(T'M)$ in $T'(T'M)$, i.e. $T'(T'M) = H(T'M) \oplus V(T'M)$. The horizontal distribution $H_u(T'M)$ is locally spanned by $\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}\}$, where $N_k^j(z,\eta)$ represent the coefficients of a (c.n.c.). Thus, we call the pair $\{\delta_k := \frac{\delta}{\delta z^k}, \dot{\partial}_k := \frac{\partial}{\partial \eta^k}\}$ the adapted frame of the (c.n.c.), which has the dual adapted base $\{\mathrm{d}z^k, \delta\eta^k := \mathrm{d}\eta^k + N_j^k \mathrm{d}z^j\}$.

One fundamental (c.n.c.) of a complex Finsler space is the Chern-Finsler (c.n.c.) ([1],[15]), with $N_j^k = g^{\bar{m}k} \frac{\partial g_{l\bar{m}}}{\partial z^j} \eta^l$, which determines the *Chern-Finsler linear connection*, locally given by the next set of coefficients ([15]) $L_{jk}^i = g^{\bar{l}i} \delta_k(g_{j\bar{l}})$, $C_{jk}^i = g^{\bar{l}i} \dot{\partial}_k(g_{j\bar{l}})$, $L_{\bar{j}k}^{\bar{i}} = 0$, $C_{\bar{j}k}^{\bar{i}} = 0$, where $D_{\delta_k} \delta_j = L_{jk}^i \delta_i$, $D_{\delta_k} \dot{\partial}_j = L_{jk}^i \dot{\partial}_i$, $D_{\dot{\partial}_k} \dot{\partial}_j = C_{jk}^i \dot{\partial}_i$, $D_{\dot{\partial}_k} \delta_j = C_{jk}^i \delta_i$ and $C_{jk}^i \eta^j = C_{jk}^i \eta^k = 0$ from (1.1). Further we will use the following notation $\bar{\eta}^j =: \eta^{\bar{j}}$ to denote a conjugate object.

The *CR structure* attempts to describe intrinsically the property of being a hypersurface in complex space; thus, a *CR manifold* can be considered as an embedded CR manifold (hypersurface and edges of wedges in complex space) or as an abstract CR manifold. Cauchy-Riemann (CR) submanifolds of almost Hermitian manifolds, introduced by A. Bejancu [3, 4, 5], were extended to the Finsler geometry by S. Dragomir in [10, 11]. Thus, a real submanifold $\tilde{M}$ of an almost Hermitian Finsler space (M, g), is a *CR-submanifold* endowed with a pair of complementary Finslerian distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ of $T\tilde{M}$, such that $\mathcal{D}$ is invariant, $J(\mathcal{D}_u) = \mathcal{D}_u$, and $\mathcal{D}^\perp$ is anti-invariant, $J(\mathcal{D}_u^\perp) \subset (T_u\tilde{M})^\perp$, for each $u \in \tilde{M}$, where J is an almost complex structure on $\tilde{M}$. It can be easily noticed that any real hypersurface $\tilde{M}$ of M is a CR-submanifold, having $\mathcal{D}_u^\perp = J(T_u\tilde{M})^\perp$ and $\mathcal{D}$ as the complementary orthogonal distribution of $\mathcal{D}^\perp$.

2 The CR geometry of the complex indicatrix

Let us take (M, F) a complex Finsler manifold, T'_zM its corresponding holomorphic tangent space and F_z the Finsler metric in an arbitrary fixed point $z \in M$. Then, (T'_zM, F_z) can be regarded as a locally complex n - dimensional Minkowski space, with (η^i) its complex coordinate system, $\eta = (\eta^i) = \eta^i \frac{\partial}{\partial z^i}|_z$. Let g be the Hermitian structure on $T'(\widetilde{T'M})$ associated to F_z, which can be extended to a complex bilinear form $\mathcal{G}$, which defines a Hermitian metric on $\widetilde{T'_zM}$, locally given as:

$$\mathcal{G} := \frac{\partial^2 F_z^2}{\partial \eta^i \partial \bar{\eta}^j} \mathrm{d}\eta^j \otimes \mathrm{d}\bar{\eta}^k = g_{j\bar{k}}(z,\eta) \mathrm{d}\eta^j \otimes \mathrm{d}\bar{\eta}^k. \tag{2.1}$$

As in [15], we can can extend, by linearity, a linear connection on M to T_CM, which is isomorphic to $V_C(T'M)$ via vertical lift. We require ∇ to be a compatible complex connection with respect to J, i.e. $\nabla J = 0$, such that ∇ conserves the holomorphic tangent space. Here J represents the natural complex structure

$$J(\dot{\partial}_k) = \mathrm{i}\dot{\partial}_k,\ J(\dot{\partial}_{\bar{k}}) = -\mathrm{i}\dot{\partial}_{\bar{k}}, \qquad \text{with} \quad \mathrm{i} := \sqrt{-1}. \tag{2.2}$$

We can choose ∇ to be the Levi-Civita connection, which is a metrical and symmetric connection and using (1.2) we get the following components:

$$\Gamma^i_{jk} = g^{\bar{h}i}C_{j\bar{h}k} =: C^i_{jk}(\eta); \quad \Gamma^{\bar{i}}_{\bar{j}k} = 0; \quad \Gamma^i_{\bar{j}k} = 0; \quad \Gamma^{\bar{i}}_{jk} = 0.$$

Since $\Gamma^i_{\bar{j}k} = \Gamma^{\bar{i}}_{\bar{j}k} = 0$, it takes that the Levi-Civita connection is Hermitian, and the non-zero coefficients satisfy $C^i_{jk} = C^i_{kj}$ and $C^i_{jk}\eta^j = C^i_{jk}\eta^k = 0$. Taking into consideration that the Levi-Civita connection considered above is equivalent to the linear Chern connection on $\pi^*T'M = span\{\frac{\partial}{\partial z^i}\}$ [2], where $\pi : T'M \to M$ is the natural projection, and since $C^i_{jk} - C^i_{kj} = 0$, we get that $(\widetilde{T'_zM}, F_z)$ is Kählerian and thus ∇ is Kählerian connection, i.e. $\nabla_X(JY) = J\nabla_X Y$.

For an arbitrary fixed point $z \in M$, the unit sphere in (T'_zM, F_z), also called the *complex indicatrix* in z is:

$$I_zM = \{\eta \in T'_zM \mid F(z,\eta) = 1\}.$$

I_zM is a strictly pseudo convex submanifold and since it has only one defining equation which involves the real valued Finsler function F, it is a real hypersurface of the holomorphic tangent bundle T'_zM, and thus a CR-hypersurface, for any $z \in M$.

Let $(u^1, ..., u^{2n-1})$ be local coordinates on I_zM and $\eta^j = \eta^j(u^1, .., u^{2n-1})$, $\forall j \in \{1, .., n\}$ the equations of inclusion $I_zM \overset{i}{\hookrightarrow} \widetilde{T'_zM}$ [11]. By taking $l^j = \frac{1}{F}\eta^j$, $l_j = g_{j\bar{k}}l^{\bar{k}}$, from $F(z, \eta(u)) = 1$, by derivation after u, we get

$$l_j \frac{\partial \eta^j}{\partial u^\alpha} + l_{\bar{j}} \frac{\partial \eta^{\bar{j}}}{\partial u^\alpha} = 0, \quad \alpha \in \{1, \dots, 2n-1\},\ j \in \{1, \dots, n\}. \tag{2.3}$$

The tangent map $i_* : T_R(I_zM) \to T_C(\widetilde{T'_zM})$ acts on tangent vectors of I_zM as

$$i_*\left(\frac{\partial}{\partial u^\alpha}\right) = X_\alpha := \frac{\partial \eta^k}{\partial u^\alpha}\frac{\partial}{\partial \eta^k} + \frac{\partial \bar{\eta}^k}{\partial u^\alpha}\frac{\partial}{\partial \bar{\eta}^k},$$

where X_α is a tangent vector of the complex indicatrix expressed in terms of tangent vectors of $T_C(T'M)$. Considering this and (2.3), we set

$$N = l^j \dot{\partial}_j + l^{\bar{j}} \dot{\partial}_{\bar{j}} \tag{2.4}$$

and thus we obtain $G_R(X_\alpha, N) = 0$, where G_R is the Riemannian metric applied to real vector fields as

$$G_R(X, Y) = \operatorname{Re} \mathcal{G}(X', \overline{Y'}).$$

Here X', $\overline{Y'}$ are the holomorphic, respectively, the anti-holomorphic part of tangent vectors X and Y of the complexified tangent bundle of $T'M$, obtained by $X' = \frac{1}{2}(X - \mathrm{i}JX)$, $\overline{Y'} = \frac{1}{2}(Y + \mathrm{i}JY)$, $\mathrm{i} = \sqrt{-1}$. Consequently, $N \in T_R(I_zM)^\perp$ and $G_R(N, N) = 1$, so that N is the unit normal vector of the indicatrix bundle.

Since each real orientable hypersurface of a Kähler manifold is a CR-submanifold with $J\mathcal{D}^\perp_x = T^\perp_x$, we get that the anti-invariant distribution $\mathcal{D}^\perp$ of the indicatrix of a complex Finsler space in a fixed point must satisfy $J\mathcal{D}^\perp = span\{N\}$, where N is the unit normal vector field to I_zM given in (2.4) and J is the complex structure from (2.2). Then we take the characteristic direction of the complex indicatrix CR structure as

$$\xi = JN = \mathrm{i}\left(l^k\dot{\partial}_k - l^{\bar{k}}\dot{\partial}_{\bar{k}}\right), \quad \mathrm{i} := \sqrt{-1},$$

which is a real tangent unit vector on I_zM, with $\xi = \bar{\xi}$, $N = -J\xi$, $\mathcal{D}^\perp = span\{\xi\}$ and $G_R(\xi,\xi) = 1$. Let then $\mathcal{D}$ be the maximal J-invariant subspace of the tangent space of I_zM, also called the Levi distribution, which is orthogonal to $\mathcal{D}^\perp$, such that

$$T_R(I_zM) = \mathcal{D} \oplus span\{\xi\}. \tag{2.5}$$

Thus, $\dim_R I_zM = 2n-1$ and $\dim_R \mathcal{D} = 2n-2$, since $\dim_C M = n$.

Considering (2.5) and $T_R(T'_zM) = T_R(I_zM) \oplus span\{N\}$ and $T_R(I_zM) = \mathcal{D} \oplus span\{\xi\}$, we can take $\mathcal{D} = T_R(\tilde{M})$, where $\tilde{M}$ is a complex hypersurface of T'_zM, with $\dim_C \tilde{M} = n-1$ and complex unit normal vector $N' = l^j\dot{\partial}_j$. Thus, we have $\mathcal{D} = \mathrm{Re}\{T'\tilde{M} \oplus T''\tilde{M}\}$ and since $T'(T'_zM) = span\{\dot{\partial}_j\}$, there exist the complex projection factors P^i_a such that

$$T'\tilde{M} = span\{Y'_a := P^j_a\dot{\partial}_j\}, \quad a \in \{1,\ldots,n-1\}.$$

Further, we denote by $\mathcal{D}' := T'\tilde{M}$, $\mathcal{D}'' := T''\tilde{M}$, and so $\mathcal{D}\otimes\mathbb{C} = \mathcal{D}' \oplus \mathcal{D}''$.

Thus, having in mind that $Y_a := Y'_a + \overline{Y'_a}$ and $JY_a = \mathrm{i}(Y'_a - \overline{Y'_a})$, we conclude that

$$\mathcal{D} = span\{Y_a := P^j_a\dot{\partial}_j + P^{\bar{j}}_{\bar{a}}\dot{\partial}_{\bar{j}},\ JY_a = \mathrm{i}(P^j_a\dot{\partial}_j - P^{\bar{j}}_{\bar{a}}\dot{\partial}_{\bar{j}})\}.$$

Moreover, since Y'_a and N' are complex tangent vectors, respectively the complex normal vector of the complex hypersurface $\tilde{M}$, the following conditions are fulfilled with respect to the Hermitian metric $\mathcal{G}$ (2.1)

$$P^j_a l_j = 0, \qquad P^{\bar{j}}_{\bar{a}} l_{\bar{j}} = 0 \qquad \text{and} \qquad l^j l_j = 1.$$

In order to analyze if the real or complex integrability conditions of the complex indicatrix distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ are fulfilled, we will need the following results regarding the Lie brackets of tangent vectors of I_zM

$$[\xi, Y_a] = 2\mathrm{Re}\,(\xi(P^j_a) - \tfrac{\mathrm{i}}{2}P^j_a)\dot{\partial}_j; \qquad [\xi, JY_a] = 2\mathrm{Re}(\mathrm{i}\xi(P^j_a) + \tfrac{1}{2}P^j_a)\dot{\partial}_j;$$
$$[Y_a, Y_b] = 2\mathrm{Re}\,(Y_a(P^j_b) - Y_b(P^j_a))\dot{\partial}_j; \qquad [Y_a, JY_b] = 2\mathrm{Re}\,(\mathrm{i}Y_a(P^j_b) - JY_b(P^j_a))\dot{\partial}_j;$$
$$[JY_a, JY_b] = 2\mathrm{Re}\{\mathrm{i}(JY_a(P^j_b) - JY_b(P^j_a))\dot{\partial}_j\}.$$

Moreover, the real metric G_R action on these vectors is

$$\begin{aligned} G_R(Y_a, Y_b) &= G_R(JY_a, JY_b) = \mathrm{Re}(g_{j\bar{k}}P^j_a P^{\bar{k}}_{\bar{b}}) =: \mathrm{Re}(g_{a\bar{b}}) \\ G_R(Y_a, JY_b) &= G_R(JY_a, Y_b) = -\mathrm{Re}(\mathrm{i}g_{j\bar{k}}P^j_a P^{\bar{k}}_{\bar{b}}) =: -\mathrm{Re}(\mathrm{i}g_{a\bar{b}}), \\ G_R(\xi, Y_a) &= G_R(\xi, JY_a) = G_R(N, Y_a) = G_R(N, JY_a) = 0, \end{aligned}$$

and, by denoting $C^m_{ab} = P^k_a P^j_b C^m_{jk}$, we have the Levi-Civita connection action as

(2.6)

$$\nabla_{Y_a} = \tfrac{1}{F}JY_a, \qquad \nabla_{JY_a}\xi = -\tfrac{1}{F}Y_a, \qquad \nabla_\xi\xi = -\tfrac{1}{F}N;$$
$$\nabla_\xi Y_a = 2\mathrm{Re}\{\xi(P^m_a)\dot{\partial}_m\}, \qquad \nabla_\xi Y_a = 2\mathrm{Re}\{\mathrm{i}\xi(P^m_a)\dot{\partial}_m\},$$
$$\nabla_{Y_a}Y_b = 2\mathrm{Re}\{(C^m_{ab} + Y_a(P^m_b))\dot{\partial}_m\}, \qquad \nabla_{Y_a}JY_b = 2\mathrm{Re}\{\mathrm{i}(C^m_{ab} + Y_a(P^m_b))\dot{\partial}_m\},$$
$$\nabla_{JY_a}Y_b = 2\mathrm{Re}\{(\mathrm{i}C^m_{ab} + JY_a(P^m_b))\dot{\partial}_m\}, \qquad \nabla_{JY_a}JY_b = 2\mathrm{Re}\{(-C^m_{ab} + \mathrm{i}JY_a(P^m_b))\dot{\partial}_m\}.$$

If we consider that I_zM a CR-hypersurface of the Kähler manifold T'_zM and if we take into consideration the above results, we can study (as in [17])

(i) the complex involutivity condition $[\Gamma(\mathcal{D}'), \Gamma(\mathcal{D}')] \subset \Gamma(\mathcal{D}')$ of $\mathcal{D}'$, which, according to [12], is equivalent to the integrability of the almost complex structure $\mathcal{D}$

$$[JX, Y] + [X, JY] \in \Gamma(\mathcal{D}) \quad \text{and} \quad [JX, JY] - [X, Y] = J([JX, Y] + [X, JY]);$$

(ii) the real integrability of $\mathcal{D}$, $h(X, JY) = h(JX, Y)$, $\forall X, Y \in \Gamma(\mathcal{D})$ (cf. [5]), which is equivalent to

$$G_R(\nabla_X JY, N) = G_R(\nabla_{JX} Y, N);$$

(iii) the integrability of the anti-invariant distribution $\mathcal{D}^\perp$ (according to [5]):

$$G_R((\nabla_\xi J)\xi, X) = 0,$$

for any $X, Y \in \Gamma(\mathcal{D})$. Thus, we can state

Theorem 2.1. *Let (M, F) be a complex Finsler manifold, $z \in M$ an arbitrary fixed point and I_zM the complex indicatrix. Then the following affirmations take place with respect to the distributions of the CR-hypersurface I_zM of T'_zM:*

(a) *the anti-invariant distribution $\mathcal{D}^\perp$ is integrable;*

(b) *even though the complex CR-structure $\mathcal{D}'$ is integrable, the real invariant distribution $\mathcal{D}$ is neither involutive, nor integrable.*

3 Almost contact structures on the complex indicatrix

In order to introduce almost contact structures on the complex indicatrix I_zM, we will make a short overview of the fundamental notions from the general theory of almost contact structures. Thus, for an odd-dimensional manifold $\tilde{M}$, $\dim_R \tilde{M} = 2n-1$, has an almost cotact structure if its structural group reduces to $U(n) \times 1$. On the other hand, in terms of structure tensors, $\tilde{M}$ has an (ϕ, ξ, θ)-*almost contact structure*, if it admits a tensor field ϕ of type $(1,1)$, the Reeb vector field ξ and a 1-form η, satisfying

$$\phi^2 = -I + \theta \otimes \xi, \quad \theta(\xi) = 1, \quad \phi\xi = 0, \quad \theta \circ \phi = 0,$$

where I is the identity transformation [7]. For a *contact manifold* the 1-form θ satisfies in addition $\theta \wedge (d\theta)^n \neq 0$.

An almost contact structure (ϕ, ξ, θ) is *normal* if

$$N \equiv N_\phi + 2d\theta \otimes \xi = 0,$$

where N_ϕ is the Nijenhuis tensor field of ϕ given as

$$N_\phi(X, Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + \phi^2[X, Y].$$

A *Sasakian manifold* is a normal contact metric manifold, and in some aspects it may be viewed as an odd-dimensional analogue of the Kähler manifold.

If the almost contact manifold $\tilde{M}$ admits a Riemannian metric g such that

$$g(\phi X, \phi Y) = g(X,Y) - \theta(X)\theta(Y),$$

for any vector fields X, Y, we say $\tilde{M}$ has an almost contact metric structure and g is called a compatible metric. Setting $Y = \xi$, we have immediately that $\theta(X) = g(X,\xi)$.

By denoting $D = \{X \in \Gamma(T\tilde{M}),\ \theta(X) = 0\}$, $\dim D = 2n$, we notice that the restriction of ϕ to D is an almost complex structure on D. In [6], A.Bejancu proved

Proposition 3.1. *Let $\tilde{M}$ be a manifold endowed with a normal almost contact structure. Then $\tilde{M}$ is a CR-manifold.*

Futher, we analyze the existence of almost contact structures on the complex indicatrix and the posibility to determine normal almost contact structures on it.

Let us consider M a complex Finsler space, $\dim_C M = n$, z an arbitrary fixed point and I_zM the complex indicatrix in z. Then, $\dim_R I_zM = 2n-1$ and, as in the pervious section, we can determine a CR-structure $(\mathcal{D}, J)$ on I_zM, such that $\mathcal{D}^{\perp} = span\{\xi\}$, with $\xi = JN$. Since $\xi \notin \mathcal{D}$, we can choose the Reeb vector of the almost contact structure to be the characteristic direction ξ and we define the 1-form $\theta(X) = G_R(X,\xi)$ for any $X \in \Gamma(T_R(I_zM))$, more precisely

$$\theta = \frac{\mathrm{i}}{2}(l_{\bar{k}} d\bar{\eta}^k - l_k d\eta^k).$$

It verifies $\theta(\xi) = 1$ and $\theta(X) = 0$, $\forall X \in \Gamma(\mathcal{D})$, so $\ker\theta = \mathcal{D}$, i.e. θ is a pseudo-Hermitian structure on M. Moreover, since I_zM is a pseudoconvex CR manifold, any 1-form θ having this properties is a contact form, such that $\theta \wedge (d\theta)^{n-1} \neq 0$.

By considering the decomposition $X = PX + \theta(X)\xi$, $\forall X \in \Gamma(T_R(I_zM))$, with $PX \in \mathcal{D}$, we define the (1,1) tensor field ϕ as

$$\phi X = J(PX) = JX + \theta(X)N, \qquad \forall\ X \in \Gamma(T_R(I_zM)).$$

We notice that $\phi X = JX$ for $X \in \Gamma(\mathcal{D})$ and it verifies $\phi^2 X = -X + \theta(X)\xi$, $\phi\xi = 0$ and $\theta(\phi X) = 0$, such that we can state

Proposition 3.2. *On the complex indicatrix I_zM of a complex Finsler space it exists a contact structure associated to the CR structure $(\mathcal{D}, J)$, determined by*

$$\phi = J + \theta \otimes N, \qquad \xi = \mathrm{i}(l^k \dot{\partial}_k - l^{\bar{k}} \dot{\partial}_{\bar{k}}), \qquad \theta = \frac{\mathrm{i}}{2}(l_{\bar{k}} d\bar{\eta}^k - l_k d\eta^k), \tag{3.1}$$

which is called the natural contact structure *of the complex indicatrix I_zM.*

An almost contact structure (ϕ, ξ, θ) is *subordonated* to the CR-structure $(I_zM, \mathcal{D})$ if it satisfies $\phi|_{\mathcal{D}} = J$ and $\ker\theta = \mathcal{D}$. Moreover, for a subordonated almost contact structure we can always contruct another almost contact structure subordonated to the same CR-structure which satisfies in addition

$$[\xi, \Gamma(\mathcal{D})] \subset \Gamma(\mathcal{D}), \quad \text{equivalent to} \quad \iota_\xi d\theta = 0 \text{ or } \mathcal{L}_\xi \theta = 0. \tag{3.2}$$

Thus, we notice that the natural contact structure (3.1) is subordonated to the CR structure $(I_zM, \mathcal{D})$, but is not unique. Another one can be constructed on I_zM by taking $\tilde{\theta}$ a pesudohermitian structure, with $\tilde{\theta}(\mathcal{D}) = 0$, the Reeb vector $\tilde{\xi}$ as $\tilde{\theta}(\tilde{\xi}) = 1$, $\iota_{\tilde{\xi}} d\tilde{\theta} = 0$, such that $T_R(I_zM) = \mathcal{D} \oplus span_R\{\tilde{\xi}\}$, and the $(1,1)$ tensor $\tilde{\phi}X = J(X - \tilde{\theta}(X)\tilde{\xi})$, for any $X \in \chi(I_zM)$.

Regarding the normality of the subordonated almost contact on I_zM, we mention the following Theorem from [14], which relates complex involutivity of the CR-structure to the normality condition

Theorem 3.3. *The complex involutivity condition for $(I_zM, \mathcal{D})$ is equivalent to $S \equiv 0$, where S is a $(1,2)$ type tensor field on I_zM defined by*

$$S(X,Y) = N_{\tilde{\phi}}(X,Y) + 2d\tilde{\theta}(X,Y)\tilde{\xi} + \tilde{\theta}(X)\tilde{\phi}(\mathcal{L}_{\tilde{\xi}}\tilde{\phi})Y - \tilde{\eta}(Y)\tilde{\phi}(\mathcal{L}_{\tilde{\xi}}\tilde{\phi})X,$$

where $\mathcal{L}$ represents the Lie derivative.

Thus, for a normal almost contact structure, $\mathcal{L}_\xi \phi = 0$ and the complex involutivity is obtained. Viceversa, besides the complex involutivity of $\mathcal{D}$ we also need $\mathcal{L}_\xi \phi = 0$ to obtain a normal contact structure. However, we have

Theorem 3.4 ([19]). *For an almost contact metric manifold the contact structure (ϕ, ξ, θ, g) is normal if and only if D' and $D'' \oplus \langle \xi \rangle^c$ (or D'' and $D' \oplus \langle \xi \rangle^c$) are integrable and $\mathcal{L}_\xi \theta = 0$, where $D = \ker \theta$ and $D \otimes \mathbb{C} = D' \oplus D''$.*

By using this Theorem, relation (3.2) and the integrability of $\mathcal{D}'$, $\mathcal{D}''$ and $\mathcal{D}^\perp$ from Theorem 2.1, we can state

Proposition 3.5. *Any almost contact metric structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ subordonated to the complex indicatrix CR structure $(I_zM, \mathcal{D})$, in particular the natural one, is normal.*

Therefore, we have

Theorem 3.6. *Let M be a complex Finsler manifold and z an arbitrary fixed point. The complex indicatrix I_zM is a Sasakian manifold.*

Obviously, the 1-form θ which defines the invariant distribution $\mathcal{D}$ is not unique. So, starting from the natural contact structure (3.1), we can determine another almost contact structure associated to the same CR-distribution on $(I_zM, \mathcal{D})$, denoted by (ϕ', ξ', θ'), as

Proposition 3.7. *Two almost contact structures associated to the same CR-structure of I_zM are related by*

$$\theta' = f\theta, \qquad \xi' = A + 1/f\xi, \qquad \phi' + \theta' \otimes JA = \phi, \tag{3.3}$$

where f is a differential function on I_zM and $A \in \Gamma(\mathcal{D})$.

If we require (ϕ', ξ', θ') to be subordonated to the CR-structure, i.e. $[\xi', X] \in \Gamma(\mathcal{D})$, $\forall X \in \Gamma(\mathcal{D})$, vector $A \in \Gamma(\mathcal{D})$ must also fulfil

$$d\theta(A, X) = \frac{1}{f^2} df(X) = \frac{1}{f^2} X(f), \quad \forall X \in \Gamma(\mathcal{D}).$$

A special case of transformation between two almost contact manifolds subordonated to the same almost complex distribution $\mathcal{D}$ is the *gauge transformation* of the 1-form θ, given as $\theta \mapsto \tilde{\theta} = \varepsilon e^f \theta$, with $f \in C^\infty(I_zM)$ and $\varepsilon = \pm 1$. It can be easily noticed that 1-forms θ and $\tilde{\theta}$ define the same distribution $\mathcal{D}$.

Proposition 3.8. *Two almost contact structures (ϕ, ξ, θ), $(\tilde{\phi}, \tilde{\xi}, \tilde{\theta})$ are subordonated to the same strict pseudoconvex CR-structure iff it exists a function $f \in C^{\infty}(I_z M)$ such that*

$$\tilde{\theta} = \varepsilon e^{f} \theta, \qquad \tilde{\xi} = \varepsilon e^{-f}(\xi + \phi A), \qquad \tilde{\phi} = \phi + \theta \otimes A,$$

with $A \in \mathcal{D}$ defined by $d\theta(\phi A, X) = df(X) = X(f), \forall X \in \Gamma(\mathcal{D})$.

Remark 3.1. In general, the complex involutivity is invariant under gauge transformations.

4 Associated connections on the complex indicatrix

In this section we will describe the existence and the action of some canonical connections closely related to the almost contact structures subordonated to the complex indicatrix CR-structure $(I_z M, \mathcal{D})$. More precisely, we present the Tanaka and Tanaka-Webster connections. With respect to these connections, the Levi distribution $\mathcal{D}$ and the complex structure J are parallel.

We start with the Tanaka connection associated to the natural contact structure (3.1), which is subordonated to the to the complex indicatrix CR-structure $(I_z M, \mathcal{D})$.

By adapting Tanaka's result [20] for the complex indicatrix case we get

Theorem 4.1. *Let (ϕ, ξ, θ) be the natural almost contact structure (3.1) subordonated to the CR-structure $(I_z M, \mathcal{D})$. Then it exists an unique linear connection $\overset{t}{D}$ such that*

$$\overset{t}{D}\phi = 0,\ \overset{t}{D}\xi = 0,\ \overset{t}{D}\theta = 0,\ \overset{t}{D}|_{\mathcal{D}} G_R = 0,\ T_{\mathcal{D}} = 0,\ \tau = -\frac{1}{2}\phi \mathcal{L}_{\xi}\phi, \tag{4.1}$$

where, $G_\theta(X, Y) = d\theta(X, JY) = G_R(X, Y)$, $\forall X, Y \in \mathcal{D}$ is the Levi metric and $\tau(X) = T(\xi, X)$, $\forall X \in \chi(I_z M)$.

Here we used that

$$d\theta = \frac{\mathrm{i}}{F}\left(g_{j\bar{k}} - \frac{1}{2} l_j l_{\bar{k}}\right) d\eta^j \wedge d\bar{\eta}^k$$

and thus $d\theta(X, Y) = \frac{1}{F} G_R(\phi X, Y)$, which are equal for $F = 1$ on $I_z M$.

In order to determine the Tanaka connection expression for the tangent vectors of $I_z M$, we take into consideration that, according to [15], a linear connection on M extends by linearity to $T_c M$, and implicitly on $I_z M$ aswell, and is well defined by the next set of coefficients:

$$\overset{t}{D}_{\dot{\partial}_k} \dot{\partial}_j = A^i_{jk} \dot{\partial}_i + A^{\bar{\mathrm{i}}}_{jk} \dot{\partial}_{\bar{\mathrm{i}}}; \quad \overset{t}{D}_{\dot{\partial}_k} \dot{\partial}_{\bar{j}} = A^i_{\bar{j}k} \dot{\partial}_i + A^{\bar{\mathrm{i}}}_{\bar{j}k} \dot{\partial}_{\bar{\mathrm{i}}};$$
$$\overset{t}{D}_{\dot{\partial}_{\bar{k}}} \dot{\partial}_j = A^i_{j\bar{k}} \dot{\partial}_i + A^{\bar{\mathrm{i}}}_{j\bar{k}} \dot{\partial}_{\bar{\mathrm{i}}}; \quad \overset{t}{D}_{\dot{\partial}_{\bar{k}}} \dot{\partial}_{\bar{j}} = A^i_{\bar{j}\bar{k}} \dot{\partial}_i + A^{\bar{\mathrm{i}}}_{\bar{j}\bar{k}} \dot{\partial}_{\bar{\mathrm{i}}},$$

where $\overline{A^i_{jk}} = A^{\bar{\mathrm{i}}}_{\bar{j}\bar{k}}$, $\overline{A^{\bar{\mathrm{i}}}_{\bar{j}k}} = A^i_{j\bar{k}}$, $\overline{A^{\bar{\mathrm{i}}}_{jk}} = A^i_{\bar{j}\bar{k}}$, $\overline{A^{\bar{\mathrm{i}}}_{j\bar{k}}} = A^i_{\bar{j}k}$, which come from $\overline{D_X Y} = D_{\bar{X}} \bar{Y}$.

Demanding $\overset{t}{D}$ to satisfy the Tanaka connection conditions (4.1), we find the following non-zero coefficients

$$\begin{aligned} A^i_{jk} &= C^i_{jk} - \tfrac{1}{F} l_j \delta^i_k - \tfrac{1}{F} l_k \delta^i_j \quad \text{and} \\ A^i_{j\bar{k}} &= \tfrac{1}{F} g_{j\bar{k}} l^i - \tfrac{2}{F} l_j l_{\bar{k}} l^i. \end{aligned}$$

Thus, Tanaka connection action on tangent vectors of I_zM is

$$\begin{aligned} \overset{t}{D}_{Y_a}\xi &= \overset{t}{D}_{JY_a}\xi = \overset{t}{D}_{\xi}\xi = 0, \\ \overset{t}{D}_{\xi}Y_a &= 2\mathrm{Re}\{\xi(P^m_a) - \tfrac{\mathrm{i}}{F}P^m_a\}\dot{\partial}_m; \\ \overset{t}{D}_{\xi}JY_a &= 2\mathrm{Re}\{\mathrm{i}\xi(P^m_a) + \tfrac{1}{F}P^m_a\}\dot{\partial}_m; \\ \overset{t}{D}_{Y_a}Y_b &= 2\mathrm{Re}\{Y_a(P^m_b) + C^m_{ab} + \tfrac{1}{F}g_{b\bar{a}}l^m\}\dot{\partial}_m; \\ \overset{t}{D}_{Y_a}JY_b &= 2\mathrm{Re}\,\mathrm{i}\{Y_a(P^m_b) + C^m_{ab} + \tfrac{1}{F}g_{b\bar{a}}l^m\}\dot{\partial}_m; \\ \overset{t}{D}_{JY_a}Y_b &= 2\mathrm{Re}\{JY_a(P^m_b) + \mathrm{i}[C^m_{ab} - \tfrac{1}{F}g_{b\bar{a}}l^m]\}\dot{\partial}_m; \\ \overset{t}{D}_{JY_a}JY_b &= 2\mathrm{Re}\{\mathrm{i}JY_a(P^m_b) - C^m_{ab} + \tfrac{1}{F}g_{b\bar{a}}l^m\}\dot{\partial}_m. \end{aligned} \tag{4.2}$$

with $C^m_{ab} = P^j_a P^k_b C^m_{jk}$, $g_{ab} = P^j_a P^k_b g_{jk}$ and $g_{b\bar{a}} = P^j_b P^{\bar{k}}_{\bar{b}} g_{j\bar{k}}$.

In the following we study the Tanaka Webster connection. Starting from the natural contact structure (3.1) subordonated to the complex indicatrix CR structure $(I_zM, \mathcal{D})$, we adjust Tanaka-Webster's result [12] for the I_zM case as

Theorem 4.2. *Let (ϕ, ξ, θ) be the natural almost contact structure (3.1) subordonated to the CR-structure $(I_zM, \mathcal{D})$. Then it exists an unique linear connection $\overset{w}{D}$ such that*

(i) $\mathcal{D}$ is prallel w.r.t. $\overset{w}{D}$, i.e. $\overset{w}{D}_X\Gamma(\mathcal{D}) \subseteq \Gamma(\mathcal{D})$, for any $X \in \chi(I_zM)$;
(ii) $\overset{w}{D}\phi = 0$, $\overset{w}{D}G_R = 0$;
(iii) the torsion T of $\overset{w}{D}$ is pure, i.e.

$$T(Z, W) = 0,\ T(Z, \bar{W}) = 2d\theta(Z, \bar{W}),\ \forall Z, W \in \Gamma(\mathcal{D}'),$$

and $\tau \circ J + J \circ \tau = 0$, with $\tau(X) = T(\xi, X)$.

The Tanaka-Webster connection is pseudo-Hermitian, analogous to the Levi-Civita connection in Riemannian geometry and to the Chern connection in the Hermitian geometry. τ tensor in also called the pseudo-Hermitian torsion and measures the deviation from the normality of the almost contact structure subordonated to the indicatrix CR-structure. However, using Proposition 3.5, we have $\tau \equiv 0$ on I_zM.

By similar arguments as before, for the Tanaka connection case, we find that Webster connection is determined by the following non-zero coefficients

$$A^i_{jk} = C^i_{jk} - \frac{1}{F}l_j\delta^i_k - \frac{1}{F}l_k\delta^i_j \ \text{ and } \ A^i_{j\bar{k}} = \frac{1}{F}g_{j\bar{k}}l^i - \frac{1}{F}l_j l_{\bar{k}} l^i.$$

Thus, we get that Webster connection action on tangent vectors of I_zM coincide with $\overset{t}{D}$ action, the only exception being for $\overset{w}{D}_\xi \xi = \frac{1}{F}N$.

Moreover, using this observation, relations (2.6), (4.2) and

$$\begin{aligned}
\theta(\nabla_{Y_a} Y_b) &= \theta(\nabla_{JY_a} JY_b) &&= \tfrac{1}{F}\mathrm{Re}(\mathrm{i}\, g_{b\bar{a}}),\\
\theta(\nabla_{JY_a} Y_b) &= -\theta(\nabla_{Y_a} JY_b) &&= \tfrac{1}{F}\mathrm{Re}(g_{b\bar{a}}),\\
\theta(\nabla_\xi Y_a) &= \theta(\nabla_\xi JY_a) &&= 0,
\end{aligned}$$

we deduce the relation between the Levi-Civita connection ∇, Tanaka connection $\overset{t}{D}$ and Webster Tanaka connection $\overset{w}{D}$ as:

$$\begin{aligned}
&\overset{w}{D}_X Y = \overset{t}{D}_X Y + \tfrac{1}{F}\theta(X)\theta(Y)N \qquad \text{and}\\
&\nabla_X Y = \overset{w}{D}_X Y + \Omega_\theta(X,Y)\xi + \theta(\nabla_X \phi Y)N + \tfrac{1}{F}[\theta(X)JY + \theta(Y)JX],
\end{aligned}$$

where $\Omega_\theta(X,Y) = G_R(X,\phi Y)$.

At the end we analyze the Bochner type curvature tensor as being the complex analogous of the conform Weyl curvature tensor, which is a pseudoconform invariant of a CR manifold. More precisely, the Bochner tensor is the fourth curvature invariant given by S.S. Chern and J. Moser in [9].

Using the Tanaka connection, the (1,2) type Bochner curvature tensor on the CR structure of complex indicatrix $(I_zM, \mathcal{D})$ is given by

$$\begin{aligned}
B(X,Y)Z = \;& R(X,Y)Z + l(Y,Z)X - l(X,Z)Y + m(Y,Z)JX\\
& -m(X,Z)JY + G_R(Y,Z)LX - G_R(X,Z)LY\\
& +G_R(JY,Z)MX - G_R(JX,Z)MY\\
& -2\{m(X,Y)JZ + G_R(JX,Y)MZ\},
\end{aligned}$$

where

$$\begin{aligned}
l(X,Y) &= -\tfrac{1}{2(n+1)}s(X,Y) + \tfrac{1}{8n(n+1)}\rho G_R(X,Y),\\
m(X,Y) &= -\tfrac{1}{2(n+1)}s(JX,Y) + \tfrac{1}{8n(n+1)}\rho G_R(JX,Y),
\end{aligned}$$

with s the usual Ricci tensor on I_zM, $\rho = \operatorname{trace} S$, with $G_R(SX,Y) = s(X,Y)$, and $G_R(LX,Y) = l(X,Y)$, $G_R(MX,Y) = m(X,Y)$, for any $X,Y \in \mathcal{D}$.

Remark 4.1. The Bochner type tensor is invariant under gauge transformation.

Moreover, we can easily verify that

$$\sum_{(X,Y,Z)} B(X,Y)Z = 0, \quad B(JX,JY) = B(X,Y), \quad B(X,Y)J = JB(X,Y),$$
$$\operatorname{trace}(X \mapsto B(X,Y)Z) = 0.$$

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Author's address:

Elena Popovici
Transilvania University, Department of Mathematics and Informatics,
Iuliu Maniu 50, Braşov 500091, Romania.
E-mail: elena.c.popovici@gmail.com