Two-time logistic map

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Abstract. The aim of this paper is three-fold: (i) To explain the meaning of "multitime". (ii) To formulate and solve problems concerning the most important non-linear multiple recurrence equation, called multitime logistic map recurrence. (iii) To define the Feigenbaum constant for a two-parameter multiple map.

M.S.C. 2010: 65Q10, 65Q30.
Key words: multitime; two-time logistic map; complex time logistic map; Feigenbaum constant.

1 Multitime concept in Physics and Mathematics

The adjective "multitime" was introduced in Physics by Dirac (1932, [4, 5]), and later was used in Mathematics by [1, 3], [10]-[12], [15, 14, 16], [17]-[22], etc. This view is an answer to the question: the time can be multidimensional?

To underline the sense of "multitime", we collect the following remarks.

1) In relativity physics problems, we use a two-time $t = (t^1, t^2)$, where t^1 means the *intrinsic time* and t^2 is the *observer time*.

2) Multitime wave functions were first considered by Dirac via m-time evolution equations

$$i\hbar \frac{\partial \psi}{\partial t^\alpha} = H_\alpha \psi.$$

The Dirac PDE system is completely integrable if and only if $[H_\alpha, H_\beta] = 0$ for $\alpha \neq \beta$. This condition is easy to achieve for non-interacting particles and becomes tricky in the presence of interaction.

We add the evolution problems with multivariate evolution parameter (called "multitime"), where is no reason to prefer one coordinate to another. Here we can include the description of torsion of prismatic bars, the maximization of the area surface for given width and diameter, etc.

3) The oscillators are very important in engineering and communications. For example, voltage-controlled oscillators, phase-locked loops, lasers etc., abound in wireless

BSG Proceedings 23. The International Conference "Differential Geometry - Dynamical Systems" DGDS-2015, October 8-11, 2015, Bucharest-Romania, pp. 75-83.

and optical systems. A new approach for analyzing frequency and amplitude modulation in oscillators was recently achieved using a novel concept, *warped time*, within a multitime PDE framework [15]. To explain this idea from our point of view, we start with a single-time wave front $y(t) = \sin\left(\frac{2}{T_1}t\right)\ \sin\left(\frac{2}{T_2}t\right)$, $T_1 = 0.02$ s; $T_2 = 1$ s, where the two tones are at frequencies $f_1 = \frac{1}{T_1} = 50Hz$ and $f_2 = \frac{1}{T_2} = 1Hz$. Here there are 50 times faster varying sinusoids of period T_1 modulated by a slowly-varying sinusoid of period T_2. Then we build a two-variable representation of $y(t)$, obtained by the rules: for the fast-varyingparts of $y(t)$, the expression $\frac{2}{T_1}\, t$ is replaced by a new variable t^1; for the slowly-varying parts, by t^2. It appears a new periodic function of two variables, $\hat{y}(t^1, t^2) = \sin t^1 \sin t^2$, motivated by the wide separated time scales. Inspection of the two-time (two-variable) wavefront $\hat{y}(t^1, t^2)$ directly provides information about the slow and fast variations of $y(t)$ more naturally and conveniently than $y(t)$ itself.

4) The known evolution laws in physical theories are single-time evolution laws (ODEs) or multitime evolution laws (PDEs). To change a single-time evolution into a multitime evolution, we have used two ideas: (i) we accepted that the time t is a C^∞ function of certain parameters, let say $t = t(s_1, ..., s_m)$; (ii) we replaced the simple operator of time derivative by multitime partial derivative along a given direction or by multitime second order differential operator in the sense of differential geometry (Hessian, Laplacian etc).

To give an example, let us show that m-flows can be generated by 1-flows and reparametrizations. We start with a C^1 non-autonomous flow $\dot{x}(\tau) = X(\tau, x(\tau))$ and a C^1 substitution of independent variable $\tau = \varphi(t)$, $t = (t^1, ..., t^m)$. Then the flow is changed into an m-flow consisting in the PDE system

$$\frac{\partial x}{\partial t^\alpha}(\varphi(t^1, ..., t^m)) = X(\varphi(t^1, ..., t^m), x(\varphi(t^1, ..., t^m)))\frac{\partial \varphi}{\partial t^\alpha}(t).$$

Conversely, a PDE system of the form

$$\frac{\partial x}{\partial t^\alpha}(t) = X(t, x(t))\varphi_\alpha(t)$$

is completely integrable iff there exists a scalar function φ such that $\varphi_\alpha(t) = \frac{\partial \varphi}{\partial t^\alpha}(t)$.

The most important cases are:

(i) the linear dependence,

$$\varphi(t) = t^1 + ... + t^m, \text{ when } \varphi_\alpha(t) = 1;$$

(ii) the volumetric dependence,

$$\varphi(t) = t^1 \cdots t^m, \text{ when } \varphi_\alpha(t) = t^1 \cdots \widehat{t^\alpha} \cdots t^m.$$

2 Multiple time dimensions

2.1 Additive decomposition of time

In traditional time series analysis it is often assumed that a time series y_t can be additively decomposed into four components: *trend, season, cycle, and irregular components*, i.e.,

$$y_t = T_t + S_t + C_t + I_t,$$

where T_t represents the *trend* in y_t at time t, S_t the *seasonal effect* at time t, C_t the *cyclical effect* at time t and I_t the *irregular effect* at time t.

2.2 Multiple time scales

In mathematics and physics, *multiple-scale analysis* (also called the *method of multiple scales*) comprises techniques used to construct uniformly valid approximations to the solutions of perturbation problems, both for small as well as large values of the independent variables.

We consider t like an evolution parameter. The method of multiple scales (MMS) is a more general approach in which we introduce one or more new "slow" time variables for each time scale of interest in the problem. For example: let $0 < \epsilon \leq 1$ be a parameter and introduce the slow scale (a strained variable) $t^1 = t\epsilon$, and a slower one , $t^2 = t\epsilon^2$. Then there appears the three-time $(t^0 = t, t^1 = t\epsilon, t^2 = t\epsilon^2)$.

2.3 Multiple time dimensions

The possibility that there might be more than one dimension of time has occasionally been discussed in physics and philosophy. Special relativity describes space-time as a manifold whose metric tensor has a negative eigenvalue. This corresponds to the existence of a "time-like" direction. A metric with multiple negative eigenvalues would correspondingly imply several time-like directions, i.e., multiple time dimensions, but there is no consensus regarding the relationship of these extra "times" to time as conventionally understood.

2.4 Philosophy of time

An Experiment with Time [8] describes an ontology in which there is an infinite hierarchy of conscious minds, each with its own dimension of time and able to view events in lower time dimensions from outside.

The conceptual possibility of multiple time dimensions has also been raised in modern analytic philosophy. The English philosopher John G. Bennett posited a *six-dimensional Universe* with the usual three spatial dimensions and three time-like dimensions that he called *time, eternity and hyparxis.* Time is the sequential chronological time that we are familiar with. The hypertime dimensions called eternity and hyparxis are said to have distinctive properties of their own. Eternity could be considered cosmological time or timeless time. Hyparxis is supposed to be characterized as an ableness-to-be and may be more noticeable in the realm of quantum processes.

The conjunction of the two dimensions of time and eternity could form a hypothetical basis for a multiverse cosmology with parallel universes existing across a plane of vast possibilities. The third time-like dimension hyparxis could allow for the theoretical existence of sci-fi possibilities such as time travel, sliding between parallel worlds and faster-than-light travel.

3 Two-time logistic map

3.1 Single-time logistic map

The single-time logistic map is a polynomial mapping (equivalently, recurrence relation) of degree 2, often cited as an archetypal example of how complex, chaotic behavior can arise from very simple non-linear dynamical equations. The map was popularized in a seminal paper [13], in part as a discrete-time demographic model analogous to the logistic equation first created by Verhulst [23]-[24] (see also [25, 9]).

Definition 3.1. Let $x(\tau)$ be a number between zero and one that represents the ratio of existing population to the maximum possible population. The single-time discrete recurrence

$$x(\tau+1) = rx(\tau)(1-x(\tau)),\ \tau \in \mathbb{N}, \tag{1}$$

is called single-time discrete logistic map.

This nonlinear difference equation is intended to capture two universality screenplay effects: (i) *reproduction* where the population will increase at a rate proportional to the current population when the population size is small; (ii) *starvation* (density-dependent mortality) where the growth rate will decrease at a rate proportional to the value obtained by taking the theoretical "carrying capacity" of the environment less the current population. However, as a demographic model the logistic map has the pathological problem that some initial conditions and parameter values lead to negative population sizes. The behavior of the evolution $x(\tau)$ is strongly dependent on the parameter r.

A typical application of the logistic equation is a common model of population growth. The wealth of information gathered on this map in the last twenty years revealed its very complicated dynamics and universality.

3.2 Multitime logistic map with imposed path

Recently, we started a study of multitime recurrences (see [10]-[12], introducing recurrences with imposed path, diagonal recurrences etc. Now, we introduce and study the *two-time logistic map.*

Definition 3.2. Let $x(t)$, $t = (t^1, t^2)$, be a number between zero and one that represents the ratio of existing population to the maximum possible population. The recurrence system (of degree two),

$$x(t+1_\alpha) = rx(t)(1-x(t)),\ t = (t^1,t^2) \in \mathbb{N}^2,\ x(t) \in \mathbb{R},\ \alpha = 1, 2, \tag{3.1}$$

(whatever parameter t^1, t^2 increases, evolution is the same), is called a *two-time (double) logistic map* with imposed path.

This is included in the general recurrence described in the next

Proposition 3.1. *We consider the function $f\colon M \to M$ and the multitime recurrence*

$$x(t+1_\alpha) = f(x(t)), \quad \forall t = (t^1, t^2, \dots, t^m) \in \mathbb{N}^m,\ \forall \alpha \in \{1, 2, \dots, m\}, \tag{3.2}$$

with unknown function $x\colon \mathbb{N}^m \to M$.

We fix $x_0 \in M$. *If* $(y(\tau))_{\tau\in\mathbb{N}}$ *is the sequence in* M *which verifies the recurrence*

$$y(\tau+1) = f(y(\tau)), \quad \forall\tau \in \mathbb{N} \tag{3.3}$$

and the condition $y(0) = x_0$, *then the solution of the recurrence* (3.2), *which satisfies* $x(0) = x_0$, *is*

$$x\colon \mathbb{N}^m \to M, \quad x(t) = y(|t|), \quad \forall t \in \mathbb{N}^m, \quad \textit{where } |t| = t^1 + t^2 + \ldots + t^m.$$

These are geometrical models on $\mathbb{N}^m$ which we carry from our existence and space.

The recurrence (3.1) is obtained setting $M = \mathbb{R}$, $f(x) = rx(1-x)$, $m = 2$. For $m = 2$, the foregoing Proposition 3.1 imposes that the time variable τ that dictates a population growth is $\tau = t^1 + t^2$, where t^1 is a *physical component* and t^2 is an *informational component* of each species. The graph of the solution $x = x(t^1, t^2)$ is a ruled surface (cylinder surface) of network type. The order after τ is moved into the order after the rulings $t^1 + t^2 = \tau$.

Generally, the parameter $t^1 + t^2 = \tau$ is moving back and forth since the pairs (t^1, t^2) are not totally ordered. The two-time nonlinear recurrence equation is intended to capture two effects: (i) reproduction where the population will increase at a rate proportional to the current population when the population size is small; (ii) starvation (density-dependent mortality) when the growth rate will decrease at a rate proportional to the value obtained by taking the theoretical "carrying capacity" of the environment less the current population.

This two-time recurrence is an archetypal example of how complex, chaotic behavior can arise from very simple non-linear bi-temporal recurrence equations.

Proposition 3.2. *If* $r = 2$ *and* $\alpha = 1, 2$, *then the solution of the recurrence* (3.1) *is*

$$x(t) = \frac{1}{2} - \frac{1}{2}(1-2x_0)^{2^{t^1+t^2}},$$

for any initial condition x_0.

Suppose $x_0 \in [0,1)$. Since $(1-2x_0) \in (-1,1)$, for any value of x_0, other than the unstable fixed point 0, the term $(1-2x_0)^{2^{t^1+t^2}}$ goes to 0 as $||(t^1,t^2)||$ goes to infinity, so $x(t)$ goes to the stable fixed point $\frac{1}{2}$.

If $x_0 \notin [0,1)$, then $x(t)$ goes to $-\infty$ as $||(t^1,t^2)||$ goes to infinity.

3.3 Solution of multitime logistic map recurrence for $r = 4$

Let m be arbitrary, not necessarily 2. Let us give complete solving of the problem

$$x(t+1_\alpha) = 4x(t)(1-x(t)),\ x(0,0,\ldots,0) = x_0, \quad \forall t \in \mathbb{N}^m,\ \forall \alpha \in \{1,2,\ldots,m\}, \tag{3.4}$$

discussing after the initial condition. In this case, the recurrence (3.3) is

$$y(\tau+1) = 4y(\tau)(1-y(\tau)),\ y(0) = x_0, \forall\tau \in \mathbb{N}. \tag{3.5}$$

There exist three cases:

i) $x_0 \in [0,1]$. Let $\varphi = \arcsin\sqrt{x_0}$. One verifies immediately that $y(\tau) = \sin^2(2^\tau\varphi)$ is the solution of the recurrence (3.5) that verifies $y(0) = x_0$. According to proposition 3.1, it follows

$$x(t) = \sin^2(2^{|t|}\varphi), \quad \forall t \in \mathbb{N}^m.$$

Now, for interpretation, let $\theta = \dfrac{\varphi}{\pi}$. For rational number θ, after a finite number of iterations, $x(t)$ maps into a periodic sequence. But almost all numbers θ are irrational, and, for irrational θ, the sequence $x(t)$ never repeats itself-it is non-periodic. This solution equation clearly proves the two key features of chaos - stretching and folding: the factor $2^{t^1+t^2}$ shows the exponential growth of stretching, which results in sensitive dependence on initial conditions, while the squared sine function keeps $x(t)$ folded within the range $[0,1]$.

ii) $x_0 < 0$. Denote by φ the unique number which verifies $\sinh\varphi = \sqrt{-x_0}$, i.e.,

$$\varphi = \ln(\sqrt{-x_0} + \sqrt{1-x_0}).$$

Obviously, $\varphi > 0$. We immediately check that $y(\tau) = -\sinh^2(2^\tau\varphi)$ is the solution of the problem (3.5). According to proposition 3.1, it follows

$$x(t) = -\sinh^2(2^{|t|}\varphi), \quad \forall t \in \mathbb{N}^m.$$

iii) $x_0 > 1$.

Then $y(1) = 4x_0(1-x_0) < 0$, and the sequence $z(\tau) = y(\tau+1)$, $\forall\tau \in \mathbb{N}$ satisfies

$$z(\tau+1) = 4z(\tau)(1-z(\tau)),\ z(0) = y(1) = 4x_0(1-x_0) < 0,\ \forall\tau \in \mathbb{N}.$$

We note that the sequence z is in the situation from case ii). We infer

$$\varphi = \ln(\sqrt{-z(0)} + \sqrt{1-z(0)}) \quad \Rightarrow \quad \varphi = \ln(2\sqrt{x_0(x_0-1)} + 2x_0 - 1).$$

Hence $z(\tau) = -\sinh^2(2^\tau\varphi)$, $\forall\tau \in \mathbb{N}$, and since $y(\tau) = z(\tau-1)$, $\forall\tau \geq 1$, it follows $y(\tau) = -\sinh^2(2^{\tau-1}\varphi)$, $\forall\tau \geq 1$. For $t \in \mathbb{N}^m$, $t \neq 0$, we have $|t| \geq 1$, which yields

$$x(t) = y(|t|) = -\sinh^2(2^{|t|-1}\varphi).$$

We conclude that, if $x_0 > 1$, then

$$x(t) = \begin{cases} -\sinh^2(2^{|t|-1}\varphi), & \text{if } t \neq 0 \\ x_0, & \text{if } t = 0, \end{cases}$$

where $\varphi = \ln(2\sqrt{x_0(x_0-1)} + 2x_0 - 1)$. Moreover, we remark that $2\sqrt{x_0(x_0-1)} + 2x_0 - 1 = (\sqrt{x_0} + \sqrt{x_0-1})^2$, and hence $\varphi = 2\ln(\sqrt{x_0} + \sqrt{x_0-1})$. Therefore, for $x_0 > 1$, we have

$$x(t) = \begin{cases} -\sinh^2(2^{|t|}\widetilde{\varphi}), & \text{if } t \neq 0 \\ x_0, & \text{if } t = 0, \end{cases}$$

where $\widetilde{\varphi} = \ln(\sqrt{x_0} + \sqrt{x_0-1})$.

3.4 The first Feigenbaum constant

We define the Feigenbaum constant as the limiting ratios of each bifurcation interval to the next between every period doubling, of a two-parameter multiple map

$$x(t+1_\alpha) = F(x(t), r),$$

where $F(x; r)$ is a function parameterized by the bifurcation parameter r. For that we have in mind the model in [2].

Definition 3.3. A function $f\colon \mathbb{N}^2 \to M$ is called double-periodic if there exists $(T_1, T_2) \in \mathbb{N}^2 \setminus \{0\}$ such that $f(t + T_\alpha \cdot 1_\alpha) = f(t), \quad \forall t \in \mathbb{Z}^2, \quad \forall \alpha \in \{1, 2\}$, i.e., $T_1 \cdot 1_1$, $T_2 \cdot 1_2$ are periods for the function f.

Definition 3.4. Let r_k be discrete values of r at the (k)-th period doubling, i.e., at r_k, the sum $T_1 + T_2$ becomes 2^k. The limit

$$\delta = \lim_{k\to\infty} \frac{r_{k-1} - r_{k-2}}{r_k - r_{k-1}} = 4.669201609...$$

is called the Feigenbaum constant.

3.5 The two-time diagonal logistic map

We have at least two kinds of diagonal logistic maps:

(i) The *double logistic map of degree two* induces a *diagonal logistic map of degree four* as follows: $x(t + \mathbf{1}) = x(t + 1_1 + 1_2)$ equals both $rx(t + 1_1)(1 - x(t + 1_1))$ and $rx(t + 1_2)(1 - x(t + 1_2))$. Finally, we obtain an induced diagonal logistic map

$$x(t + \mathbf{1}) = r^2 x(t)(1 - x(t))(1 - rx(t)(1 - x(t))).$$

The initial conditions on axes, for this induced diagonal logistic map of degree four, are induced by the initial condition $x(0, 0) = x_0$ of double logistic map of degree two. For $r = 2$, we find the 1-parameter solution $x(t) = \frac{1}{2} - \frac{1}{2}\, a^{2^{t^1+t^2}}$.

(ii) A direct *diagonal logistic map of degree two* is $x(t + \mathbf{1}) = rx(t)(1 - x(t))$. For $r = 2$, we find a 2-parameters solution $x(t) = \frac{1}{2} - \frac{1}{2}\, a^{2^{t^1}} b^{2^{t^2}}$.

3.6 The areal two-time logistic map

Since the nonautonomous two-time recurrence

$$x(t + \mathbf{1}) = \frac{1}{r} - \frac{1}{r}(1 - 2x(t))^{2^{t^1+t^2+1}},\ t = (t^1, t^2) \in \mathbb{N}^2,\ x(t) \in \mathbb{R},$$

with $r = 2$, has the solution $x(t) = \frac{1}{2} - \frac{1}{2} a^{2^{t^1 t^2}}$, we shall use the name *areal* which is coming from the product $t^1 t^2$.

The two-time hyperbolic recurrence

$$\ln(1 - 2x(t)) \ln(1 - 2x(t + \mathbf{1})) = r \ln(1 - 2x(t + 1_1)) \ln(1 - 2x(t + 1_2))$$

is equivalent to

$$y(t)y(t + \mathbf{1}) = ry(t + 1_1)y(t + 1_2),$$

via $y(t) = \ln(1 - 2x(t))$, $t = (t^1, t^2)$.

If $r = 2^k$, $k \in \mathbb{R}$, then a solution of the areal recurrence is $y(t) = 2^{k\, t^1 t^2}$.

3.7 The complex time logistic map

Let us accept the complex time $z = t^1 + it^2$, as shown in the model of scalar waves. The observer has access to the real part t^1 and the information flow is placed in imaginary part t^2. The "imaginary time" is undetectable directly by experiment, but involved in the evolution of physical systems (we should somehow detect it in the structure of real phenomena, patterns, behavior of living systems etc). According Drăgănescu's theory [6]-[7], the components (t^1, t^2) belongs to the so called *ortho-existence*.

Mathematically, we organize the plane Ot^1t^2 as complex plane $z = t^1 + it^2$. We introduce the recurrence (with imposed path)

$$x(z+1) = x(z+i) = rx(z)(1 - x(z)),$$

with real unknown sequence $x(z)$.

We use the real number

$$\varphi(z) = \frac{(1-i)z + (1+i)\bar{z}}{2} = \frac{e^{-\frac{i\pi}{4}} z + e^{\frac{i\pi}{4}} \bar{z}}{\sqrt{2}}.$$

Then we outline the following cases

(i) for $r = 2$, we obtain a real solution $x(z) = \frac{1}{2} - \frac{1}{2}(1 - 2x_0)^{2^{\varphi(z)}}$;

(ii) for $r = 4$, we find a real solution $x(z) = \sin^2(2^{\varphi(z)} \arcsin\sqrt{x_0})$.

Acknowledgments. Partially supported by University Politehnica of Bucharest, by UNESCO Chair in Geodynamics, "Sabba S. Ştefănescu" Institute of Geodynamics, Romanian Academy and by Academy of Romanian Scientists.

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