Hyperbolic median and its applications

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Abstract. The hyperbolic median is the point which minimizes the weighted sum of the geodesic distances to the given points on the hyperbolic plane $\mathbb{H}^2$. We analyze the existence and uniqueness of the hyperbolic median. Furthermore, we derive three important properties of the hyperbolic median which are the geometric plasticity of geodesic polygons, the dynamic plasticity and the generalized plasticity of geodesic quadrilaterals in $\mathbb{H}^2$.

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1 Introduction

In 1643, P. de Fermat stated a problem in the field of geometric optimization:

Given three points in the Euclidean plane, find a (unique) point having the minimal sum of distances to these three points.

A complete survey regarding the characterization of the solutions and various generalizations of Fermat's problem (or weighted Fermat-Torricelli problem) is given in [1, Chapter 2].

In 1997, W. Thurston defined the hyperbolic median and generalized the Fermat problem to the hyperbolic plane H^2 with constant negative Gaussian curvature $-K$, for $K > 0$.

Definition 1.1. [4, Exercise 2.5.21,p. 96] The hyperbolic median for a weighted collection of points on the hyperbolic plane $\mathbb{H}^2$ is any point that minimizes the weighted sum of geodesic distances to the points.

Problem 1.2. [4, Exercise 2.5.21,p. 96] *Analyze the existence and uniqueness of the hyperbolic median on* $\mathbb{H}^2$ *and describe its qualitative properties.*

In the paper, we shall solve Thurston's problem of the weighted hyperbolic median on the hyperbolic plane H^2 by applying results which have been obtained in [9],[2],[7],[8] and [10]. We prove that the weighted hyperbolic median uniquely exists

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(Section 2) and we give the three properties of the hyperbolic median which are: (a) the geometric plasticity, (b) the dynamic plasticity and (c) the generalized plasticity (Section 3).

2 Existence and uniqueness of the hyperbolic median

Let A_i be n given points in $\mathbb{H}^2$, which do not belong to a geodesic arc, a positive real number (weight) B_i corresponds to A_i, $d(A_i, A_j)$ be the geodesic distance from A_i to A_j and $E(A_i, A_j)$ be the Euclidean distance from A_i to A_j.

We restrict the definition of a totally convex set and strict convexity of a function on a Riemannian manifold given in [6] to the hyperbolic plane H^2.

Definition 2.1. [6, Definition 2.1,p. 59] A subset W is said to be totally convex if W contains every geodesic γ_{xy} of H^2 whose endpoints x and y are in W.

Definition 2.2. [6, Definition 2.1,p. 60] Let W be a totally convex set in H^2 and $f : W \to \mathbb{R}$. If $(1-t)f(X) + tf(Y) - f(\gamma_{XY}(t)) > 0$ and $t \in (0,1)$, then the function is called strictly convex.

We proceed by proving the strict convexity of the weighted sum of hyperbolic geodesic distances from n given points A_i in $\mathbb{H}^2$ by applying an Euclidean approach which has been introduced in [9, Proof of Proposition 1 (Uniqueness)] for three fixed points on a C^2 complete surface in $\mathbb{R}^3$ and has been generalized for four points on a CAT(0) surface in [10, Proposition 3, Remark 1].

Theorem 2.1. *The function $f(P) = \sum_{i=1}^{n} B_i d(A_i, P)$ is strictly convex.*

Proof. Let $X, Y \in \mathbb{H}^2$. We shall prove that:

$$\sum_{i=1}^{n} B_i((1-t_0)d(A_i, X) + t_0 d(A_i, Y) - d(A_i, \gamma_{XY}(t_0))) > 0,$$

for every $t_0 \in (0,1)$. We have

$$\sum_{i=1}^{n} B_i((1-t_0)d(A_i, X) + t_0 d(A_i, Y) - d(A_i, \gamma_{XY}(t_0))) >$$

$$\sum_{i=1}^{n} B_i((1-t_0)E(A_i, X) + t_0 E(A_i, Y) - E(A_i, \gamma_{XY}(t_0))) > 0$$

taking into account that:

(a) $\triangle(A_i XY)_{\mathbb{R}^2}$ is a comparison triangle of $\triangle A_i XY$ on H^2, with the same side lengths,

(b) $d(A_i, \gamma_{XY}(t_0)) < E(A_i, \gamma_{XY}(t_0))$, and

(c) $\{A_i X_{t_0} \gamma_{XY}(t_0) Y_{1-t_0}\}_{\mathbb{R}^2}$ is a parallelogram in $\mathbb{R}^2$ (Fig. 1). □

Remark 2.3. We applied the same method which was introduced in [9], in order to prove the uniqueness of the median (weighted Fermat-Torricelli point) on a C^2 surface.

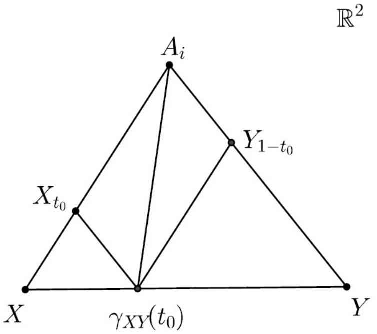

Figure 1: Uniqueness of the hyperbolic median

We need the following two lemmas, in order to prove that the weighted hyperbolic median P uniquely exists.

Lemma 2.2. *[5] Let W be a totally convex set with the nonvoid interior and $f : W \to \mathbb{R}$, then f is continuous on $intW$.*

Lemma 2.3. *[6, Corollary 7.4,p. 91] If f is strictly convex, then f has at most one minimum point on W.*

Theorem 2.4. *The hyperbolic median A_0 uniquely exists.*

Proof. The existence and uniqueness is a direct application of lemmas 2.2, 2.3 and taking into account from Theorem 2.1 that the objective function $f(P)$ is strictly convex. □

Remark 2.4. We note that the existence and uniqueness of the weighted median for n dimensional simply connected manifolds with non-positive sectional curvature (Hadamard manifolds) was also proved in [3] by using Jacobi vector fields.

3 Three properties of the hyperbolic median

The qualitative properties of the hyperbolic median are three:

(1) the geometric plasticity of geodesic hyperbolic polygons which was introduced in [10];

(2) the dynamic plasticity of geodesic hyperbolic quadrilaterals which was introduced in [8], and

(3) the generalized plasticity of geodesic quadrilaterals.

3.1 The geometric plasticity of the hyperbolic median

Let $\triangle A_1A_2A_3$ be a geodesic triangle in $\mathbb{H}^2$, let the points A_1, A_3 be fixed, and let $d(A_i, A_j) \equiv a_{ij}$.

We proceed by giving the first variational formula of the length of the geodesic arc w.r. to (a) the length of the geodesic arc and (b) a variable angle which is formed between two geodesic arcs.

Theorem 3.1. *[8, 9, 2] The derivative of the length of the geodesic arc a_{23} w.r.t. the length of the geodesic arc a_{12} is given by:*
$\frac{da_{23}}{da_{12}} = \cos\angle A_1A_2A_3$.

Theorem 3.2. *[7] The derivative of the length of the geodesic arc a_{23} w.r. to the angle $\angle A_2A_1A_3$ is given by:*
$\frac{da_{23}}{d\angle A_2A_1A_3} = \frac{\sinh(\sqrt{-K}a_{12})}{\sqrt{-K}}\sin\angle A_1A_2A_3$.

We denote by $\vec{U}_{ij}$ the unit tangent vector of the geodesic arc A_iA_j at A_i, for $i = 0, 1, 2, ..., n$. The combination of the two formulas of Theorems 3.1 and 3.2 yield the weighted balancing condition of unit tangent vectors $\vec{U}_{0i}$ at the hyperbolic median A_0.

Thus, we obtain the following:

Theorem 3.3. *The following (I), (II), (III) conditions are equivalent in* $\mathbb{H}^2$:
(I) $\|\sum_{i=1,i\neq j}^{n} B_i\vec{U}_{ji}\| > B_j$, *for every* $j = 1, 2, 3, ..., n$,
(II) The hyperbolic median A_0 *does not belong in* $\{A_1, A_2, ..., A_n\}$,
(III) $\sum_{i=1}^{n} B_i\vec{U}_{0i} = \vec{0}$.

Remark 3.1. Theorem 3.3 is proved for equal weights in [3] and for a geodesic triangle on a smooth surface in [9],[2].

The geometric plasticity principle of geodesic polygons in H^2 is given by Theorem 3.4.

Theorem 3.4. *[10, Proposition 8] Suppose that* $\|\sum_{i=1,i\neq j}^{n} B_i\vec{U}_{ji}\| > B_j$, *for every* $j = 1, 2, 3, ..., n$ *and* A_0 *is connected with every vertex* A_i *and we select a point* $A_{i'}$ *with non negative weight* B_i, *which lie on the geodesic arc* A_iA_0, *such that:*

$$\|\sum_{i'=1,i'\neq j'}^{n} B_i\vec{U}_{j'i'}\| > B_j$$

Then, the hyperbolic median $A_{0'}$ *of* $\{A_{1'}A_{2'}...A_{n'}\}$ *is identical with* A_0.

Remark 3.2. The geometric plasticity principle was introduced for the Euclidean plane in [8] and proved in the K-plane (S^2, H^2) for geodesic quadrilaterals in [10].

3.2 The dynamic plasticity of the hyperbolic median

The second property of the hyperbolic median is the dynamic plasticity of geodesic quadrilaterals which is derived by the inverse problem of the weighted hyperbolic median (or inverse weighted Fermat-Torricelli problem).

Problem 3.3 (The inverse problem of the weighted hyperbolic median)**.** Given a point A_0 which does not belong in $\{A_1A_2A_3...A_n\}$ in $\mathbb{H}^2$, does there exist a unique set of positive weights B_i, such that

$$\sum_{i=1}^{n} B_i = c = const,$$

for which A_0 minimizes

$$f(A_0) = \sum_{i=1}^{n} B_i a_{0i}.$$

In 2008, we gave a positive answer to the inverse problem of the hyperbolic median in [7, Propositions 3.2,3.5].

Theorem 3.5. *[7, Propositions 3.2,3.5]The solution of the inverse problem of the hyperbolic median for a geodesic triangle* $\triangle A_iA_jA_k$ *in* $\mathbb{H}^2$ *is given by:*

$$\frac{B_i}{B_j} = \frac{\sin \angle A_kA_0A_j}{\sin \angle A_iA_0A_k}.$$

Proof. It is a direct application of the angular first variation formula in $\mathbb{H}^2$. □

Let $(B_i)_{1234}$ be the weight which corresponds to the vertex A_i of a geodesic convex quadrilateral $A_1A_2A_3A_4$ and $(B_j)_{jkl}$ the weight which corresponds to the vertex A_j of the geodesic triangle $\triangle A_jA_kA_l$, $j,k,l = 1,2,3,4$.

By differentiating the objective function $f(A_0)$ w.r. to some specific variable angles and taking into account the solution of Theorem 3.5 for the geodesic triangles $\triangle A_1A_2A_3$, $\triangle A_1A_2A_4$ and $\triangle A_1A_3A_4$, we derive the equations of dynamic plasticity of geodesic quadrilaterals.

The equations of dynamic plasticity are given in the following theorem which was proved in [8] and [10].

Theorem 3.6. *[8, Proposition 4.4],[10, Theorem 1] The following equations point out the dynamic plasticity of the weighted convex quadrilateral* $A_1A_2A_3A_4$ *in* $\mathbb{H}^2$:

$$\left(\frac{B_2}{B_1}\right)_{1234} = \left(\frac{B_2}{B_1}\right)_{123}\left(1 - \left(\frac{B_4}{B_1}\right)_{1234}\left(\frac{B_1}{B_4}\right)_{134}\right)$$

$$\left(\frac{B_3}{B_1}\right)_{1234} = \left(\frac{B_3}{B_1}\right)_{123}\left(1 - \left(\frac{B_4}{B_1}\right)_{1234}\left(\frac{B_1}{B_4}\right)_{124}\right)$$

and $\displaystyle\sum_{i=1}^{4}(B_i)_{1234} = c.$

3.3 The generalized plasticity of the hyperbolic median

The generalized plasticity of geodesic quadrilaterals is the third property of the weighted hyperbolic median where the first and the second property may occur simultaneously, such that the weighted hyperbolic median remains invariant.

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