On holomorphic jets bundle $j^{(2,0)}M$ with Randers metric

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Abstract. The geometry of Randers spaces has been investigated by G. Randers using Riemannian techniques in [12], by R. Ingarden in [7], by R. Miron in [9, 10], etc. A higher order approach of the geometry of Randers spaces was made by M. Roman in [13]. The complex version of Randers metrics was studied by N. Aldea and G. Munteanu in [3]. In this paper, using the theory of the holomorphic jet bundle $J^{(2,0)}M$, we define and study the geometrical theory of a Randers space of order two. We define a complex Randers metric and we determine circumstances in which this metric is a Finsler one.

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1 Introduction

Let M be a complex manifold, $dim_{\mathbf{C}}M = n$. We briefly revise the decomposition of the complexified tangent bundle $T_{\mathbf{C}}M = T'M \oplus T''M$, where $T'M$ is an holomorphic vector bundle over M and $T''M$ is a vector bundle over M, but not an holomorphic one.

If M is considered as a real manifold, its second order jet manifold $J^2(M)$ is a fiber bundle over M, [14]. We have the decomposition of the complexified space of $J^2(M)$: $J^2_{\mathbf{C}}M = J^{(2,0)}(M) \oplus J^{(1,1)}(M) \oplus J^{(0,2)}(M)$, [8], where the terms are fiber bundles over the complex manifold M, the first one being an holomorphic bundle which contains the holomorphic second order jets on M in sense of [15].

In a previous paper, [16], the structure of the holomorphic bundle $J^{(2,0)}M$ for the 2-jets on the complex manifold M was studied. Let us remember that on the complex manifold $J^{(2,0)}M$, on a local chart, the coordinates are denoted by $Z = (z^k, \eta^k, \zeta^k)$, $k = \overline{1,n}$, and the changes in the local basis on M will transform according to the

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following rules:

$$(1.1)\qquad \begin{aligned} z'^i &= z'^i(z); \\ \eta'^i &= \frac{\partial z'^i}{\partial z^j}\eta^j; \\ 2\zeta'^i &= \frac{\partial \eta'^i}{\partial z^j}\eta^j + 2\frac{\partial \eta'^i}{\partial \eta^j}\zeta^j \end{aligned}$$

and that $\frac{\partial z'^i}{\partial z^j} = \frac{\partial \eta'^i}{\partial \eta^j} = \frac{\partial \zeta'^i}{\partial \zeta^j}$; $\frac{\partial \eta'^i}{\partial z^j} = \frac{\partial \zeta'^i}{\partial \eta^j}$. The local basis in the holomorphic bundle $T'(J^{(2,0)}M)$ is $\left\{\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \eta^i}, \frac{\partial}{\partial \zeta^i}\right\}$ and in $T''(J^{(2,0)}M)$ is its conjugates. The changes of the local basis are made according to the following rules:

$$(1.2)\qquad \begin{aligned} \frac{\partial}{\partial z^j} &= \frac{\partial z'^i}{\partial z^j}\frac{\partial}{\partial z'^i} + \frac{\partial \eta'^i}{\partial z^j}\frac{\partial}{\partial \eta'^i} + \frac{\partial \zeta'^i}{\partial z^j}\frac{\partial}{\partial \zeta'^i}; \\ \frac{\partial}{\partial \eta^j} &= \frac{\partial \eta'^i}{\partial \eta^j}\frac{\partial}{\partial \eta'^i} + \frac{\partial \zeta'^i}{\partial \eta^j}\frac{\partial}{\partial \zeta'^i}; \\ \frac{\partial}{\partial \zeta^j} &= \frac{\partial \zeta'^i}{\partial \zeta^j}\frac{\partial}{\partial \zeta'^i}, \end{aligned}$$

and similar for the conjugate basis that coresponds in $T_z^{''}(J^{(2,0)}M)$.

The two structures, that play a special role in defining the linear and nonlinear connection on $J^{(2,0)}M$ are: the natural complex structure J and the almost second order tangent structure F.

A complex nonlinear connection, (c.n.c.), is given by $H(J^{(2,0)}M)$ which is supplementary to $W(J^{(2,0)}M)$ in $T'(J^{(2,0)}M)$, where $W_z(J^{(2,0)}M)$ is spanned by $\left\{\frac{\partial}{\partial \eta^j}, \frac{\partial}{\partial \zeta^j}\right\}$ in a local chart. With $V(J^{(2,0)}M)$ we denote the vertical bundle spanned by $\left\{\frac{\partial}{\partial \zeta^j}\right\}$. By conjugation, we obtain the decomposition for $T_C(J^{(2,0)}M)$. A characterization for the nonlinear connection using the k-tangent structure on the k-osculating bundle is in the paper [5]. A local basis in $H_z(J^{(2,0)}M)$ is denoted by $\frac{\delta}{\delta z^j} = \frac{\partial}{\partial z^j} - \overset{(1)}{N_j^i}\frac{\partial}{\partial \eta^i} - \overset{(2)}{N_j^i}\frac{\partial}{\partial \zeta^i}$ and it is called adapted basis of the (c.n.c.). With $F(\frac{\delta}{\delta z^j}) =: \frac{\delta}{\delta \eta^j} = \frac{\partial}{\partial \eta^j} - \overset{(1)}{N_j^i}\frac{\partial}{\partial \zeta^i}$ is obtained a local adapted basis in $W_z(J^{(2,0)}M)$. The transformations of coordinates (1.1) on $J^{(2,0)}M$ produces the transformations of the coefficients $\overset{(1)}{N_j^i}$ and $\overset{(2)}{N_j^i}$ of the (c.n.c.) in the form:

$$\begin{aligned} \overset{(1)}{N'^i_k}\frac{\partial z'^k}{\partial z^j} &= \frac{\partial z'^i}{\partial z^k}\overset{(1)}{N_j^k} - \frac{\partial \eta'^i}{\partial z^j}; \\ \overset{(2)}{N'^i_k}\frac{\partial z'^k}{\partial z^j} &= \frac{\partial z'^i}{\partial z^k}\overset{(2)}{N_j^k} + \frac{\partial \eta'^i}{\partial z^k}\overset{(1)}{N_j^k} - \frac{\partial \zeta'^i}{\partial z^j}. \end{aligned}$$

The adapted basis will change as follow: $\frac{\delta}{\delta z^j} = \frac{\partial z'^i}{\partial z^j}\frac{\delta}{\delta z'^i}$ and $\frac{\delta}{\delta \eta^j} = \frac{\partial z'^i}{\partial z^j}\frac{\delta}{\delta \eta'^i}$. Obviously $\frac{\delta}{\delta \zeta^j} = \frac{\partial z'^i}{\partial z^j}\frac{\delta}{\delta \zeta'^i}$, so these fields are changing as those on the base manifold M. Generally, the geometrical objects which are changed by $\frac{\partial z'^i}{\partial z^j}$ or by their conjugates

$\frac{\partial \bar{z}'^i}{\partial \bar{z}^j}$, are called d- tensor fields. The adapted basis on $T''(J^{(2,0)}M)$ is obtained by conjugation. The joint between the dual co-basis $\{dz^i, \delta\eta^i = d\eta^i + \overset{(1)}{M^i_j} dz^j, \delta\zeta^i = d\zeta^i + \overset{(1)}{M^i_j} d\eta^j + \overset{(2)}{N^i_j} dz^j\}$ and the adapted basis is given by the rules:

$$\overset{(1)}{M^i_j} = \overset{(1)}{N^i_j} \ ; \ \overset{(2)}{M^i_j} = \overset{(2)}{N^i_j} + \overset{(1)}{N^i_k}\overset{(1)}{N^k_j},$$

where $\overset{(1)}{M^i_j}$ and $\overset{(2)}{M^i_j}$ are changing by the following rules (see [16]):

$$\frac{\partial z'^i}{\partial z^k} \overset{(1)}{M^k_j} = \overset{(1)}{M'^i_k} \frac{\partial z'^k}{\partial z^j} + \frac{\partial \eta'^i}{\partial z^j};$$
$$\frac{\partial z'^i}{\partial z^k} \overset{(2)}{M^i_j} = \overset{(2)}{M'^i_k} \frac{\partial z'^k}{\partial z^j} + \overset{(1)}{M'^i_k} \frac{\partial \eta'^k}{\partial z^j} + \frac{\partial \zeta'^i}{\partial z^j}.$$

The formulas which make the joint between $\overset{(1)}{N^i_j}, \overset{(2)}{N^i_j}$ and $\overset{(1)}{M^i_j}, \overset{(2)}{M^i_j}$ are: $\overset{(1)}{M^i_j} = \overset{(1)}{N^i_j}$ and $\overset{(2)}{M^i_j} = \overset{(2)}{N^i_j} + \overset{(1)}{N^i_k}\overset{(1)}{N^k_j}$. The notion of complex nonlinear connection is connected with the complex spray notion, which is defined as a field $S \in T'(J^{(2,0)}M)$ with property $F \circ S = \overset{(2)}{\mathcal{L}}$, where $\overset{(2)}{\mathcal{L}} = \eta^i \frac{\partial}{\partial \eta^i} + 2\zeta^i \frac{\partial}{\partial \zeta^i}$ is a Liouville field. The spray S has the coefficients G^i, thus $S = \eta^i \frac{\partial}{\partial z^i} + 2\zeta^i \frac{\partial}{\partial \eta^i} - 3G^i(z,\eta,\zeta)\frac{\partial}{\partial \zeta^i}$, and they are transformed by the rule:

$$3G'^i = 3\frac{\partial z'^i}{\partial z^j} G^j - \left(\eta^j \frac{\partial \zeta'^i}{\partial z^j} + 2\zeta^j \frac{\partial \zeta'^i}{\partial \eta^j} \right).$$

Therefore, a normal complex nonlinear connection, N-(c.l.c.), is a derivative law on $T_{\mathbf{C}}(J^{(2,0)}M)$ and depending on the adapted frame, is defined by the following coefficients $D\Gamma = (L^i_{jk}, L^{\bar{i}}_{\bar{j}k}, F^i_{jk}, F^{\bar{i}}_{\bar{j}k}, C^i_{jk}, C^{\bar{i}}_{\bar{j}k})$ which are changing as follow:

$$L'^i_{jk} = \frac{\partial z'^i}{\partial z^r} \frac{\partial z^p}{\partial z'^j} \frac{\partial z^q}{\partial z'^k} L^r_{pq} + \frac{\partial z'^i}{\partial z^p} \frac{\partial^2 z^p}{\partial z'^j \partial z'^k}$$

and the others are d-tensors.

2 Complex Randers spaces

Let us consider a metric Hermitian structure $g(z)$ with respect to J_M on the base manifold M, that mince that g is bilinear on the sections of $T_C M$ and $\overline{g(X,\overline{Y})} = g(Y,\bar{X})$. Locally, if we consider the chart (U, z^k), then $g = g_{i\bar{j}}(z) dz^i \otimes d\bar{z}^j$ is well defined by $g_{i\bar{j}}(z) := g(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j})$ with $\overline{g_{i\bar{j}}} = g_{j\bar{\imath}}$. Because g defines a metric structure, the Hermitian matrix $(g_{i\bar{j}})$ has $\det(g_{i\bar{j}}) \neq 0$, with the inverse $(g^{\bar{j}i})$ such that $g_{i\bar{j}} g^{\bar{j}k} = \delta^k_i$.

The Hermitian metric tensor $g_{i\bar{j}}$ could be considered as a $d-$tensor on $J^{(2,0)}M$, still denoted by $g_{i\bar{j}}$, such that $(g_{i\bar{j}} \circ \pi)(u) = g_{i\bar{j}}(z)$, where $\pi(u) = z$.

One, like for real Osc^2M, let us consider the first Cristoffel symbol of the metric $g(z)$,

$$\Gamma^i_{jk} := \tfrac{1}{2}g^{\bar{m}i}\{\tfrac{\partial g_{k\bar{m}}}{\partial z^j} + \tfrac{\partial g_{j\bar{m}}}{\partial z^k}\} = \Gamma^i_{kj};$$
$$\Gamma^i_{j\bar{k}} := \tfrac{1}{2}g^{\bar{m}i}\{\tfrac{\partial g_{j\bar{m}}}{\partial \bar{z}^k} - \tfrac{\partial g_{j\bar{k}}}{\partial \bar{z}^m}\};$$
$$\Gamma^i_{\bar{j}k} := \tfrac{1}{2}g^{\bar{m}i}\{\tfrac{\partial g_{k\bar{m}}}{\partial \bar{z}^j} - \tfrac{\partial g_{k\bar{j}}}{\partial \bar{z}^m}\},$$

and their conjugates, because the Hermitian structure is integrable. The coefficients Γ^i_{jk} are changed as follows $\Gamma'^i_{rs}\frac{\partial z'^r}{\partial z^j}\frac{\partial z'^s}{\partial z^k} = \Gamma^r_{jk}\frac{\partial z'^i}{\partial z^r} - \frac{\partial^2 z'^i}{\partial z^j \partial z^k}$ and all the others are d-tensors.

We consider the Liouville fields:

$$\begin{aligned} z^i &:= z^i, \\ \eta^i &:= \eta^i, \\ \xi^i &:= \zeta^i + \frac{1}{2}\Gamma^i_{jk}\eta^j\eta^k. \end{aligned} \tag{2.1}$$

Taking into account (2.1), (1.1) and (1.2) it is a straightforward computation to prove that these are d-vector fields.

Recall from [1, 2, 6, 11] that a *complex Finsler structure* is a pair (M, F), where $F : T'M \to \mathbb{R}^+$ is a smooth function except zero section satisfying the homogeneity condition $F(z, \lambda\eta) = |\lambda|F(z, \eta)$ for $\forall\lambda \in \mathbb{C}$ and the metric tensor $g_{j\bar{k}} := \frac{\partial^2 L}{\partial\eta^j\partial\bar{\eta}^k}$ with $L := F^2$ defines a Hermitian bilinear form. The main example of a complex Finsler structure with Randers metric tensor is $F = \alpha + |\beta|$, where $\alpha = \sqrt{a_{i\bar{j}}(z)\eta^i\bar{\eta}^j}$ and $\beta = b_i(z)\eta^i$, [3].

If we consider the Hermitian matrix nondegenerated, i.e. $\det\big(g_{j\bar{k}}(z, \eta)\big) \neq 0$, then a more general study is obtained, that of the *complex Lagrange spaces*, [11].

In [19] we extended the definition to $J^{(2,0)}M$.

A *second order complex Finsler space* is a pair (M, F), where $F : J^{(2,0)}M \to \mathbf{R}^+$ is a $(1, 1)$-homogeneous smooth function different from the 0-section, i.e. $F(z, \lambda\eta, \lambda^2\zeta) = |\lambda|^2 F(z, \eta, \zeta)$, $\forall\lambda \in \mathbb{C}^*$ and the metric tensor $g_{i\bar{j}}(z, \eta, \zeta) = \frac{\partial^2 L}{\partial\zeta^i\partial\bar{\zeta}^j}$ determine a Hermitian bilinear form.

Let us note that the complex Finsler (actually purely Hermitian) metric tensor $g_{i\bar{j}}(z)$ can define a Hermitian metric structure on $J^{(2,0)}M$ by means of the second order complex Lagrangian $\mathcal{L} = g_{i\bar{j}}(z)\xi^i\bar{\xi}^j$ with the same metric tensor $g_{i\bar{j}}(z)$, i.e. $\frac{\partial^2\mathcal{L}}{\partial\xi^i\partial\bar{\xi}^j} = g_{i\bar{j}}(z)$. This process was studied by us in [18] and it is called the prolongation of complex Finsler structure.

Let us consider now the prolongation functions on $J^{(2,0)}M$, still denoted by α and β,

$$\alpha(z, \eta, \zeta) = \sqrt{a_{i\bar{j}}(z)\xi^i\bar{\xi}^j}; \qquad \beta(z, \eta, \zeta) = b_i(z)\xi^i(z, \eta, \zeta),$$

where $a := a_{i\bar{j}}(z)dz^i \otimes d\bar{z}^j$ is a purely Hermitian positive metric and $b_i(z)$ are potential forms on the base manifold M, such that $b_i(z) = b_i(\pi(u))$.

Definition 2.1. The function $F : J^{(2,0)}M \to \mathbb{R}$ given by $F := \alpha + |\beta|$, i.e.

$$F(z, \eta, \zeta) = \sqrt{a_{i\bar{j}}(z)\xi^i\bar{\xi}^j} + |b_i(z)\xi^i| \tag{2.2}$$

is a complex Randers metric. The pair $(M, \alpha + |\beta|)$ is a complex Randers space.

Our goal in the sequel is to find the circumstances in which the function (2.2) is a second order complex Finsler metric.

The complex Randers metric $F := \alpha + |\beta|$ is positive and smooth on $J^{(2,0)}M\backslash\{0\}$, due to the presence of $|\beta|$. Moreover, α and β are homogeneous thus $F(z, \lambda\eta, \lambda^2\zeta) = |\lambda|^2 F(z, \eta, \zeta)$, $\forall \lambda \in \mathbb{C}$ and this property implies

$$\frac{\partial \alpha}{\partial \xi^i}\xi^i = \frac{1}{2}\alpha;\ \ \frac{\partial|\beta|}{\partial \xi^i}\xi^i = \frac{1}{2}|\beta|;\ \ \mathcal{L}_\alpha = \mathcal{L}_{|\beta|} = 2\mathcal{F};\ \ \alpha\mathcal{L}_\alpha + |\beta|\mathcal{L}_{|\beta|} = 2\mathcal{F}^{\,2};$$

$$\alpha\mathcal{L}_{\alpha\alpha} + |\beta|\mathcal{L}_{\alpha|\beta|} = \mathcal{L}_\alpha;\ \ \alpha\mathcal{L}_{\alpha|\beta|} + |\beta|\mathcal{L}_{|\beta||\beta|} = \mathcal{L}_{|\beta|};$$

$$\alpha^2\mathcal{L}_{\alpha\alpha} + 2\alpha|\beta|\mathcal{L}_{\alpha|\beta|} + |\beta|^2\mathcal{L}_{|\beta||\beta|} = 2\mathcal{L},$$

where $\mathcal{L}_\alpha := \frac{\partial \mathcal{L}}{\partial \alpha}, \mathcal{L}_{|\beta|} := \frac{\partial \mathcal{L}}{\partial |\beta|}, \mathcal{L}_{\alpha\alpha} := \frac{\partial^2 \mathcal{L}}{\partial \alpha^2}, \mathcal{L}_{\alpha|\beta|} := \frac{\partial^2 \mathcal{L}}{\partial \alpha |\beta|}$, etc.

Theorem 2.1. *a)The fundamental tensor field is given by*

$$g_{i\bar{j}} = \frac{\mathcal{F}}{\alpha}h_{i\bar{j}} + \frac{\mathcal{F}}{2|\beta|}b_i b_{\bar{j}} + \frac{1}{2\mathcal{L}}\xi_i\xi_{\bar{j}}, \tag{2.3}$$

where $\frac{\partial \alpha}{\partial \xi^i} = \frac{1}{2\alpha}l_i$, $\frac{\partial|\beta|}{\partial \xi^i} = \frac{\bar{\beta}}{2|\beta|}b_i$ *and* $h_{i\bar{i}} := a_{i\bar{j}} - \frac{1}{2\alpha^2}l_i l_{\bar{j}}$.
b)The function $\mathcal{L} := \mathcal{F}^{\,2} = (\alpha + |\beta|)^2$ *is a differentiable Lagrangian of second order;*

Proof. a) If we consider

$$b^i = a^{\bar{j}i}b_{\bar{j}}, \xi_i = \frac{\partial \mathcal{L}}{\partial \xi^i} = \mathcal{L}_\alpha\frac{\partial \alpha}{\partial \xi^i} + \mathcal{L}_{|\beta|}\frac{\partial|\beta|}{\partial \xi^i} = \frac{\mathcal{F}}{\alpha}l_i + \frac{\mathcal{F}\bar{\beta}}{|\beta|}b_i \text{ and } g_{i\bar{j}} = \frac{\partial \eta_i}{\partial \bar{\eta}^j},$$

we obtain (2.3).

b) First, we must prove that F is a second order complex Finsler metric. Let $c_1, c_2, ..., c_n$ be the complex numbers with $c_{\bar{i}} = \bar{c}_i$, $\forall i = \overline{1,n}$ and $D_{i\bar{j}}$ a non-singular $n \times n$ complex matrix with $(D^{\bar{j}i})$ its inverse. We consider $c^i = D^{\bar{j}i}c_{\bar{j}}$ and similar for the conjugates; $c^2 = c^i c_i = c^{\bar{i}}c_{\bar{i}}$. Following the ideas from [4] and [3] for the matrix $H_{i\bar{j}} = D_{i\bar{j}} + c_i c_{\bar{j}}$, we obtain $\det(H_{i\bar{j}}) = (1 \pm c^2)\det(D_{i\bar{j}})$. If $1 + c^2 \neq 0$ than the matrix $H_{i\bar{j}}$ is invertible and its inverse is $H^{\bar{j}i} = D^{\bar{j}i} \pm \frac{1}{1 \pm c^2}c^i c^{\bar{j}}$. For the complex Randers metric F we obtain

$$g^{\bar{j}i} = \frac{\alpha}{F}a^{\bar{j}i} + \frac{|\beta|(\alpha||b||^2 + |\beta|)}{L\gamma}\xi^i\xi^{\bar{j}} - \frac{\alpha^3}{F\gamma}b^i b^{\bar{j}} - \frac{\alpha}{F\gamma}(\bar{\beta}\xi^i b^{\bar{j}} + \beta b^i \xi^{\bar{j}})$$

and $\det(g_{i\bar{j}}) = \left(\frac{F}{\alpha}\right)^n \frac{\gamma}{2\alpha|\beta|}\det(a_{i\bar{j}})$, where $||b||^2 := a^{\bar{j}i}b_i b_{\bar{j}}$ and $\gamma := L + \alpha^2(||b||^2 - 1)$. Because α and β are of C^∞-class on $\widetilde{J^{(2,0)}M}$ and continuous on the null section of the projection $\pi : J^{(2,0)}M \to M$ we obtain that L has also these properties. We know that ξ^i is 2-homogeneous on the fibres of $\widetilde{J^{(2,0)}M}$. Taking into account all those formulas we can see that F is a second order complex Finsler metric if $\gamma > 0$ (in this case the fundamental metric tensor $g_{i\bar{j}}(z, \eta, \zeta)$ is positive definite). □

For the differentiable Lagrangian $L(z,\eta,\zeta) = F^2(z,\eta,\zeta)$ we consider its Lie derivative with respect to a vector field X on $\widetilde{J^{(2,0)}M}$, denoted by

$$X(L(z,\eta,\zeta)) = \mathcal{L}_X L(z,\eta,\zeta).$$

Therefore, with respect to the Liouville vector fields $\overset{1}{\Gamma}$ and $\overset{2}{\Gamma}$ we get the scalar fields on $\widetilde{J^{(2,0)}M}$

$$\overset{1}{I}(L) = \mathcal{L}_{\overset{1}{\Gamma}} L, \quad \overset{2}{I}(L) = \mathcal{L}_{\overset{2}{\Gamma}} L,$$

which are called the *main invariants* of the Lagrangian L. The expanded expressions of the main invariants are follows:

$$\overset{1}{I}(L) = \eta^i \frac{\partial L}{\partial \zeta^i}, \quad \overset{2}{I}(L) = \eta^i \frac{\partial L}{\partial \eta^i} + 2\zeta^i \frac{\partial L}{\partial \zeta^i}.$$

Let us consider a smooth parametrized curve $c : [a,b] \to J^{(2,0)}M$ represented in a domain U of a local chart by $c(t) = (z^k(t), \eta^k(t), \zeta^k(t))$, where $\eta^k(t) = \frac{dz^k}{dt}$ and $\zeta^k(t) = \frac{1}{2}\frac{d^2z^k}{dt^2}$, $\forall t \in [a,b]$. The integral of action of the Lagrangian L along c is defined by

$$I(c) = \int_a^b L\left(z(t), \frac{dz}{dt}, \frac{1}{2}\frac{d^2z}{dt^2}\right) dt.$$

We can prove the *Zermelo conditions*:

Theorem 2.2. *The necessary conditions that the integral of action I(c) does not depend on the parametrization of the curve c are the following ones:*

$$\overset{1}{I}(L) = 0, \quad \overset{2}{I}(L) = L.$$

Proof. The integral of action $I(c)$ does not depend on the parametrization of the curve c if we have:

$$\tilde{L}\left(\tilde{z}, \frac{d\tilde{z}}{d\tilde{t}}, \frac{1}{2}\frac{d^2\tilde{z}}{d\tilde{t}^2}\right)\frac{d\tilde{t}}{dt} = L\left(z, \frac{dz}{d\tilde{t}}\frac{d\tilde{t}}{dt}, \frac{1}{2}\frac{d}{dt}\left(\frac{dz}{d\tilde{t}}\frac{d\tilde{t}}{dt}\right)\right),$$

where $\tilde{t} = \tilde{t}(t)$, $t \in [0,1]$ is a differentiable diffeomorphism. If we derive this equation with respect to $\frac{d\tilde{t}}{dt}$ and then, with respect to $\frac{d^2\tilde{t}}{dt^2}$, taking $\tilde{t} = t$, we obtain (2.11). $\square$

3 Connections in the complex Randers spaces

In this section, we examine what are the circumstances in which a (c.n.c.) exists. Moreover, we want to see if we can obtain a canonical nonlinear connection from a Chern-Lagrange connection.

The complex spray has the local expression

$$S = \eta^i \frac{\partial}{\partial z^i} + 2\zeta^i \frac{\partial}{\partial \eta^i} - 3G^i(z,\eta,\zeta)\frac{\partial}{\partial \zeta^i},$$

where G^i are the coefficients of the spray and they follow the rule

$$3G'^i = 3\frac{\partial z'^i}{\partial z^j}G^j - \left(\eta^j \frac{\partial \zeta'^i}{\partial z^j} + 2\zeta^j \frac{\partial \zeta'^i}{\partial \eta^j}\right).$$

Proposition 3.1 ([19]). *If S is a complex spray with coefficients G^i, then*

$$\overset{(1)}{M_j^i} = \frac{\partial G^i}{\partial \zeta^j} \quad , \quad \overset{(2)}{M_j^i} = \frac{\partial G^i}{\partial \eta^j}$$

are the dual coefficients of a (c.n.c.), and then

$$\overset{(1)}{N_j^i} = \overset{(1)}{M_j^i}; \quad \overset{(2)}{N_j^i} = \overset{(2)}{M_j^i} - \overset{(1)}{M_k^i}\overset{(1)}{M_j^k}$$

give the coefficients of a (c.n.c.).

Conversely, the following result holds:

Proposition 3.2 ([19]). *If $\overset{(1)}{M_j^i}$ and $\overset{(2)}{M_j^i}$ define a (c.n.c.), then a complex spray on $J^{(2,0)}M$ is given by:* $3G^i = \overset{(2)}{M_j^i}\,\eta^j + 2\,\overset{(1)}{M_j^i}\,\zeta^j$.

In a previous paper, [19], we concluded that:

Theorem 3.3 ([19]). *The pair $\overset{(1)}{M_j^i}$, $\overset{(2)}{M_j^i}$ determines the dual coefficients of a (c.n.c.), named the Chern-Lagrange (c.n.c.), where*

$$\overset{(1)CL}{M_j^i} = g^{\overline{m}i}\frac{\partial^2 L}{\partial \eta^j \partial \overline{\zeta}^m} \quad ; \quad \overset{(2)CL}{M_j^i} = g^{\overline{m}i}\frac{\partial^2 L}{\partial z^j \partial \overline{\zeta}^m}. \tag{3.1}$$

Taking into account the complex Lagrange space (M, L), we obtain:

Theorem 3.4. *The set of the dual coefficients are given by:*

$$\begin{aligned}
\overset{(1)CL}{M_j^i} = \;& \Big(\frac{\alpha a^{\bar{j}i}l_i}{4F\bar{\xi}^m} + \frac{1}{4\bar{\xi}^m} + \frac{\alpha^2 a^{\bar{j}i}b_i}{2\bar{\xi}^m} + \frac{c|\beta|}{L_\gamma}\xi^i + \frac{c|\beta|a_{i\bar{j}}F}{4\alpha L_\gamma}\xi^i + \frac{\alpha c|\beta|b_i}{2L_\gamma}\xi^i - \frac{\alpha^3 l_i b^i b^{\bar{j}}}{4F_\gamma\bar{\xi}^m} \\
& - \frac{\alpha^2 a_{i\bar{j}}Fb^ib^{\bar{j}}}{4F_\gamma\bar{\xi}^m} - \frac{\alpha^4 b^{\bar{j}}}{2F_\gamma\bar{\xi}^m} - \frac{\alpha\bar{\beta}\xi^i b^{\bar{j}}}{4F_\gamma\bar{\xi}^m} - \frac{a_{i\bar{j}}F\bar{\beta}b^{\bar{j}}\xi^i}{4F_\gamma\bar{\xi}^m} - \frac{\alpha^2\bar{\beta}\xi^i}{2F_\gamma\bar{\xi}^m} - \frac{\alpha\beta b^i}{4F_\gamma} - \frac{a_{i\bar{j}}F\beta b^i}{4F_\gamma} \\
& - \frac{\alpha^2\beta}{2F_\gamma}\Big)\frac{\partial\eta^i}{\partial z^j} + \Big(\frac{\alpha a^{\bar{j}i}l_i}{4F\bar{\xi}^m} + \frac{\alpha^2 a^{\bar{j}i}b_i}{2F\bar{\xi}^m} - \frac{\alpha^3 l_i b^i b^{\bar{j}}}{4F_\gamma\bar{\xi}^m} - \frac{\alpha^4 b^{\bar{j}}}{2F_\gamma\bar{\xi}^m} - \frac{\alpha\bar{\beta}l_i\xi^i b^{\bar{j}}}{4F_\gamma\bar{\xi}^m} - \frac{\alpha^2\bar{\beta}\xi^i}{2F_\gamma\bar{\xi}^m}\Big)\Gamma^i_{mk}\eta^m \\
& + \Big(\frac{\alpha a^{\bar{j}i}l_i}{4F\bar{\xi}^m} + \frac{\alpha^2 a^{\bar{j}i}b_i}{2F\bar{\xi}^m} - \frac{\alpha^3 l_i b^i b^{\bar{j}}}{4F_\gamma\bar{\xi}^m} - \frac{\alpha^4 b^{\bar{j}}}{2F_\gamma\bar{\xi}^m} - \frac{\alpha\bar{\beta}l_i\xi^i b^{\bar{j}}}{4F_\gamma\bar{\xi}^m} - \frac{\alpha^2\bar{\beta}\xi^i}{2F_\gamma\bar{\xi}^m}\Big)\Gamma^i_{mk}\eta^k \\
& + \Big(\frac{1}{4\bar{\xi}^m} + \frac{cl_i|\beta|}{4L_\gamma}\xi^i + \frac{cFa_{i\bar{j}}|\beta|}{4\alpha L_\gamma}\xi^i + \frac{\alpha cb_i|\beta|}{2L_\gamma}\xi^i - \frac{\alpha^2 a_{i\bar{j}}Fb^ib^{\bar{j}}}{4F_\gamma\bar{\xi}^m} - \frac{\beta Fa_{i\bar{j}}\xi^i b^{\bar{j}}}{4F_\gamma\bar{\xi}^m} \\
& - \frac{\alpha\beta b^i l_i}{4F_\gamma} - \frac{\beta a_{i\bar{j}}Fb^i}{4F_\gamma} - \frac{\alpha^2\beta}{2F_\gamma}\Big)\Gamma^i_{jk}\eta^k + \Big(\frac{1}{4\bar{\xi}^m} + \frac{cl_i|\beta|}{4L_\gamma}\xi^i + \frac{cFa_{i\bar{j}}|\beta|}{4\alpha L_\gamma}\xi^i \\
& + \frac{\alpha cb_i|\beta|}{2L_\gamma}\xi^i - \frac{\alpha^2 a_{i\bar{j}}Fb^ib^{\bar{j}}}{4F_\gamma\bar{\xi}^m} - \frac{\bar{\beta}Fa_{i\bar{j}}\xi^i b^{\bar{j}}}{4F_\gamma\bar{\xi}^m} - \frac{\alpha\beta b^i}{4F_\gamma} - \frac{\beta a_{i\bar{j}}Fb^i}{4F_\gamma} - \frac{\alpha^2\beta}{2F_\gamma}\Big)\Gamma^i_{jk}\eta^j,
\end{aligned}$$

$$\begin{aligned}\overset{(2)CL}{M_j^i} = {} & [\tfrac{1}{2F}a^{\bar{j}i} + \tfrac{a^{\bar{j}i}}{2} + \tfrac{|\beta|c(1+F)}{2\alpha L_\gamma}(\xi^i)^2\bar{\xi}^j - \tfrac{\alpha^3(1+F)b^i b^{\bar{j}}\xi^i}{2\alpha F_\gamma} - (\bar{\beta}\xi^i b^{\bar{j}} \\ & +\beta b^i\bar{\xi}^j)\tfrac{\alpha(1+F)}{2\alpha F_\gamma}\xi^i]\tfrac{\partial a_{i\bar{k}}}{\partial z^j} + [\tfrac{1}{2F} + \tfrac{\alpha^2 a^{\bar{j}i}b_i}{F} + \tfrac{1}{2} + \tfrac{a_{i\bar{k}}}{2\alpha} + \tfrac{|\beta|c\alpha b_i}{2L_\gamma}\xi^i \\ & -\tfrac{\alpha^3(1+F)b^i b^{\bar{j}}a_{i\bar{k}}}{2\alpha F_\gamma} - \tfrac{\alpha^4||b||b_i}{2F_\gamma\bar{\xi}^m} - \tfrac{\alpha^2}{2F_\gamma}(\bar{\beta}\xi^i\tfrac{b^{\bar{j}}}{\bar{\xi}^m} + \beta b^i)b_i - \tfrac{\alpha(1+F)}{F_\gamma}(\bar{\beta}\xi^i b^{\bar{j}} \\ & +\beta b^i\bar{\xi}^j)\tfrac{a_{i\bar{k}}}{2\alpha}]\tfrac{\partial\zeta i}{\partial z^j} + [\tfrac{1}{4F}a^{\bar{j}i}\Gamma^i_{mk} + \tfrac{\alpha a^{\bar{j}i}b_i}{2F}\Gamma^i_{mk} + \tfrac{a^{\bar{j}i}}{4}\Gamma^i_{mk} + \tfrac{|\beta|c(1+F)}{4\alpha L_\gamma}\xi^i\bar{\xi}^j\Gamma^i_{mk} \\ & +\tfrac{|\beta|c\alpha\xi^i}{4L_\gamma}b_i\Gamma^i_{jk} - \tfrac{\alpha^3(1+F)b^i b^{\bar{j}}}{4\alpha F_\gamma}\Gamma^i_{mk} - \tfrac{\alpha^4||b||b_i}{4F_\gamma\bar{\xi}^m}\Gamma^i_{mk} - \tfrac{\alpha^2}{4F_\gamma}(\bar{\beta}\xi^i\tfrac{b^{\bar{j}}}{\bar{\xi}^m} + \beta b^i)b_i\Gamma^i_{mk} \\ & -\tfrac{\alpha(1+F)}{4\alpha F_\gamma}(\bar{\beta}\xi^i b^{\bar{j}} + \beta b^i\bar{\xi}^j)\Gamma^i_{mk}]\tfrac{\partial\eta^m}{\partial z^j}\eta^k + [\tfrac{1}{4F}a^{\bar{j}i}\Gamma^i_{mk} + \tfrac{\alpha a^{\bar{j}i}b_i}{2F}\Gamma^i_{mk} + \tfrac{a^{\bar{j}i}}{4}\Gamma^i_{mk} \\ & +\tfrac{|\beta|c(1+F)}{4\alpha L_\gamma}\xi^i\bar{\xi}^j\Gamma^i_{mk} + \tfrac{|\beta|c\alpha\xi^i}{4L_\gamma}b_i\Gamma^i_{jk} - \tfrac{\alpha^2(1+F)b^i b^{\bar{j}}}{4F_\gamma}\Gamma^i_{mk} - \tfrac{\alpha^4||b||}{4F_\gamma\bar{\xi}^m}\Gamma^i_{mk} \\ & -\tfrac{\alpha^2}{4F_\gamma}(\bar{\beta}\xi^i\tfrac{b^{\bar{j}}}{\bar{\xi}^m} + \beta b^i)b_i\Gamma^i_{jk} - \tfrac{(1+F)}{4F_\gamma}(\bar{\beta}\xi^i b^{\bar{j}} + \beta b^i\bar{\xi}^j)\Gamma^i_{mk}]\tfrac{\partial\eta^k}{\partial z^j}\eta^m + [\tfrac{\alpha^2}{2F\bar{\xi}^m}a^{\bar{j}i}\zeta^i \\ & +\tfrac{\alpha a^{\bar{j}i}}{2F}\Gamma^i_{mk}\eta^m\eta^k + \tfrac{c|\beta|\alpha\xi^i}{2L_\gamma}\zeta^i + \tfrac{1}{2}\Gamma^i_{jk}\eta^j\eta^k - \tfrac{\alpha^4||b||}{2F_\gamma\bar{\xi}^m}\zeta^i - \tfrac{\alpha^4||b||}{4F_\gamma\bar{\xi}^m}\Gamma^i_{mk}\eta^m\eta^k \\ & -\tfrac{\alpha^2}{2F_\gamma}(\bar{\beta}\xi^i\tfrac{b^{\bar{j}}}{\bar{\xi}^m} + \beta b^i)\zeta^i - \tfrac{\alpha^2}{4F_\gamma}\Gamma^i_{jk}\eta^j\eta^k]\tfrac{\partial b_i}{\partial z^j}.\end{aligned}$$

Proof. We compute these coefficients, taking into account that $L = (\alpha + |\beta|)^2$ and the relations (3.1). □

In [17], we pointed out the connection between the canonical complex spray and the Chern-Lagrange (c.n.c). We showed that due to the relations $\frac{\partial \overset{(1)CL}{M_k^i}}{\partial\eta^j} = \frac{\partial \overset{(1)CL}{M_j^i}}{\partial\eta^k} = \frac{\partial \overset{(2)CL}{M_j^i}}{\partial\zeta^k} = \frac{\partial \overset{(2)CL}{M_k^i}}{\partial\zeta^j}$, the canonical spray is derived from the Chern-Lagrange (c.n.c).

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