On the Lindley family distribution: quantile function, limiting distributions of sample minima and maxima and entropies

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Abstract. We propose in this paper an overview of several types of Lindley families distributions. More precisely, we give a treatment of the mathematical properties of new distributions named respectively gamma Lindley, Lindley Pareto, pseudo-Lindley, quasi Lindley, two parameters Lindley and power Lindley distributions. The properties studied include the quantile function, entropy, and limiting distribution of extreme order statistics. Simulations studies and data-driven applications are also reported.

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Key words: Lindley distribution; gamma Lindley distribution; pseudo Lindley distribution; quasi Lindley distribution, two parameters Lindley distribution, power Lindley distribution.

1 Introduction

There are several distributions for modeling lifetime data. Among the known parametric models, the most popular are the gamma, lognormal and the Lindley distributions. The Lindley distribution is more popular than the gamma and lognormal distributions because it is flexible and modeling different types of lifetime data.

Moreover, mixture models provide a mathematical-based, flexible and meaningful approach for the wide variety of classification requirements: Unsupervised or supervised classification, data features, a number of classes selection, etc. Fields in which mixture models have been successfully applied are numerous and specific software is now available.

Although the Lindley distribution has drawn little attention in the statistical literature over the great popularity of the well-known exponential distribution, the considered form is closed [17], according to [12], the Lindley distribution is important for studying stress–strength reliability modeling. Moreover, some researchers have proposed new classes of distributions based on modifications of the Lindley distribution, including also their properties. The main idea is always directed by embedding

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former distributions to more flexible structures. Sankaran introduced in [19] the discrete Poisson-Lindley distribution by combining the Poisson and Lindley distributions. Ghitany et al. [9] investigated most of the statistical properties of the Lindley distribution, showing that this distribution may provide a better fitting than the exponential distribution. Mahmoudi and Zakerzadeh proposed in [16] an extended version of the compound Poisson distribution which was obtained by compounding the Poisson distribution with the generalized Lindley distribution. Louzada et al. proposed in [18] a complementary exponential geometric distribution by combining the geometric and the exponential distributions. That flexibility is the reason behind making the practitioners using it as the first choice to fit their data.

Recently, Zeghdoudi and Nedjar [25, 26, 28, 29], and Zeghdoudi and Lazri [24] introduced several distributions, called gamma Lindley, based on mixtures of gamma $(2, \theta)$ and one-parameter Lindley (θ) distributions, pseudo Lindley, based on mixtures of gamma $(2, \theta)$ and exponential(θ) distributions, Lindley Pareto distributions.

In this work, we give a treatment of the mathematical properties of new distributions named respectively Gamma Lindley, Lindley Pareto, pseudo-Lindley, quasi Lindley, two parameters Lindley and power Lindley distributions. The studied properties include the quantile function, maximum likelihood estimation, and entropy. Simulation studies and data driven applications are also considered.

2 The quantile functions

2.1 The Lambert W function

The Lambert W function has attracted a great deal of attention beginning with Lambert in 1758 and Euler in 1779. The name of *Lambert W function* has become a standard, after its implementation in the computer algebra system Maple in the 1980s and subsequent publication by Corless et al. [6] of a comprehensive survey of the history, theory and applications of this function. The Lambert W function is a multivalued complex function defined as the solution of the equation:

$$W(z)exp(W(z)) = z, \tag{2.1}$$

where z is a complex number. If z is a real number such that $z \geq -1/e$, then $W(z)$ becomes a real function and there are two possible real branches. The real branch taking on values in $(-\infty, -1]$ is called the negative branch and denoted by W_{-1}. The real branch taking on values in $[-1, \infty)$ is called the principal branch and denoted by W_0. Both real branches of W are depicted in Fig. 1. We emphasize that (2.1) has two possible solutions if $z \in (-1/e, 0]$, and a unique solution if $z \geq 0$. For our results in this note, we shall use the negative branch W_{-1}, which satisfies the following elementary properties: $W_{-1}(-1/e) = -1$, $W_{-1}(z)$ is decreasing as z increases to 0 and $W_{-1}(z) \to -\infty$ as $z \to 0$.

Lemma 2.1. *Let a, b and c complex numbers. The solution of the equation $z+ab^z = c$ with respect to $z \in \mathbb{C}$ is*

$$z = c - \frac{1}{\log(b)} W\left(ab^c \log(b)\right),$$

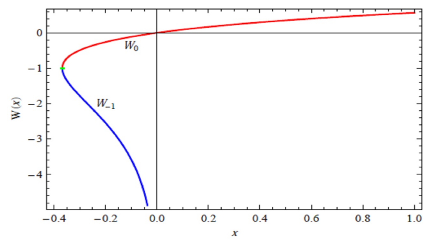

Figure 1: Lambert W function graph

where W denotes the Lambert W function. For details of the proof see [11]. The quantile function of X is $Q_X(u) = F_X^{-1}(u)$, $0 < u < 1$.

In the following, we give the explicit expressions for Q_X in terms of the Lambert W function.

Theorem 2.2. *For any $\theta \succ 0, \beta \succ \frac{\theta}{(1+\theta)}$, the quantile function of the Gamma Lindley distribution X is*

$$Q_X(u) = -\frac{\beta(1+\theta)}{\theta(\beta(1+\theta)-\theta)} - \frac{1}{\theta} W_{-1}\left(\frac{\beta(1+\theta)(u-1)}{\beta(1+\theta)-\theta} e^{-\frac{\beta(1+\theta)}{\beta(1+\theta)-\theta}}\right), 0 \prec u \prec 1,$$

where W_{-1} denotes the negative branch of the Lambert W function.

Proof. For any fixed $\theta \succ 0, \beta \succ \frac{\theta}{(1+\theta)}$, let $u \in (0,1)$. We have to solve the equation $F_X(x) = u$ with respect to x, for $x \succ 0$. We have to solve the following equation:

$$[\theta [\beta(1+\theta)-\theta] x + \beta(1+\theta)] e^{-\theta x} = \beta(1+\theta)(1-u). \tag{2.2}$$

Multiplying by $\frac{e^{-\frac{\beta(1+\theta)}{\beta(1+\theta)-\theta}}}{\beta(1+\theta)-\theta}$ at both sides of equation (2.2), we obtain:

$$-\frac{\theta [\beta(1+\theta)-\theta] x + \beta(1+\theta)}{\beta(1+\theta)-\theta} e^{-\left[\frac{\theta[\beta(1+\theta)-\theta]x+\beta(1+\theta)}{\beta(1+\theta)-\theta}\right]} = \frac{\beta(1+\theta)(u-1)}{\beta(1+\theta)-\theta} e^{-\frac{\beta(1+\theta)}{\beta(1+\theta)-\theta}}. \tag{2.3}$$

From equation (2.3), we see that $-\frac{\theta[\beta(1+\theta)-\theta]x+\beta(1+\theta)}{\beta(1+\theta)-\theta}$ is the Lambert W function of the real argument $\frac{\beta(1+\theta)(u-1)}{\beta(1+\theta)-\theta} e^{-\frac{\beta(1+\theta)}{\beta(1+\theta)-\theta}}$. Then, we have

$$W\left(\frac{\beta(1+\theta)(u-1)}{\beta(1+\theta)-\theta} e^{-\frac{\beta(1+\theta)}{\beta(1+\theta)-\theta}}\right) = -\frac{\theta [\beta(1+\theta)-\theta] x + \beta(1+\theta)}{\beta(1+\theta)-\theta}, 0 < u < 1. \tag{2.4}$$

Moreover, for any $\theta, \beta \succ 0$ and $x \succ 0$ it is immediate that $\frac{\theta[\beta(1+\theta)-\theta]x+\beta(1+\theta)}{\beta(1+\theta)-\theta} \succ 0$, and it can also be checked that

$$-\frac{\theta [\beta(1+\theta)-\theta] x + \beta(1+\theta)}{\beta(1+\theta)-\theta} e^{-\left[\frac{\theta[\beta(1+\theta)-\theta]x+\beta(1+\theta)}{\beta(1+\theta)-\theta}\right]} = \frac{\beta(1+\theta)(u-1)}{\beta(1+\theta)-\theta} e^{-\frac{\beta(1+\theta)}{\beta(1+\theta)-\theta}} \in \left(\frac{-1}{e}, 0\right),$$

since $u \in (0,1)$. Therefore, by taking into account the properties of the negative branch of the Lambert W function, equation (2.4) becomes

$$W_{-1}\left(\frac{\beta(1+\theta)(u-1)}{\beta(1+\theta)-\theta} e^{-\frac{\beta(1+\theta)}{\beta(1+\theta)-\theta}}\right) = -\frac{\theta [\beta(1+\theta)-\theta] x + \beta(1+\theta)}{\beta(1+\theta)-\theta},$$

which completes the proof. $\square$

2.2 The quantile function of the Lindley Pareto distribution

Theorem 2.3. *The quantile function of the L-P distribution X is*

$$x_\gamma = \alpha \left(-\frac{1}{\theta} - \frac{1}{\theta} \mathrm{LambertW}(-1, (\gamma-1)(\theta+1)\exp(-(\theta+1))) \right)^{\frac{1}{k}}, 0 < \gamma < 1,$$

where $\theta, \alpha, k > 0$ and LambertW *denotes the negative branch of the Lambert W function ($W(z)\exp(W(z)) = z$), and z is a complex number.*

Proof. For any fixed $\theta, \alpha, k > 0$, let $\gamma \in (0,1)$. We have to solve the equation $F_X(x) = \gamma$ with respect to x, for $x > \alpha$, namely:

$$-\left(\frac{\theta}{\alpha^k}x^k + 1\right)\exp\left(-\theta\left(\frac{x^k}{\alpha^k} - 1\right)\right) = (\gamma - 1)(\theta + 1). \tag{2.5}$$

Multiplying by $e^{-(\theta+1)}$ both sides of equation (2.5), we obtain:

$$-\left(\frac{\theta}{\alpha^k}x^k + 1\right)\exp\left(-\left(\frac{\theta}{\alpha^k}x^k + 1\right)\right) = (\gamma - 1)(\theta + 1)e^{-(\theta+1)},$$

and we see that $-\left(1 + \frac{x^k}{\alpha^k}\theta\right)$ is the Lambert W function of the real argument $(\gamma - 1)(\theta + 1)e^{-(\theta+1)}$. Thus, we have

$$\mathrm{LambertW}\left((\gamma - 1)(\theta + 1)e^{-(\theta+1)}\right) = -\left(\frac{\theta}{\alpha^k}x^k + 1\right). \tag{2.6}$$

Moreover, for any $\theta, \alpha, k > 0$ and $x > \alpha$ it is immediate that $\left(1 + \frac{\theta}{\alpha^k}x^k\right) > 1$ and it can also be checked that $-\left(\frac{\theta}{\alpha^k}x^k + 1\right)e^{-\left(\frac{\theta}{\alpha^k}x^k+1\right)} = (\gamma - 1)(\theta + 1)e^{-(\theta+1)} \in \left(\frac{-1}{e}, 0\right)$ since $\gamma \in (0,1)$. Therefore, by taking into account the properties of the negative branch of the Lambert W function, equation (2.6) becomes

$$\mathrm{LambertW}\left(-1, (\gamma-1)(\theta+1)e^{-(\theta+1)}\right) = -\left(\frac{\theta}{\alpha^k}x^k + 1\right).$$

Again, solving for x, we get

$$x_\gamma = \alpha\left(-\frac{1}{\theta} - \frac{1}{\theta}\mathrm{LambertW}\left(-1, (\gamma - 1)(\theta + 1)e^{-(\theta+1)}\right)\right)^{\frac{1}{k}}.$$

□

2.3 The quantile function of the pseudo Lindley distribution

Theorem 2.4. *For any $\theta \succ 0, \beta \succ 1$, the quantile function of the pseudo-Lindley distribution X is*

$$Q_X(u) = -\frac{\beta}{\theta} - \frac{1}{\theta}W_{-1}\left(\beta e^{-\beta}(u-1)\right), \quad 0 < u < 1,$$

where W_{-1} denotes the negative branch of the Lambert W function.

Proof. For any fixed $\theta \succ 0, \beta \succ 1$, let $u \in (0,1)$. We have to solve the equation $F_X(x) = u$ with respect to x, for x$\succ$ 0, namely:

$$(\beta + \theta x)e^{-\theta x} = \beta(1-u). \tag{2.7}$$

Multiplying by $-e^{-\beta}$ both sides of equation (2.7), we obtain:

$$-(\beta + \theta x)\, e^{-(\beta+\theta x)} = \beta(u-1)e^{-\beta}. \tag{2.8}$$

From (2.8), we see that $-(\beta + \theta x)$ is the Lambert W function of the real argument $\beta(u-1)e^{-\beta}$. Then, we have

$$W\left(\beta e^{-\beta}(u-1)\right) = -(\beta + \theta x)\,. \tag{2.9}$$

Moreover, for any $\theta, \beta \succ 0$ and $x \succ 0$ it is immediate that $(\beta + \theta x) \succ 0$ and it can also be checked that $-(\beta + \theta x)\, e^{-(\beta+\theta x)} = \beta(u-1)e^{-\beta} \in \left(\frac{-1}{e}, 0\right)$, since $u \in (0,1)$. Therefore, by taking into account the properties of the negative branch of the Lambert W function, (2.9) becomes

$$W_{-1}\left(\beta e^{-\beta}(u-1)\right) = -(\beta + \theta x)\,,$$

which in turn implies the result. □

2.4 The quantile function of the quasi Lindley distribution

Theorem 2.5. *For any $\theta, \alpha > \frac{1}{e} - 1$, the quantile function of the quasi Lindley distribution X is*

$$Q_X(u) = -\frac{1}{\theta} - \frac{\alpha}{\theta} - \frac{1}{\theta} W_{-1}\left((u-1)(\alpha+1)\exp\left(-1-\alpha\right)\right), \quad 0 < u < 1,$$

where W_{-1} denotes the negative branch of the Lambert W function.

Proof. For any fixed $\theta, \alpha > \frac{1}{e} - 1$, let $u \in (0,1)$. We have to solve the equation $F_X(x) = u$ with respect to x, for $x > 0$, namely:

$$(1 + \alpha + \theta x)\exp\left(-\theta x\right) = (1-u)(\alpha+1). \tag{2.10}$$

Multiplying by $\exp\left(-(1+\alpha)\right)$ both sides of the equation (2.10), we obtain:

$$-(1 + \alpha + \theta x)\exp\left(-(1 + \alpha + \theta x)\right) = (u-1)(\alpha+1)\exp\left(-(1+\alpha)\right). \tag{2.11}$$

From this equation, we see that $-(1 + \alpha + \theta x)$ is the Lambert W function of the real argument $(u-1)(\alpha+1)\exp\left(-(1+\alpha)\right)$. Then, we have

$$W\left[(u-1)(\alpha+1)\exp\left(-(1+\alpha)\right)\right] = -(1 + \alpha + \theta x)\,.$$

Moreover, for any $\theta, \alpha > \frac{1}{e} - 1$ and $x > 0$, it is immediate that $(1 + \alpha + \theta x) > 0$, and it can also be checked that

$$-(1 + \alpha + \theta x)\exp\left(-(1 + \alpha + \theta x)\right) = (u-1)(\alpha+1)\exp\left(-(1+\alpha)\right) \in \left(\frac{-1}{e}, 0\right),$$

since $u \in (0,1), \alpha > \frac{1}{e} - 1$. Therefore, by taking into account the properties of the negative branch of the Lambert W function, equation (2.11) becomes

$$W_{-1}\left[(u-1)(\alpha+1)\exp\left(-(1+\alpha)\right)\right] = -(1 + \alpha + \theta x)\,,$$

which in turn implies the result. □

$\alpha = 0.5$	$\theta = 0.1$	$\theta = 0.5$	$\theta = 5$
Q_1	6.4566779	1.291	0.1291
Q_2= *median*	13.268424	2.6536	0.2653
Q_3	23.2146244	4.6429	0.4642
mean	16.6667	3.3333	0.3333
mod	5	1	0.1

$\alpha = 1$	$\theta = 0.1$	$\theta = 0.5$	$\theta = 5$
Q_1	5.18	1.036	0.1036
Q_2= *median*	11.462	2.2923	0.2292
Q_3	21.0546	4.211	0.4210
mean	15	3	0.3
mod	0	0	0

$\alpha = 1.5$	$\theta = 0.1$	$\theta = 0.5$	$\theta = 5$
Q_1	4.5483	0.9096	0.0909
Q_2= *median*	10.4137	2.0827403	0.2082
Q_3	19.6664	3.9332802	0.3933
mean	14	2.8	0.28
mod	0	0	0

$\alpha = 2$	$\theta = 0.1$	$\theta = 0.5$	$\theta = 5$
Q_1	4.1817	0.8363	0.0836
Q_2= *median*	9.7441	1.9488	0.1948
Q_3	18.7097	3.7419	0.3742
mean	13.3333	2.6666	0.2666
mod	0	0	0

2.5 The quantile function of the two parameters Lindley distribution

Theorem 2.6. *For any $\theta, \alpha > 0$, the quantile function of the two parameter Lindley distribution X is*

$$Q_X(u) = -\frac{\theta+\alpha}{\theta\alpha} - \frac{1}{\theta} W\left[-\frac{1}{\alpha}(1-u)(\theta+\alpha)\exp\left(-\frac{\theta+\alpha}{\alpha}\right)\right], \quad 0 \ < u < 1,$$

where W_{-1} denotes the negative branch of the Lambert W function.

Proof. For any fixed $\theta, \alpha > 0$, let $u \in (0,1)$. We have to solve the equation $F_X(x) = u$ with respect to x, for $x > 0$, namely the equation:

$$\theta + \alpha + \alpha\theta x = (1-u)(\theta+\alpha)\exp(\theta x). \tag{2.12}$$

Multiplying by $\frac{1}{\alpha}$ and adding $-\frac{1}{\alpha}(1-u)(\theta+\alpha)\exp(\theta x) - \frac{\theta+\alpha}{\alpha}$ to both sides of equation (2.12), we obtain:

$$\theta x - -\frac{1}{\alpha}(1-u)(\theta+\alpha)\exp(\theta x) = -\frac{\theta+\alpha}{\alpha}.$$

From this, we see that $-\theta x - \frac{\theta+\alpha}{\alpha}$ is the Lambert W function of the real argument $-\frac{1}{\alpha}(1-u)(\theta+\alpha)\exp\left(-\frac{\theta+\alpha}{\alpha}\right)$. Then, we have

$$-\left(\theta x + \frac{\theta+\alpha}{\alpha}\right) = W\left[-\frac{1}{\alpha}(1-u)(\theta+\alpha)\exp\left(-\frac{\theta+\alpha}{\alpha}\right)\right], \quad 0 < u < 1. \tag{2.13}$$

Moreover, for any $\theta, \alpha > 0$ and $x > 0$, it is immediate that $\theta x + \frac{\theta+\alpha}{\alpha} > 0$, and it can also be checked that

$$-\left(\theta x + \frac{\theta+\alpha}{\alpha}\right)\exp\left(-\left(\theta x + \frac{\theta+\alpha}{\alpha}\right)\right) = -\frac{1}{\alpha}(1-u)(\theta+\alpha)\exp\left(-\frac{\theta+\alpha}{\alpha}\right) \in \left(\frac{-1}{e}, 0\right),$$

since $u \in (0,1)$. Therefore, by taking into account the properties of the negative branch of the Lambert W function, (2.13) becomes

$$W_{-1}\left[-\frac{1}{\alpha}(1-u)(\theta+\alpha)\exp\left(-\frac{\theta+\alpha}{\alpha}\right)\right] = -\left(\theta x + \frac{\theta+\alpha}{\alpha}\right),$$

which in turn implies the result. $\square$

$\alpha = 0.5$	$\theta = 0.1$	$\theta = 0.5$	$\theta = 5$
Q_1	16.0739	4.5289	2.2000
Q_2= *median*	14.2670	4.3171	2.2001
Q_3	12.9984	4.1455	2.2006
mean	18.3333	3	0.2181
mod	8	0	0

$\alpha = 1$	$\theta = 0.1$	$\theta = 0.5$	$\theta = 5$
Q_1	15.1649	3.7193	1.2022
Q_2= *median*	13.3055	3.4110	1.2014
Q_3	12.0129	3.1834	1.2007
mean	19.0909	3.3333	0.2333
mod	9.9	1	0

$\alpha = 1.5$	$\theta = 0.1$	$\theta = 0.5$	$\theta = 5$
Q_1	14.85	3.4444	0.8755
Q_2= *median*	12.9798	3.1040	0.8725
Q_3	11.6825	2.8602	0.8695
mean	19.375	3.5	0.2461
mod	9.3333	1.3333	0

$\alpha = 2$	$\theta = 0.1$	$\theta = 0.5$	$\theta = 5$
Q_1	14.6899	3.3023	0.7172
Q_2= *median*	12.8159	2.9480	0.7111
Q_3	11.5169	2.6976	0.7054
mean	19.5238	3.6	0.2571
mod	9.95	1.5	0

2.6 The quantile function of the power Lindley distribution

Theorem 2.7. *For any $\beta, \alpha > 0$, the quantile function of the power Lindley distribution X is*

$$Q_X(u) = \left[-\frac{\beta+1}{\beta} - \frac{1}{\beta}W_{-1}\left[-(1-u)(\beta+1)\exp\left(-(\beta+1)\right)\right]\right]^{\frac{1}{\alpha}}, \quad 0 < u < 1,$$

where W_{-1} denotes negative branch of the Lambert W function.

Proof. For any fixed $\beta, \alpha > 0$, let $u \in (0,1)$. We have to solve the equation $F_X(x) = u$ with respect to x, for $x > 0$, namely

$$\beta x^\alpha - (1-u)(\beta+1)\exp\left(\beta x^\alpha\right) = -(\beta+1). \tag{2.14}$$

From equation (2.14), we obtain:

$$\beta x^\alpha = -(\beta+1) - W\left(-(1-u)(\beta+1)\exp(-(\beta+1))\right). \tag{2.15}$$

From equation (2.15), we see that $-\left(\beta x^\alpha + (\beta+1)\right)$ is the Lambert W function of the real argument $-(1-u)(\beta+1)\exp\left(-(\beta+1)\right)$. Then, we have

$$-\left(\beta x^\alpha + (\beta+1)\right) = W\left[-(1-u)(\beta+1)\exp\left(-(\beta+1)\right)\right], 0 < u < 1. \tag{2.16}$$

Moreover, for any $\theta, \alpha > 0$ and $x > 0$ it is immediate that $\theta x + \frac{\theta+\alpha}{\alpha} > 0$ and it can also be checked that

$$-\left(\beta x^\alpha + (\beta+1)\right)\exp\left(-\beta x^\alpha - (\beta+1)\right) = -(1-u)(\beta+1)\exp(-(\beta+1)) \in \left(\frac{-1}{e}, 0\right),$$

since $u \in (0,1)$. Therefore, by taking into account the properties of the negative branch of the Lambert W function, equation (2.16) becomes

$$-(\beta x^\alpha + (\beta+1)) = W_{-1}[-(1-u)(\beta+1)\exp(-(\beta+1))],$$

which in turn implies the result. □

$\alpha = 0.5$	$\beta = 0.1$	$\beta = 0.5$	$\beta = 5$
Q_1	75.8333	1.6675	0.0047
Q_2= *median*	251.4840	7.0420	0.0269
Q_3	675.5133	21.556	0.1057
mean	254.8406	7.420	0.0369

$\alpha = 1$	$\beta = 0.1$	$\beta = 0.5$	$\beta = 5$
Q_1	8.7082	1.2913	0.0686
Q_2= *median*	15.8582	2.6536	0.1642
Q_3	25.9902	4.6429	0.3252
mean	15.582	2.536	0.2042

$\alpha = 1.5$	$\beta = 0.1$	$\beta = 0.5$	$\beta = 5$
Q_1	4.2327	1.1858	0.1676
Q_2= *median*	6.3120	1.9167	0.2999
Q_3	8.7742	2.7830	0.4729
mean	6.20	2.167	0.321

$\alpha = 2$	$\beta = 0.1$	$\beta = 0.5$	$\beta = 5$
Q_1	2.9509	1.13636	0.26204
Q_2= *median*	3.9822	1.62901	0.40532
Q_3	5.0981	2.15474	0.57028
mean	3.8822	1.2901	0.43532

3 Limiting distributions of sample minima and maxima

If $X_1, \dots, X_n$ is a random sample, and if $\overline{X} = \dfrac{X_1 + \cdots + X_n}{n}$ denotes the sample mean then by the usual central limit theorem $\dfrac{\sqrt{n}\left(\overline{X} - E(X)\right)}{\sqrt{Var(X)}}$ approaches the standard normal distribution as $n \to \infty$. Sometimes one would be interested in the asymptotic of the extreme values $X_{n:n} = \max(X_1, \dots, X_n)$ and $X_{1:n} = \min(X_1, \dots, X_n)$.

We can derive the asymptotic distributions of the sample minimum $X_{1:n}$ and the sample maximum $X_{n:n}$ by using [1, Th. 8.3.2, 8.3.3, and 8.3.6], so we have the following results.

3.1 Limiting distributions of sample minima and maxima of the gamma Lindley distribution

The asymptotic distributions of the sample minimum $X_{1:n}$ and the sample maximum $X_{n,n}$ of the Gamma Lindley distribution are

$$\begin{cases} P\left\{a_n(X_{n:n} - b_n) \leq x\right\} \xrightarrow{\mathcal{D}} \exp(-\exp(-\theta x)), \\ P\left\{c_n(X_{1:n} - d_n) \leq x\right\} \xrightarrow{\mathcal{D}} 1 - \exp(-x), \end{cases}$$

where the norming constants $a_n, b_n, c_n > 0$ and d_n are given in [14, Th. 1.6.2].

Proof. For all $x > 0$, by using l'Hospital rule, it follows that,

$$\begin{aligned} \lim_{t\to 0} \tfrac{F(tx)}{F(t)} &= x \lim_{t\to 0} \tfrac{f(tx)}{f(t)} \\ &= x \lim_{t\to 0} \frac{\dfrac{\theta^2((\beta + \beta\theta - \theta)xt + 1)e^{-\theta xt}}{\beta(1+\theta)}}{\dfrac{\theta^2((\beta + \beta\theta - \theta)t + 1)e^{-\theta t}}{\beta(1+\theta)}} \\ &= x. \end{aligned}$$

Also,

$$\begin{aligned} \lim_{t\to\infty} \tfrac{1-F(t+x)}{1-F(t)} &= \lim_{t\to\infty} \frac{\frac{((\theta\beta+\beta-\theta)(\theta(t+x)+1)+\theta)\exp(-\theta(x+t))}{\beta(1+\theta)}}{\frac{((\theta\beta+\beta-\theta)(\theta t+1)+\theta)\exp(-\theta t)}{\beta(1+\theta)}} \\ &= \lim_{t\to\infty} \tfrac{((\theta\beta+\beta-\theta)(\theta(t+x)+1)+\theta)\exp(-\theta(x+t))}{((\theta\beta+\beta-\theta)(\theta t+1)+\theta)\exp(-\theta t)} \\ &= \lim_{t\to\infty} \tfrac{((\theta\beta+\beta-\theta)(\theta(t+x)+1)+\theta)\exp(-\theta x)}{((\theta\beta+\beta-\theta)(\theta t+1)+\theta)}, \end{aligned}$$

and hence $\lim_{t\to\infty} \frac{1-F(t+x)}{1-F(t)} = \exp(-\theta x)$. □

3.2 Limiting distributions of sample minima and maxima of the Lindley Pareto distribution

The asymptotic distributions of the sample minimum $X_{1:n}$ and the sample maximum $X_{n,n}$ of the Lindley Pareto distribution are

$$\begin{cases} P\left\{a_n(X_{n:n}-b_n)\le x\right\} \stackrel{\mathcal{D}}{\to} \exp(-\exp(-\theta(\frac{x}{\alpha})^k)), \\ P\left\{c_n(X_{1:n}-d_n)\le x\right\} \stackrel{\mathcal{D}}{\to} 1-\exp(-x^{2k}), \end{cases}$$

where the norming constants $a_n, b_n, c_n > 0$ and d_n are given in [14, Th. 1.6.2].

Proof. By using the l'Hospital rule, it follows that, for all $x > 0$,

$$\begin{aligned} \lim_{t\to 0}\frac{F(tx)}{F(t)} &= x\lim_{t\to 0}\frac{f(tx)}{f(t)} \\ &= x\lim_{t\to 0}\frac{\frac{k\theta^2\exp(\theta)}{(\theta+1)\alpha^{2k}}(tx)^{2k-1}\exp(-\theta(\frac{tx}{\alpha})^k)}{\frac{k\theta^2\exp(\theta)}{(\theta+1)\alpha^{2k}}t^{2k-1}\exp(-\theta(\frac{t}{\alpha})^k)} = x^{2k}. \end{aligned}$$

Also

$$\begin{aligned} \lim_{t\to\infty}\frac{1-F(t+x)}{1-F(t)} &= \lim_{t\to\infty}\frac{\frac{1}{(\theta+1)\alpha^k}(\alpha^k+(t+x)^k\theta)\exp(-\theta\frac{(t+x)^k}{\alpha^k}-1)}{\frac{1}{(\theta+1)\alpha^k}(\alpha^k+t^k\theta)\exp(-\theta\frac{t^k}{\alpha^k}-1)} \\ &= \lim_{t\to\infty}\frac{(\alpha^k+(t+x)^k\theta)\exp(-\frac{\theta}{\alpha^k}((t+x)^k-t^k))}{(\alpha^k+t^k\theta)}, \end{aligned}$$

whence $\lim_{t\to\infty}\frac{1-F(t+x)}{1-F(t)} = \exp(-\theta(\frac{x}{\alpha})^k)$. □

3.3 Limiting distributions of sample minima and maxima of the pseudo Lindley distribution

The asymptotic distributions of the sample minimum $X_{1:n}$ and the sample maximum $X_{n,n}$ of the pseudo Lindley distribution are

$$\begin{cases} P\left\{a_n(X_{n:n}-b_n)\le x\right\} \stackrel{\mathcal{D}}{\to} \exp(-\exp(-\theta x)), \\ P\left\{c_n(X_{1:n}-d_n)\le x\right\} \stackrel{\mathcal{D}}{\to} 1-\exp(-x), \end{cases}$$

where the norming constants $a_n, b_n, c_n > 0$ and d_n are given in [14, Th. 1.6.2].

Proof. By using the l'Hospital rule, it follows that, for all $x > 0$,

$$\begin{aligned} \lim_{t\to 0}\frac{F(tx)}{F(t)} &= x\lim_{t\to 0}\frac{f(tx)}{f(t)} \\ &= x\lim_{t\to 0}\frac{\frac{\theta(\beta-1+\theta tx)\exp(-\theta tx)}{\beta}}{\frac{\theta(\beta-1+\theta t)\exp(-\theta t)}{\beta}} = x \end{aligned}$$

Also

$$\begin{aligned} \lim_{t\to\infty}\frac{1-F(t+x)}{1-F(t)} &= \lim_{x\to\infty}\frac{\frac{\beta+\theta(t+x)}{\beta}\exp(-\theta(t+x))}{\frac{\beta+\theta t}{\beta}\exp(-\theta t)} \\ &= \lim_{x\to\infty}\frac{\beta+\theta(t+x)}{\beta+\theta t}\exp(-\theta x) = \exp(-\theta x). \end{aligned}$$

□

3.4 Limiting distributions of sample minima and maxima of the quasi Lindley distribution

The asymptotic distributions of the sample minimum $X_{1:n}$ and the sample maximum $X_{n,n}$ of the quasi Lindley distribution are

$$\begin{cases} P\left\{a_n(X_{n:n}-b_n)\leq x\right\} \overset{\mathcal{D}}{\rightarrow} \exp(-\exp(-\theta x)), \\ P\left\{c_n(X_{1:n}-d_n)\leq x\right\} \overset{\mathcal{D}}{\rightarrow} 1-\exp(-x), \end{cases}$$

where the norming constants $a_n, b_n, c_n > 0$ and d_n are given in [14, Th. 1.6.2].

Proof. By using the l'Hospital rule, it follows that for all $x > 0$, we have

$$\begin{aligned} \lim_{t\to 0}\tfrac{F(tx)}{F(t)} &= x\lim_{t\to 0}\tfrac{f(tx)}{f(t)} = x\lim_{t\to 0}\frac{\frac{\theta(\alpha+tx\theta)}{\alpha+1}\exp(-\theta tx)}{\frac{\theta(\alpha+t\theta)}{\alpha+1}\exp(-\theta t)} \\ &= x\lim_{t\to 0}\tfrac{(\alpha+tx\theta)\exp(-\theta tx)}{(\alpha+t\theta)\exp(-\theta t)} = x. \end{aligned}$$

Also,

$$\begin{aligned} \lim_{x\to\infty}\frac{1-F(t+x)}{1-F(t)} &= \lim_{x\to\infty}\frac{\frac{1+\alpha+\theta(t+x)}{\theta+1}\exp(-\theta(t+x))}{\frac{1+\alpha+\theta t}{\theta+1}\exp(-\theta t)} \\ &= \lim_{x\to\infty}\frac{1+\alpha+\theta(t+x)}{1+\alpha+\theta t}\exp(-\theta x) = \exp(-\theta x). \end{aligned}$$

□

3.5 Limiting distributions of sample minima and maxima of the two parameters Lindley distribution

The asymptotic distributions of the sample minimum $X_{1:n}$ and the sample maximum $X_{n,n}$ of the wo Parameters Lindley distribution are

$$\begin{cases} P\left\{a_n(X_{n:n}-b_n)\leq x\right\} \overset{\mathcal{D}}{\rightarrow} \exp(-\exp(-\theta x)), \\ P\left\{c_n(X_{1:n}-d_n)\leq x\right\} \overset{\mathcal{D}}{\rightarrow} 1-\exp(-x), \end{cases}$$

where the norming constants $a_n, b_n, c_n > 0$ and d_n are given in [14, Th. 1.6.2].

Proof. By using the l'Hospital rule, it follows that, for all $x > 0$,

$$\begin{aligned} \lim_{t\to 0}\frac{F(tx)}{F(t)} &= x\lim_{t\to 0}\frac{f(tx)}{f(t)} \\ &= x\lim_{t\to 0}\frac{\frac{\theta^2(1+\alpha tx)}{\theta+\alpha}\exp\left(-\theta tx\right)}{\frac{\theta^2(1+\alpha t)}{\theta+\alpha}\exp\left(-\theta t\right)} = x. \end{aligned}$$

Also,

$$\begin{aligned} \lim_{t\to\infty}\frac{1-F(t+x)}{1-F(t)} &= \lim_{t\to\infty}\frac{\frac{\theta+\alpha+\alpha\theta(t+x)}{\theta+\alpha}\exp(-\theta(t+x))}{\frac{\theta+\alpha+\alpha\theta t}{\theta+\alpha}\exp(-\theta t)} \\ &= \lim_{t\to\infty}\frac{\theta+\alpha+\alpha\theta(t+x)}{\theta+\alpha+\alpha\theta t}\exp(-\theta x) = \exp(-\theta x). \end{aligned}$$

□

3.6 Limiting distributions of sample minima and maxima of the power Lindley distribution

The asymptotic distribution of the sample minimum $X_{1:n}$ of the power Lindley distribution is

$$P\left\{c_n(X_{1:n}-d_n)\leq x\right\}\overset{\mathcal{D}}{\rightarrow}1-\exp(-x),$$

where the norming constants $c_n>0$ and d_n are given in [14, Th. 1.6.2].

Proof. By using the l'Hospital rule, it follows that, for all $x>0$,

$$\begin{aligned}
\lim_{t\to 0}\frac{F(tx)}{F(t)} &= x\lim_{t\to 0}\frac{f(tx)}{f(t)}\\
&= x\lim_{t\to 0}\frac{\frac{\alpha\beta^2}{\beta+1}\left(1+(tx)^{\alpha}\right)(tx)^{\alpha-1}\exp\left(-\beta(tx)^{\alpha}\right)}{\frac{\alpha\beta^2}{\beta+1}\left(1+t^{\alpha}\right)t^{\alpha-1}\exp\left(-\beta t^{\alpha}\right)}\\
&= x\lim_{t\to 0}\frac{\left(1+(tx)^{\alpha}\right)x^{\alpha-1}\exp(-\beta(tx)^{\alpha})}{\left(1+t^{\alpha}\right)\exp(-\beta t^{\alpha})}=x^{\alpha}.
\end{aligned}$$

Also, $\lim\limits_{t\to\infty}\frac{1-F(t+x)}{1-F(t)}=\lim\limits_{t\to\infty}\frac{(1+\frac{\beta(t+x)^{\alpha}}{\beta+1})\exp(-\beta(x+t)^{\alpha})}{(1+\frac{\beta t^{\alpha}}{\beta+1})\exp(-\beta t^{\alpha})}=0.$ □

4 Entropies

In many fields of science such as communications, physics and probability, entropy is an important concept to measure the amount of uncertainty associated with a random variable X. Several entropy measures and information indices are available but among them the most popular entropy measure called Rényi entropy is defined as

$$\Im(\zeta)=\frac{1}{1-\zeta}\log\left(\int\limits_{\mathbb{R}^+}f^{\zeta}(x)dx\right),\ \text{for }\zeta>1.$$

As well, the Shannon entropy is defined by $E\left\{-\ln(f(X))\right\}$. This is a special case derived from $\lim\limits_{\zeta\to 1}\Im(\zeta)$.

4.1 Rényi entropy of the Gamma Lindley distribution

The Gamma Lindley distribution with parameters θ and β is defined by

$$f(x)=\frac{\theta^2((\beta+\beta\theta-\theta)x+1)e^{-\theta x}}{\beta(1+\theta)},\quad x,\theta,\beta>0.$$

Then we have

$$\int f^{\zeta}(x)dx=\frac{\theta^{2\zeta}e^{\frac{\zeta}{\beta+\beta\theta-\theta}}}{\beta^{\zeta}(1+\theta)^{\zeta}}E_{-\zeta}\left(\frac{\zeta}{\beta+\beta\theta-\theta}\right),$$

where

$$E_{-\zeta}\left(\frac{\zeta}{\beta+\beta\theta-\theta}\right)=\int_{0}^{\infty}u^{\zeta}e^{\frac{-\zeta}{\beta+\beta\theta-\theta}u}du.$$

Thus, the Rényi entropy of the $GaL(\theta,\beta)$ distribution is given by

$$\Im\left(\zeta\right)=\frac{1}{1-\zeta}\log\left[\frac{\theta^{2\zeta}e^{\frac{\zeta}{\beta+\beta\theta-\theta}}}{\beta^{\zeta}(1+\theta)^{\zeta}}E_{-\zeta}\left(\frac{\zeta}{\beta+\beta\theta-\theta}\right)\right].$$

4.2 Rényi entropy of the Lindley Pareto distribution

The Lindley Pareto distribution with parameters θ,α, and k is defined by its density function

$$f(x)=\frac{k\theta^{2}e^{\theta}}{\left(\theta+1\right)\alpha^{2k}}x^{2k-1}\exp\left(-\theta\left(\frac{x}{\alpha}\right)^{k}\right),\quad x>\alpha.$$

Then

$$\begin{aligned}\int_{\alpha}^{\infty}f^{\zeta}(x)dx &= \int_{\alpha}^{\infty}\tfrac{k^{\zeta}\theta^{2\zeta}e^{\zeta\theta}}{(\theta+1)^{\zeta}\alpha^{2\zeta k}}x^{\zeta(2k-1)}\exp\left(-\theta\zeta\left(\frac{x}{\alpha}\right)^{k}\right)dx\\ &= \tfrac{k^{\zeta}\theta^{2\zeta}e^{\zeta\theta}}{(\theta+1)^{\zeta}\alpha^{\zeta}}\int_{\alpha}^{\infty}\left(\frac{x}{\alpha}\right)^{\zeta(2k-1)}\exp\left(-\theta\zeta\left(\frac{x}{\alpha}\right)^{k}\right)dx,\end{aligned}$$

and substituting $x=\alpha y$ $\left(\text{i.e } y=\frac{x}{\alpha}\right)$, the above expression reduces to

$$\begin{aligned}&= \tfrac{k^{\zeta}\theta^{2\zeta}e^{\zeta\theta}}{(\theta+1)^{\zeta}\alpha^{\zeta}}\int_{1}^{\infty}y^{\zeta(2k-1)}\exp\left(-\theta\zeta y^{k}\right)dy\\ &= \tfrac{k^{\zeta}\theta^{2\zeta}e^{\zeta\theta}}{(\theta+1)^{\zeta}\alpha^{\zeta-1}}\tfrac{\Gamma\left(\frac{2k\zeta-\zeta+1}{k},\theta\zeta\right)}{k(\theta\zeta)^{\frac{2k\zeta-\zeta+1}{k}}}\\ \int_{\alpha}^{\infty}f^{\zeta}(x)dx &= \tfrac{k^{\zeta-1}\theta^{2\zeta}e^{\zeta\theta}}{\theta^{\frac{1-\zeta}{k}}\zeta^{\frac{2k\zeta-\zeta+1}{k}}(\theta+1)^{\zeta}\alpha^{\zeta-1}}\Gamma\left(\tfrac{2k\zeta-\zeta+1}{k},\theta\zeta\right).\end{aligned}$$

Thus the Rényi entropy of the $LP(\theta,\alpha,k)$ distribution is given by

$$\Im\left(\zeta\right)=\tfrac{1}{1-\zeta}\ln\left(\tfrac{k^{\zeta}\theta^{2\zeta}e^{\zeta\theta}}{\theta^{\frac{1-\zeta}{k}}\zeta^{\frac{2k\zeta-\zeta+1}{k}}(\theta+1)^{\zeta}\alpha^{\zeta-1}}\tfrac{\Gamma\left(\frac{2k\zeta-\zeta+1}{k},\theta\zeta\right)}{k}\right),\quad\zeta>0,\zeta\neq1.$$

4.3 Rényi entropy of the pseudo Lindley distribution

The pseudo Lindley distribution with parameters θ and β is defined by its density function

$$f(x;\theta,\beta)=\frac{\theta\left(\beta-1+\theta x\right)e^{-\theta x}}{\beta},\ x,\theta,\beta>0.$$

Then

$$\int_{\mathbb{R}^{+}}f^{\zeta}(x;\theta,\beta)dx=\left(\frac{\theta}{\beta}\right)^{\zeta}\int_{\mathbb{R}^{+}}\left(\beta-1+\theta x\right)^{\zeta}\exp\left(-\zeta\theta x\right)dx,$$

and by substituting $x = \frac{1}{\theta} z$ and using power series expansion $(a+b)^n = \sum_{j=0}^{\infty} \binom{n}{j} a^j b^{n-j}$, the above expression reduces to

$$\begin{aligned}
\int_{\mathbb{R}^+} f^{\zeta}(x;\theta,\beta)dx &= \frac{1}{\theta}\left(\frac{\theta}{\beta}\right)^{\zeta} \int_{\mathbb{R}^+} (\beta-1+z)^{\zeta} \exp(-\zeta z)dz \\
&= \frac{1}{\theta}\left(\frac{\theta}{\beta}\right)^{\zeta} \int_{\mathbb{R}^+} \sum_{j=0}^{\infty} \binom{\zeta}{j} (\beta-1)^j \, z^{\zeta-j} \exp(-\zeta z)dz \\
&= \frac{1}{\theta}\left(\frac{\theta}{\beta}\right)^{\zeta} \sum_{j=0}^{\infty} \binom{\zeta}{j} (\beta-1)^j \int_{\mathbb{R}^+} z^{\zeta-j} \exp(-\zeta z)dz \\
&= \frac{1}{\theta}\left(\frac{\theta}{\beta}\right)^{\zeta} \sum_{j=0}^{\infty} \binom{\zeta}{j} (\beta-1)^j \, \frac{\Gamma(\zeta-j)}{\zeta^{\zeta-j}}.
\end{aligned}$$

Thus, the Rényi entropy of the $PsL(\theta,\beta)$ distribution is given by

$$\Im(\zeta) = \frac{1}{1-\zeta} \ln\left(\frac{1}{\theta}\left(\frac{\theta}{\beta}\right)^{\zeta} \sum_{j=0}^{\infty} \binom{\zeta}{j} (\beta-1)^j \, \frac{\Gamma(\zeta-j)}{\zeta^{\zeta-j}}\right).$$

4.4 Rényi entropy of the quasi Lindley distribution

The quasi Lindley distribution with parameters α and θ is defined by its density function

$$f(x;\alpha,\theta) = \frac{\theta(\alpha+x\theta)}{\alpha+1} \exp(-\theta x); \quad x>0, \theta>0, \alpha>-1.$$

Then

$$\int_{\mathbb{R}^+} f^{\zeta}(x;\alpha,\theta)dx = \left(\frac{\theta}{\alpha+1}\right)^{\zeta} \int_{\mathbb{R}^+} (\alpha+x\theta)^{\zeta} \exp(-\zeta\theta x)\,dx,$$

and substituting $x = \frac{1}{\theta} z$ and using the power series expansion $(a+b)^n = \sum_{j=0}^{\infty} \binom{n}{j} a^j b^{n-j}$, the above expression reduces to

$$\begin{aligned}
\int_{\mathbb{R}^+} f^{\zeta}(x;\alpha,\theta)dx &= \frac{1}{\theta}\left(\frac{\theta}{\alpha+1}\right)^{\zeta} \int_{\mathbb{R}^+} (\alpha+z)^{\zeta} \exp(-\zeta z)\,dz \\
&= \frac{1}{\theta}\left(\frac{\theta}{\alpha+1}\right)^{\zeta} \int_{\mathbb{R}^+} \sum_{j=0}^{\infty} \binom{\zeta}{j} \alpha^j z^{\zeta-j} \exp(-\zeta z)\,dz \\
&= \frac{1}{\theta}\left(\frac{\theta}{\alpha+1}\right)^{\zeta} \sum_{j=0}^{\infty} \binom{\zeta}{j} \alpha^j \int_{\mathbb{R}^+} z^{\zeta-j} \exp(-\zeta z)\,dz \\
&= \frac{1}{\theta}\left(\frac{\theta}{\alpha+1}\right)^{\zeta} \sum_{j=0}^{\infty} \binom{\zeta}{j} \alpha^j \frac{\Gamma(\zeta-j)}{\zeta^{\zeta-j}}.
\end{aligned}$$

Thus, the Rényi entropy of the $QL(\alpha, \theta)$ distribution is given by

$$\Im(\zeta) = \frac{1}{1-\zeta} \log \left[\frac{1}{\theta} \left(\frac{\theta}{\alpha+1} \right)^{\zeta} \sum_{j=0}^{\infty} \binom{\zeta}{j} \alpha^j \frac{\Gamma(\zeta - j)}{\zeta^{\zeta-j}} \right].$$

4.5 Rényi entropy of the two parameters Lindley distribution

The two parameters Lindley distribution with parameters α and θ is defined by its density function

$$f(x; \alpha, \theta) = \frac{\theta^2 (1+\alpha x)}{\theta+\alpha} \exp(-\theta x); \quad x > 0, \theta > 0, \alpha > -\theta.$$

Then

$$\int_{\mathbb{R}^+} f^{\zeta}(x; \alpha, \theta) dx = \left(\frac{\theta^2}{\theta+\alpha} \right)^{\zeta} \int_{\mathbb{R}^+} (1+\alpha x)^{\zeta} \exp(-\zeta \theta x)\, dx,$$

and substituting $x = \frac{1}{\alpha} z$ and by using power series expansion $(a+b)^n = \sum_{j=0}^{\infty} \binom{n}{j} a^j b^{n-j}$, the above expression reduces to

$$\begin{aligned}
\int_{\mathbb{R}^+} f^{\zeta}(x; \alpha, \theta) dx &= \frac{1}{\alpha} \left(\frac{\theta^2}{\theta+\alpha} \right)^{\zeta} \int_{\mathbb{R}^+} (1+z)^{\zeta} \exp\left(-\frac{\zeta\theta}{\alpha} z \right) dz \\
&= \frac{1}{\alpha} \left(\frac{\theta^2}{\theta+\alpha} \right)^{\zeta} \int_{\mathbb{R}^+} \sum_{j=0}^{\infty} \binom{\zeta}{j} z^j \exp\left(-\frac{\zeta\theta}{\alpha} z \right) dz \\
&= \frac{1}{\alpha} \left(\frac{\theta^2}{\theta+\alpha} \right)^{\zeta} \sum_{j=0}^{\infty} \binom{\zeta}{j} \int_{\mathbb{R}^+} z^j \exp\left(-\frac{\zeta\theta}{\alpha} z \right) dz.
\end{aligned}$$

If $\alpha > 0$, we have

$$\int_{\mathbb{R}^+} f^{\zeta}(x; \alpha, \theta) dx = \frac{1}{\alpha} \left(\frac{\theta^2}{\theta+\alpha} \right)^{\zeta} \sum_{j=0}^{\infty} \binom{\zeta}{j} \frac{\Gamma(j)}{\left(\frac{\zeta\theta}{\alpha} \right)^j}.$$

Thus, the Rényi entropy of the $two-parameter\ LD(\alpha, \theta)$ distribution is

$$\Im(\zeta) = \frac{1}{1-\zeta} \log \left[\frac{1}{\alpha} \left(\frac{\theta^2}{\theta+\alpha} \right)^{\zeta} \sum_{j=0}^{\infty} \binom{\zeta}{j} \frac{\Gamma(j)}{\left(\frac{\zeta\theta}{\alpha} \right)^j} \right].$$

If $-\theta < \alpha < 0$, we have

$$\Im(\zeta) = \frac{1}{1-\zeta} \log \left[\frac{1}{\alpha} \left(\frac{\theta^2}{\theta+\alpha} \right)^{\zeta} \sum_{j=0}^{\infty} \binom{\zeta}{j} \frac{\Gamma(j)}{\left(-\frac{\zeta\theta}{\alpha} \right)^j} \right], \quad j \neq \frac{1}{2n}, n \in \mathbb{N}^*.$$

4.6 Rényi entropy of the power Lindley distribution

The power Lindley distribution with parameters α and β is defined by its density function

$$f(x;\alpha,\beta) = \frac{\alpha\beta^2}{\beta+1}\left(1+x^\alpha\right)x^{\alpha-1}\exp\left(-\beta x^\alpha\right); x>0, \alpha>0, \beta>0.$$

Then

$$\int\limits_{\mathbb{R}^+} f^\zeta(x;\alpha,\theta)dx = \left(\frac{\alpha\beta^2}{\beta+1}\right)^\zeta \int\limits_{\mathbb{R}^+} \left(1+x^\alpha\right)^\zeta x^{\zeta(\alpha-1)}\exp(-\zeta\beta x^\alpha)dx,$$

and substituting $x = z^{\frac{1}{\alpha}}$ and using power series expansion $(a+b)^n = \sum\limits_{j=0}^{\infty}\binom{n}{j}a^j b^{n-j}$, the above expression reduces to

$$\begin{aligned}
\int\limits_{\mathbb{R}^+} f^\zeta(x;\alpha,\theta)dx &= \frac{1}{\alpha}\left(\frac{\alpha\beta^2}{\beta+1}\right)^\zeta \int\limits_{\mathbb{R}^+}\sum_{j=0}^{\infty}\binom{\zeta}{j} z^j z^{\zeta\frac{\alpha-1}{\alpha}} z^{\frac{1-\alpha}{\alpha}}\exp(-\zeta\beta z)dz. \\
&= \frac{1}{\alpha}\left(\frac{\alpha\beta^2}{\beta+1}\right)^\zeta \sum_{j=0}^{\infty}\binom{\zeta}{j}\int\limits_{\mathbb{R}^+} z^{j+\zeta y-y}\exp(-\zeta\beta z)dz,
\end{aligned}$$

where $y = \frac{\alpha-1}{\alpha}$.

- If $y > -\frac{j}{\zeta-1}$, then

$$\int\limits_{\mathbb{R}^+} f^\zeta(x;\alpha,\theta)dx = \frac{1}{\alpha}\left(\frac{\alpha\beta^2}{\beta+1}\right)^\zeta \sum_{j=0}^{\infty}\binom{\zeta}{j}\frac{\Gamma(j+\zeta y-y)}{(\zeta\beta)^{j+\zeta y-y}}.$$

Thus, the Rényi entropy of the $Power-Lindley$ $PL(\alpha,\beta)$ distribution is

$$\Im(\zeta) = \frac{1}{1-\zeta}\log\left[\frac{1}{\alpha}\left(\frac{\alpha\beta^2}{\beta+1}\right)^\zeta \sum_{j=0}^{\infty}\binom{\zeta}{j}\frac{\Gamma(j+\zeta y-y)}{(\zeta\beta)^{j+\zeta y-y}}\right].$$

- In the opposite case, we infer

$$\Im(\zeta) = \frac{1}{1-\zeta}\log\left[\frac{1}{\alpha}\left(\frac{\alpha\beta^2}{\beta+1}\right)^\zeta \sum_{j=0}^{\infty}\binom{\zeta}{j}\frac{\Gamma(-j-\zeta y+y)}{(\zeta\beta)^{-j-\zeta y+y}}\right].$$

5 Illustration

According to [13, pp. 204, 263] - we consider two series of real data. The first one represents the failure times (mm) for a sample of fifteen electronic components in an acceleration life test: 1.4, 5.1, 6.3, 10.8, 12.1, 18.5, 19.7, 22.2, 23, 30.6, 37.3, 46.3, 53.9, 59.8, 66.2. The second set of data are the number of cycles to failure for 25 100-cm specimens of year, tested at a particular strain level: 15, 20, 38, 42, 61, 76, 86, 98, 121, 146, 149, 157, 175, 176, 180, 180, 198, 220, 224, 251, 264, 282, 321, 325, 653. The results of these data are presented in the following table

Table 1 Comparison between distributions

Data	*Distribution*	β	θ	γ	$-LL$	$K-S$	AIC	BIC
*Serie*1	*power Lindley*	1.203	0.064	0.083	64.080	0.095	134.16	136.28
	PsL	1.063	0.0684		**62.075**	**0.082**	**128.15**	**129.57**
n=15	*GaLD*	1.129	0.0684		64.015	0.094	132.03	133.45
m=27.546	*QLD*	4.016	-0.99		1504	0.93	3012	3013.4
s=20.06	*TwoPLD*	0.070	1.110			0.196		
	Gamma	1.442	0.052		64.197	0.102	132.39	133.81
	Weibull	1.306	0.034		64.026	0.450	132.05	133.47
	Lognormal	1.061	2.931		65.626	0.163	135.25	136.67
*Serie*2	*power Lindley*	1.505	0.012	0.018	152.369	0.137	310.74	314.39
	PsL	1.086	0.010		**150.232**	**0.128**	**304.464**	**306.9**
n=25	*GaL*	0. 05	0.010		152.132	0.129	308.26	310.7
m=178.32	*QLD*	0.010	8.514		1045.9	0.94	2131.8	2156.2
$s = 131.1$	*TwoPLD*	0.010	0.125			0.232		
	Gamma	1.794	0.010		152.371	0.135	308.74	311.18
	Weibull	1.414	0.005		152.440	0.697	308.88	310.7
	Lognormal	0.891	4.880		154.092	0.155	312.18	314.62

According to Table 1, we can observe that the pseudo Lindley distribution provides the smallest -LL, AIC, and BIC values, as compared to gamma Lindley, Lindley Pareto, quasi Lindley, two parameters Lindley, gamma, Weibull, Lognormal and power Lindley distributions, and hence best fits the data among all the models considered.

6 Conclusions

We introduce the new distributions gamma Lindley, Lindley Pareto, pseudo Lindley, quasi Lindley, two parameter Lindley and power Lindley. Then, we study some of their properties, as: the function quantile, the limiting distributions of sample minima and maxima, and the Rényi entropy. Our future concerns involve using censured data. We will be able in our future research to propose and investigate other distributions, as:

- the size biased gamma Lindley distribution;
- the size biased pseudo Lindley distribution.

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