Invariant, anti-invariant and slant submanifolds of a para-Kenmotsu manifold

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Abstract. Properties of an invariant, anti-invariant and timelike-slant isometrically immersed submanifold M of a para-Kenmotsu manifold $(\bar{M}, \varphi, \xi, \eta, g)$ are studied. In particular, we prove that if M is an invariant submanifold, then it is a para-Kenmotsu manifold, too. We also provide a characterization for a timelike-slant submanifold M of $\bar{M}$ in terms of a g-skew-symmetric operator on TM.

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1 Introduction

The theory of submanifolds has the origin in the study of the geometry of plane curves initiated by Fermat. Since then it has been evolving in different directions of differential geometry and mechanics, especially. It is still an active and vast research field playing an important role in the development of modern differential geometry. The modeling spaces of dynamical systems always carry different canonical geometrical objects: affine connections, differential forms, tensor fields etc. A natural question arising is when the submanifold inherits the geometrical structures of the ambient manifold. In this spirit, we shall consider a certain kind of isometrically immersed submanifolds of para-Kenmotsu manifolds, namely, *slant* submanifolds. First time, the notion of slant submanifold appeared for complex manifolds in Chen's book [7]. Remark that the slant submanifolds have been studied in different other context: for contact [14], LP-contact [13], K-contact [15], Kähler [6], Sasakian [3], Lorentzian [18], Kenmotsu [10], almost product Riemannian manifold [19] etc.

We shall begin recalling the basic properties of a para-Kenmotsu structure and prove some immediate consequences of the Gauss and Weingarten equations for an isometrically immersed submanifold in a para-Kenmotsu manifold $(\bar{M}, \varphi, \xi, \eta, g)$. In the main section we characterize the invariant, anti-invariant and timelike-slant submanifolds of $\bar{M}$.

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2 Para-Kenmotsu manifold revisited

Let us recall the main aspects of para-Kenmotsu geometry. On a $(2n+1)$-dimensional smooth manifold $\bar{M}$, consider a tensor field φ of $(1,1)$-type, a vector field ξ, a 1-form η and a pseudo-Riemannian metric g of signature $(n+1,n)$.

Definition 2.1. [22] We say that (φ,ξ,η,g) defines an *almost paracontact metric structure* on $\bar{M}$ if:

$$\varphi\xi=0,\quad \eta\circ\varphi=0,\quad \eta(\xi)=1,\quad \varphi^2=I_{\Gamma(T\bar{M})}-\eta\otimes\xi,\quad g(\varphi\cdot,\varphi\cdot)=-g+\eta\otimes\eta$$

and φ induces on the $2n$-dimensional distribution $\mathcal{D}:=\ker\eta$ an almost paracomplex structure P i.e. $P^2=I_{\Gamma(T\bar{M})}$ and the eigensubbundles $\mathcal{D}^+$, $\mathcal{D}^-$, corresponding to the eigenvalues 1, -1 of P respectively, have equal dimension n; hence $\mathcal{D}=\mathcal{D}^+\oplus\mathcal{D}^-$.

In this case, $(\bar{M},\varphi,\xi,\eta,g)$ is called *almost paracontact metric manifold*, φ *the structure endomorphism*, ξ *the characteristic vector field*, η *the paracontact form* and g *compatible metric*.

Examples of almost paracontact metric structures can be found in [11] and [8]. From the definition it follows that η is the g-dual of the unitary vector field ξ:

$$\eta(X)=g(X,\xi),\quad g(\xi,\xi)=1 \tag{2.1}$$

and φ is a g-skew-symmetric operator:

$$g(\varphi X,Y)=-g(X,\varphi Y). \tag{2.2}$$

Remark that the canonical distribution $\mathcal{D}$ is φ-invariant since $\mathcal{D}=Im\varphi$. Moreover, ξ is orthogonal to $\mathcal{D}$ and therefore the tangent bundle splits orthogonally:

$$T\bar{M}=\mathcal{D}\oplus\langle\xi\rangle. \tag{2.3}$$

An analogue of Kenmotsu manifold [12] in paracontact geometry will be further considered.

Definition 2.2. [16] We say that the almost paracontact metric structure (φ,ξ,η,g) is *para-Kenmotsu* if the Levi-Civita connection $\bar{\nabla}$ associated to g satisfies $(\bar{\nabla}_X\varphi)Y=g(\varphi X,Y)\xi-\eta(Y)\varphi X$, for any X, $Y\in\Gamma(T\bar{M})$.

Example 2.3. Let $\bar{M}=\{(x,y,z)\in\mathbb{R}^3 : z\neq 0\}$, where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. Set:

$$\varphi:=\frac{\partial}{\partial y}\otimes dx+\frac{\partial}{\partial x}\otimes dy,\quad \xi:=-\frac{\partial}{\partial z},\quad \eta:=-dz,$$

$$g:=dx\otimes dx-dy\otimes dy+dz\otimes dz.$$

Then (φ,ξ,η,g) defines a para-Kenmotsu structure on $\mathbb{R}^3$.

Note that the para-Kenmotsu structure was introduced by J. Wełyczko in [21] for 3-dimensional normal almost paracontact metric structures.

In the next Proposition we shall point out immediate properties of this structure.

Proposition 2.1. *[2] On a para-Kenmotsu manifold $(\bar{M}, \varphi, \xi, \eta, g)$, the following relations hold:*

$$\bar{\nabla}\xi = I_{\Gamma(T\bar{M})} - \eta \otimes \xi \tag{2.4}$$

$$\eta(\bar{\nabla}_X \xi) = 0, \tag{2.5}$$

$$R_{\bar{\nabla}}(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.6}$$

$$\eta(R_{\bar{\nabla}}(X,Y)W) = -\eta(X)g(Y,W) + \eta(Y)g(X,W), \tag{2.7}$$

$$\bar{\nabla}\eta = g - \eta \otimes \eta, \tag{2.8}$$

$$L_\xi \varphi = 0, \quad L_\xi \eta = 0, \quad L_\xi g = 2(g - \eta \otimes \eta), \tag{2.9}$$

where $R_{\bar{\nabla}}$ is the Riemann curvature tensor field and $\bar{\nabla}$ is the Levi-Civita connection associated to g. Moreover, $\mathcal{D}$ is involutive, η is closed and the Nijenhuis tensor field of φ vanishes identically.

Remark that a particular notion of almost contact metric structure which includes the Sasakian, Kenmotsu, cosymplectic, quasi-Sasakian and trans-Sasakian ones, namely, the generalized quasi-Sasakian structure, was defined by C. Călin in [5]. Analogue, we can see the para-Kenmotsu structure as being a particular case of a similar more general notion in almost paracontact geometry.

3 Submanifolds of para-Kenmotsu manifolds

3.1 Isometrically immersed submanifolds

We shall focuse on a certain kind of isometrically immersed submanifolds of para-Kenmotsu manifolds, namely, *timelike-slant* submanifolds. Different other related notions have also been studied: slant submanifolds [7], semi-slant submanifolds [17] which represent a generalized version of CR-submanifolds, pseudo-slant and bi-slant submanifolds as generalization of the previous ones [4].

Recall first that for an isometrically immersed submanifold M of a pseudo-Riemannian manifold $(\bar{M}, g)$, denoting also by g the pseudo-Riemannian metric on M induced from $(\bar{M}, g)$ and by h the second fundamental form of M, Gauss formula holds:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{3.1}$$

where $\bar{\nabla}$ and ∇ are the Levi-Civita connections on $(\bar{M}, g)$ and (M, g) respectively and also the Weingarten formula corresponding to M:

$$\bar{\nabla}_X U = -A_U X + D_X U, \tag{3.2}$$

where A_U is the shape operator (or the Weingarten map) in the direction of the normal vector field U defined by $g(A_U X, Y) = g(h(X,Y), U)$, for X, $Y \in \Gamma(TM)$, $U \in \Gamma(TM^{\perp})$ and D stands for the covariant differentiation operator with respect to the canonical connection in the normal bundle.

The curvature of $\bar{\nabla}$ and ∇ are connected by Gauss equation:

$$R_{\bar{\nabla}}(X,Y)Z = R_{\nabla}(X,Y)Z - A_{h(Y,Z)}X - A_{h(X,Z)}Y + (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z), \tag{3.3}$$

for X, Y, $Z \in \Gamma(TM)$.

Denote by:

$$T : \Gamma(TM) \to \Gamma(TM), \quad TX := (\varphi X)^T, \tag{3.4}$$

$$N : \Gamma(TM) \to \Gamma(TM^{\perp}), \quad NX := (\varphi X)^N, \tag{3.5}$$

where $\varphi X = (\varphi X)^T + (\varphi X)^N$ with $(\varphi X)^T \in \Gamma(TM)$ and $(\varphi X)^N \in \Gamma(TM^{\perp})$ and by:

$$t : \Gamma(TM^{\perp}) \to \Gamma(TM), \quad tU := (\varphi U)^t, \tag{3.6}$$

$$n : \Gamma(TM^{\perp}) \to \Gamma(TM^{\perp}), \quad nU := (\varphi U)^n, \tag{3.7}$$

where $\varphi U = (\varphi U)^t + (\varphi U)^n$ with $(\varphi U)^t \in \Gamma(TM)$ and $(\varphi U)^n \in \Gamma(TM^{\perp})$.

Proposition 3.1. *If M is an isometrically immersed submanifold of the para-Kenmotsu manifold $(\bar{M}, \varphi, \xi, \eta, g)$ such that ξ is tangent to M, then:*

1. *for any X, $Y \in \Gamma(TM)$, we have:*
   (a) $(\nabla_X T)Y = t(h(X,Y)) + A_{NY}X + g(TX,Y)\xi - \eta(Y)TX$,
   (b) $D_X NY - N(\nabla_X Y) = n(h(X,Y)) - h(X,TY) - \eta(Y)NX$;
2. *for any $X \in \Gamma(TM)$, $U \in \Gamma(TM^{\perp})$, we have:*
   (a) $\nabla_X tU - t(D_X U) = A_{nU}X - T(A_U X) + g(NX,U)\xi$,
   (b) $(D_X n)U = -h(X,tU) - N(A_U X)$.

Proof. For any X, $Y \in \Gamma(T\bar{M})$, we know that $(\bar{\nabla}_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X$. Then:

1. for X, $Y \in \Gamma(TM)$:

$$(\bar{\nabla}_X \varphi)Y = [g(TX,Y)\xi - \eta(Y)TX] - \eta(Y)NX$$

and

$$(\bar{\nabla}_X \varphi)Y = \bar{\nabla}_X TY + \bar{\nabla}_X NY - \varphi(\nabla_X Y + h(X,Y)) =$$

$$= [(\nabla_X T)Y - A_{NY}X - t(h(X,Y))] + [h(X,TY) + D_X NY - N(\nabla_X Y) - n(h(X,Y))];$$

2. for $X \in \Gamma(TM)$, $U \in \Gamma(TM^{\perp})$:

$$(\bar{\nabla}_X \varphi)U = g(NX, U)\xi$$

and

$$(\bar{\nabla}_X \varphi)U = \bar{\nabla}_X tU + \bar{\nabla}_X nU - \varphi(-A_U X + D_X U) =$$

$$= [\nabla_X tU - A_{nU} X + T(A_U X) - t(D_X U)] + [h(X, tU) + D_X nU - n(D_X U) + N(A_U X)].$$

□

Remark that the maps T and n are g-skew-symmetric:

$$g(TX, Y) + g(X, TY) = 0, \quad X, Y \in \Gamma(TM), \tag{3.8}$$

$$g(nU, V) + g(U, nV) = 0, \quad U, V \in \Gamma(TM^{\perp}) \tag{3.9}$$

and

$$g(NX, U) + g(X, tU) = 0, \quad X \in \Gamma(TM), U \in \Gamma(TM^{\perp}). \tag{3.10}$$

3.2 Invariant and anti-invariant submanifolds

3.2.1 Invariant submanifolds

Let M be a submanifold of $\bar{M}$. Then M is called *invariant* submanifold of $\bar{M}$ if $\varphi(T_x M) \subset T_x M$, for any $x \in M$. It follows $\varphi(T_x M^{\perp}) \subset T_x M^{\perp}$, for any $x \in M$. Indeed, for any $U \in \Gamma(TM^{\perp})$, $g(X, \varphi U) = -g(\varphi X, U) = 0$, for any $X \in \Gamma(TM)$.

Proposition 3.2. *An isometrically immersed invariant submanifold M of a para-Kenmotsu manifold $(\bar{M}, \varphi, \xi, \eta, g)$ such that ξ is tangent to M, is para-Kenmotsu manifold, too.*

Proof. Let X, $Y \in \Gamma(TM)$. Then φY, $\nabla_X Y$, $\varphi(\nabla_X Y)$, $\nabla_X \varphi Y \in \Gamma(TM)$ and so:

$$(\nabla_X \varphi)Y := \nabla_X \varphi Y - \varphi(\nabla_X Y) \in \Gamma(TM).$$

From Gauss formula we have:

$$(\nabla_X \varphi)Y := \nabla_X \varphi Y - \varphi(\nabla_X Y) = \bar{\nabla}_X \varphi Y - h(X, \varphi Y) - \varphi(\bar{\nabla}_X Y) + \varphi(h(X, Y)) :=$$

$$:= (\bar{\nabla}_X \varphi)Y - h(X, \varphi Y) + \varphi(h(X, Y)) = g(\varphi X, Y)\xi - \eta(Y)\varphi X - h(X, \varphi Y) + \varphi(h(X, Y)).$$

Therefore

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X \tag{3.11}$$

$$h(X, \varphi Y) = \varphi(h(X, Y)), \tag{3.12}$$

for any X, $Y \in \Gamma(TM)$. □

Proposition 3.3. *If M is an isometrically immersed invariant submanifold of the para-Kenmotsu manifold $(\bar{M}, \varphi, \xi, \eta, g)$ such that ξ is tangent to M, then for any X, $Y \in \Gamma(TM)$:*

$$h(X, \varphi Y) = \varphi(h(X, Y)) = h(\varphi X, Y), \tag{3.13}$$

$$h(\varphi X, \varphi Y) = h(X, Y), \tag{3.14}$$

$$h(X, \xi) = 0, \tag{3.15}$$

$$\nabla_X \xi = X - \eta(X)\xi, \tag{3.16}$$

$$(\nabla_X h)(Y, \xi) = -h(Y, \nabla_X \xi) = -h(X, Y), \tag{3.17}$$

$$R_\nabla(X, Y)\xi = \eta(X)Y - \eta(Y)X. \tag{3.18}$$

Proof. From the symmetry of h follows:

$$h(X, \varphi Y) = \varphi(h(X, Y)) = \varphi(h(Y, X)) = h(Y, \varphi X) = h(\varphi X, Y)$$

and

$$h(\varphi X, \varphi Y) = \varphi(h(\varphi X, Y)) = \varphi^2(h(X, Y)) = h(X, Y) - \eta(h(X, Y))\xi = h(X, Y),$$

since $\xi \in \Gamma(TM)$ and $h(X, Y) \in \Gamma(TM^\perp)$. Therefore $h(X, \xi) = 0$ and $\nabla_X \xi = \bar{\nabla}_X \xi = X - \eta(X)\xi$.

Computing the covariant derivative of h, we obtain:

$$(\nabla_X h)(Y, \xi) := \nabla_X h(Y, \xi) - h(\nabla_X Y, \xi) - h(Y, \nabla_X \xi) = -h(Y, \nabla_X \xi) =$$
$$= -h(Y, X) = -h(X, Y).$$

From Gauss equation we get:

$$R_\nabla(X, Y)\xi = R_{\bar{\nabla}}(X, Y)\xi + A_{h(Y,\xi)}X + A_{h(X,\xi)}Y - (\nabla_X h)(Y, \xi) + (\nabla_Y h)(X, \xi) =$$
$$= R_{\bar{\nabla}}(X, Y)\xi + h(X, Y) - h(Y, X) = R_{\bar{\nabla}}(X, Y)\xi = \eta(X)Y - \eta(Y)X.$$

□

3.2.2 Anti-invariant submanifolds

Let M be a submanifold of $\bar{M}$. Then M is called *anti-invariant* submanifold of $\bar{M}$ if $\varphi(T_xM) \subset T_xM^\perp$, for any $x \in M$.

Proposition 3.4. *If M is an isometrically immersed anti-invariant submanifold of the para-Kenmotsu manifold $(\bar{M}, \varphi, \xi, \eta, g)$ such that ξ is tangent to M, then for any X, $Y \in \Gamma(TM)$:*

$$t(h(X, Y)) = -A_{\varphi Y}X + g(\varphi Y, X)\xi, \tag{3.19}$$

$$n(h(X, Y)) = D_X \varphi Y - \varphi(\nabla_X Y) + \eta(Y)\varphi X. \tag{3.20}$$

Proof. Let X, $Y \in \Gamma(TM)$. Then $\nabla_X Y \in \Gamma(TM)$, φX, φY, $\varphi(\nabla_X Y) \in \Gamma(TM^{\perp})$ and so:

$$\begin{aligned} g(\varphi X, Y)\xi - \eta(Y)\varphi X = (\bar{\nabla}_X \varphi)Y := \bar{\nabla}_X \varphi Y - \varphi(\bar{\nabla}_X Y) = \\ = -A_{\varphi Y} X + D_X \varphi Y - \varphi(\nabla_X Y) - \varphi(h(X,Y)). \end{aligned}$$

Therefore

$$\begin{aligned} t(h(X,Y)) &= -A_{\varphi Y} X + g(\varphi Y, X)\xi, \\ n(h(X,Y)) &= D_X \varphi Y - \varphi(\nabla_X Y) + \eta(Y)\varphi X. \end{aligned}$$

□

Corollary 3.5. *If M is an isometrically immersed anti-invariant submanifold of the para-Kenmotsu manifold $(\bar{M}, \varphi, \xi, \eta, g)$ such that ξ is tangent to M, then for any X, $Y \in \Gamma(TM)$:*

$$h(X, \xi) = 0, \tag{3.21}$$

$$\nabla_X \xi = X - \eta(X)\xi, \tag{3.22}$$

$$(\nabla_X h)(Y, \xi) = -h(Y, \nabla_X \xi) = -h(X, Y), \tag{3.23}$$

$$R_{\nabla}(X, Y)\xi = \eta(X)Y - \eta(Y)X. \tag{3.24}$$

Proof. $h(X,\xi) = \bar{\nabla}_X \xi - \nabla_X \xi = X - \eta(X)\xi - \nabla_X \xi \in \Gamma(TM) \cap \Gamma(TM^{\perp})$, so $h(X,\xi) = 0$ and $\nabla_X \xi = \bar{\nabla}_X \xi = X - \eta(X)\xi$. Following the same steps in the computations from Proposition 3.3 we obtain the last two relations. □

3.3 Timelike-slant submanifolds

The operator T will essentially be involved in characterizing the timelike-slant submanifolds.

Assume that

$$TM = \nu \oplus \langle \xi \rangle \tag{3.25}$$

such that for any timelike vector field X of ν, the tangential part of φX is a non zero spacelike vector field. Denote by ν_t the distribution generated by the timelike vector fields of ν and by ν_s the distribution generated by the spacelike vector fields of ν.

We say that ν is *timelike-slant distribution* if for any non zero section X of ν_t, the angle $\theta(X)$ between φX and TM (which agrees with the angle between φX and TX) is constant. In this case, we call $\theta =: \theta(X)$ the *timelike-slant angle* and M *timelike-slant submanifold.*

Notice that for any nonzero section X of ν_t, we have $g(\varphi X, \varphi X) > 0$ and $g(TX, TX) > 0$. Indeed, $g(X, \xi) = 0$ and so $g(\varphi X, \varphi X) = -g(X, X) > 0$. Then, the cosine of the timelike-slant angle θ can be expressed as:

$$\cos(\theta(X)) := \frac{g(\varphi X, TX)}{\sqrt{g(\varphi X, \varphi X)}\sqrt{g(TX, TX)}} = \frac{||TX||}{||\varphi X||}, \tag{3.26}$$

for any nonzero section X of ν_t.

Properties of timelike-slant submanifolds will be stated in the next Propositions.

Proposition 3.6. *Let M be an isometrically immersed submanifold of the para-Kenmotsu manifold $(\bar{M}, \varphi, \xi, \eta, g)$ satisfying (3.25). If M is timelike-slant with the timelike-slant angle θ, then:*

$$g(TX, TY) = -\cos^2(\theta)[g(X, Y) - \eta(X)\eta(Y)] \tag{3.27}$$

and

$$g(NX, NY) = -\sin^2(\theta)[g(X, Y) - \eta(X)\eta(Y)], \tag{3.28}$$

for any X, $Y \in \Gamma(\nu_t)$.

Proof. Taking $X+Y$ in $g(TX, TX) = \cos^2(\theta)g(\varphi X, \varphi X)$ we easily obtain $g(TX, TY) = \cos^2(\theta)g(\varphi X, \varphi Y) = -\cos^2(\theta)[g(X, Y) - \eta(X)\eta(Y)]$. Also, $g(NX, NY) = g(\varphi X, \varphi Y) - g(TX, TY) = -\sin^2(\theta)[g(X, Y) - \eta(X)\eta(Y)]$. □

A characterization in terms of the T-operator of a timelike-slant submanifold of a para-Kenmotsu manifold is now given:

Proposition 3.7. *Let M be an isometrically immersed submanifold of the para-Kenmotsu manifold $(\bar{M}, \varphi, \xi, \eta, g)$ satisfying (3.25). Then M is timelike-slant if and only if there exists a real number $\lambda \in [0, 1]$ such that*

$$T^2 = \lambda(I_{\Gamma(\nu_t)} - \eta \otimes \xi). \tag{3.29}$$

Proof. If M is timelike-slant, θ is constant and using the Proposition 3.6 we get:

$$g(T^2X, Y) = g(\varphi(TX), Y) = -g(TX, \varphi Y) = -g(TX, TY) =$$

$$= \cos^2(\theta)[g(X, Y) - \eta(X)\eta(Y)] = \cos^2(\theta)g(X - \eta(X)\xi, Y),$$

for any X, $Y \in \Gamma(\nu_t)$.

Conversely, if there exists a real number $\lambda \in [0, 1]$ such that $T^2 = \lambda(I_{\Gamma(\nu_t)} - \eta \otimes \xi)$ follows $\cos^2(\theta(X)) = \lambda$, so λ does not depend on X. □

From Proposition 3.7 follows immediately:

Corollary 3.8. *If M is a timelike-slant submanifold of the para-Kenmotsu manifold $(\bar{M}, \varphi, \xi, \eta, g)$ satisfying (3.25) and having the timelike-slant angle θ, then $\cos^2(\theta)$ is the only eigenvalue of $T^2|_{\nu_t}$.*

Proposition 3.9. *Let M be an isometrically immersed submanifold of the para-Kenmotsu manifold $(\bar{M}, \varphi, \xi, \eta, g)$ satisfying (3.25). If M is timelike-slant with the timelike-slant angle θ, then:*

$$(\nabla_X T^2)Y - (\nabla_Y T^2)X = \cos^2(\theta)R_{\overline{\nabla}}(X, Y)\xi, \tag{3.30}$$

for any X, $Y \in \Gamma(\nu_t)$.

Proof. We have:

$$T^2(\nabla_X Y) = \cos^2(\theta)(\nabla_X Y - \eta(\nabla_X Y)\xi)$$

and

$$T^2 Y = \cos^2(\theta)(Y - \eta(Y)\xi).$$

Differentiating covariantly the last relation with respect to X we obtain:

$$\nabla_X T^2 Y = \cos^2(\theta)[\nabla_X Y - g(\nabla_X Y, \xi)\xi - g(Y, \nabla_X \xi)\xi - g(Y, \xi)\nabla_X \xi] =$$

$$= \cos^2(\theta)[\nabla_X Y - \eta(\nabla_X Y)\xi - g(X, Y)\xi + 2\eta(X)\eta(Y)\xi - \eta(Y)X].$$

It follows:

$$(\nabla_X T^2)Y := \nabla_X T^2 Y - T^2(\nabla_X Y) = -\cos^2(\theta)\{[g(X, Y) - 2\eta(X)\eta(Y)]\xi + \eta(Y)X\}$$

and

$$(\nabla_X T^2)Y - (\nabla_Y T^2)X = \cos^2(\theta)[\eta(X)Y - \eta(Y)X] = \cos^2(\theta)R_{\bar{\nabla}}(X, Y)\xi.$$

□

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