On some concircular mappings of Kähler manifols

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Abstract. Along the line of conformal geodesic mappings between Riemannian manifolds and conformally holomorphically projective mappings on Kähler spaces, we introduce the notion of concircular holomorphically projective mappings between Kähler manifolds. Using the properties of J-traceless component of the Weyl conformal tensor, we obtain a rigidity condition such that a concircular HP-mapping is a concircular one. Also, some families of concircular metrics are built, considering techniques of certain ${}^{J}W$-Kähler-Riemann type flows.

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1 Introduction

The invariance of the curvature type tensors under certain transformations of metrics plays a central role in Riemannian and Kählerian geometry and has deep geometric significance.

Concircular transformations of metrics, a special kind of conformal change of metrics, were studied by Yano. Using the decomposition of Lakoma and Jukl [13], Fubanashi, H. Kim, Y. Kim and Pak [7] proved that the J-traceless component ${}^{J}W$ (in the sense of Mikeš [15]) of the conformal curvature tensor field for a Kähler manifold is invariant under the concircular mappings.

Also, the H-projective curvature tensor field ${}^{H}P$ is invariant with respect to the HP-mappings. These mappings, which preserve the H-planar curves on Kähler manifolds, introduced by Otsuki and Tashiro, were studied by Hinterleitner [9] in terms of differentiability. Kiyohara and Topalov [12] classified the compact connected Kähler manifolds admitting pairs of H-projectively equivalent metrics.

Čap, Gover, Macbeth [2] studied the Einstein metrics in projective geometry. Chudá and Mikeš [4] studied the conformal geodesic mappings between Riemannian manifolds. The Kähler case was generalized by Chudá and Shiha [6] for conformally

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holomorphically projective mappings. Along this line, we consider the notion of concircular holomorphically projective mappings. Using properties of invariance for ${}^{J}W$ and ${}^{H}P$, we determine certain conditions such that a concircular HP-mapping is a concircular one.

The geometric flows, as a class of important geometric partial differential equations, have been highlighted in many fields of theoretical research and practical applications. The Ricci flow [8] has been proven to be a very useful tool in understanding the topology of arbitrary Riemannian manifolds. The Kähler-Ricci flow [3] is the analogue of the Ricci flow in the complex case. Motivated by these ideas, along the line of constant sectional curvature manifolds, we extend this concept. The last section is devoted to a special class of concircular related metrics produced by some ${}^{J}W$-Kähler-Riemann flows.

2 Conformal transformation of metrics

A diffeomorphism f between the Riemannian manifolds $V_n = (M, g)$ and $\overline{V}_n = (\overline{M}, \bar{g})$ is called a conformal mapping, if f preserves angles between all (smooth) curves on $V_n (n \geq 2)$. Equivalently, a mapping f is conformal if and only if $\bar{g} = e^{2u} g$, [11] where u is a nowhere zero function on M and we will suppose $M = \overline{M}$.

The Levi-Civita connections are related by

$$\overline{\nabla}_X Y = \nabla_X Y + X(u)Y + Y(u)X - g(X,Y) grad \quad (u).$$

Let $B \in \mathcal{T}^{0,2}(M)$, $B_{ij} = u_{i,j} - u_i u_j$, $u_i = \frac{\partial u}{\partial x^i}$, $i, j = \overline{1, n}$, be the tensor field of the conformal change. If $B = \frac{1}{n}\, Tr(B) g$, then the conformal change is called *concircular transformation.*

A concircular transformation carries all the circles of the manifold into circles (a curve in a Riemannian manifold is called circle when the first curvature is constant and all the other curvatures are identically zero).

Let (M, g, J) be a Kähler manifold of real dimension n, with complex structure J, Kähler metric g and R, S, ρ be the curvature tensor field, the Ricci tensor field and the scalar curvature.

For any (1,3)-tensor field of curvature type A, there exists an unique J-decomposition:

$$\begin{aligned}
&A^a_{dcb} = {}^{J}A^a_{dcb} + \delta^a_d C_{cb} + \delta^a_c D_{db} + \delta^a_b E_{dc} + J^a_d G_{cb} + J^a_c H_{db} + J^a_b I_{dc},\\
&C_{cb} = \tfrac{1}{n} A_{cb}; D_{cb} = -\tfrac{1}{n} A_{cb} = -C_{cb}; E_{cb} = 0; G_{cb} = \tfrac{1}{n} J^t_b A_{ct} = -G_{bc};\\
&H_{cb} = -\tfrac{1}{n} J^t_b A_{ct} = -G_{cb}; I_{cb} = \tfrac{2}{n} J^t_b A_{ct} = 2G_{cb};\\
&{}^{J}A^a_{dcb} = A^a_{dcb} - \tfrac{1}{n}(\delta^a_d A_{cb} - \delta^a c A_{db} + J^a_d J^t_b A_{ct} - J^a_c J^t_b A_{dt} + 2J^a_b J^t_c A_{dt}).
\end{aligned}$$

One should notice that ${}^{J}A$ is J-traceless by means of Mikeš' analysis [15].

Let A be a (p,q)-tensor field. It is called traceless if $A^{\dots a_{i-1} t a_{i+1} \dots}_{\dots b_{j-1} t b_{j+1} \dots} = 0$.

A traceless (p,q)-tensor field A is called J-traceless if $J^s_t A^{\dots a_{i-1} t a_{i+1} \dots}_{\dots b_{j-1} s b_{j+1} \dots} = 0$.

The Weyl conformal curvature tensor W is invariant under the conformal change, $n \geq 4$, where

$$W^a_{dcb} = R^a_{dcb} + \frac{1}{n}(S^a_d g_{cb} - S^a_c g_{db} + \delta^a_d S_{cb} - \delta^a_c S_{db} -$$

$$-U^a_d J_{cb} + U^a_c J_{db} - J^a_d U_{cb} + J^a_c U_{db} + 2U_{dc} J^a_b + 2J_{dc} U^a_b)+$$

$$+\frac{(n+4)\rho}{n^2(n+2)}(J^a_d J_{cb} - J^a_c J_{db} - 2J_{dc} J^a_b) - \frac{(3n+4)\rho}{n^2(n+2)}(\delta^a_d g_{cb} - \delta^a_c g_{db})$$

and $U_{ab} = J^t_c S_{tb}, U^a_b = U_{bt} g^{ta}$.

The J-traceless component of the Weyl conformal curvature tensor on a Kähler manifold, by means of Mikeš' analysis [15], is given by the following formula

$$^J W^a_{dcb} = R^a_{dcb} - \frac{\rho}{n(n+2)}(\delta^a_d g_{cb} - g_{db}\delta^a_c + J^a_d J_{cb} - J^a_c J_{db} - 2J_{dc} J^a_b)+$$

$$+\frac{1}{n}[(S^a_d - \frac{\rho}{n}\delta^a_d)g_{cb} + \delta^a_d(S_{cb} - \frac{\rho}{n}g_{cb}) - (S^a_c - \frac{\rho}{n}\delta^a_c)g_{db} - \delta^a_c(S_{db} - \frac{\rho}{n}g_{db})-$$

$$-J^t_d(S^a_t - \frac{\rho}{n}\delta^a_t)J_{cb} - J^a_d J^t_c(S_{bt} - \frac{\rho}{n}g_{bt}) + J^t_c(S^a_t - \frac{\rho}{n}\delta^a_t)J_{db}+$$

$$+J^a_c J^t_d(S_{bt} - \frac{\rho}{n}g_{bt}) + 2J^t_d(S_{tc} - \frac{\rho}{n}g_{tc})J^a_b + 2J_{dc}J^t_b(S^a_t - \frac{\rho}{n}\delta^a_t)-$$

$$-\frac{2(n-4)}{n^2}\delta^a_d(S_{cb} - \frac{\rho}{n}g_{cb}) - \delta^a_c(S_{db} - \frac{\rho}{n}g_{db}) + J^a_d J^t_b(S_{ct} - \frac{\rho}{n}g_{ct})-$$

$$-J^a_c J^t_b(S_{dt} - \frac{\rho}{n}g_{dt}) + 2J^a_b J^t_c(S_{dt} - \frac{\rho}{n}g_{dt})].$$

A geometric meaning of the J-traceless component $^J W$ of the conformal Weyl curvature tensor W is given by:

Theorem A [7] *The J-traceless component $^J W$ of the conformal curvature tensor W on a Kähler manifold of real dimension $n \geq 4$ is invariant under concircular change of metrics.*

3 H-projective equivalent metrics

Let (M, J) be a complex manifold. If Kähler metrics g and $\bar{g}$ are projective equivalent (i.e. if their unparametrised geodesic coincide), then the associated Levi-Civita connections coincide i.e. $\nabla = \overline{\nabla}$ and there are only trivial examples of projective Kähler metrics.

Otsuki and Tashiro introduced another notion in the complex case. Therefore, in H-projective geometry, the unparametrized geodesics are replaced by the "generalized complex geodesics", known as h-planar curves.

Let (M, g, J) be a Kähler manifold and ∇ the Levi-Civita connection.

A regulat curve $\gamma : I \mapsto M$ is called H-planar with respect to g if satisfies $\nabla_{\gamma'}\gamma' = \alpha\gamma' + \beta J(\gamma')$, for some functions $\alpha, \beta : I \mapsto \mathbf{R}$.

Particularly, in the case $\beta = 0$, an H-planar curve is a geodesic.

Two Kähler metrics g and $\bar{g}$ are called H-projectively equivalent if each H-planar curve of g is H-planar with respect to $\bar{g}$ and viceversa.

∇ and $\overline{\nabla}$ are H-projectively equivalent iff there exists a (real) 1-form θ such that

$$\overline{\nabla}_X Y - \nabla_X Y = \theta(X)Y + \theta(Y)X - \theta(JX)JY - \theta(JY)JX.$$

A bi-holomorphic mapping $f : M \mapsto M$ is called H-projective transformation if f^*g is H-projectively equivalent to g. Equivalently, we can require that f preserves the set of H-planar curves.

The holomorphically projective tensor, ${}^H P$ which is defined by the following form,

$${}^H P^h_{ijk} = R^h_{ijk} + \frac{1}{n+2}(\delta^h_k S_{ij} - \delta^h_j S_{ik} - J^h_k S_{it} J^t_j + J^h_j S_{it} J^t_k + 2 J^h_i S_{jt} J^t_k)$$

is invariant with respect to holomorphically projective mappings, i.e.
${}^H P = \overline{{}^H P}$.

The relation between the HP-mappings and the homotheties is given by the following result:

Theorem B [5] *Let f be a holomorphically projective mapping between Kähler manifolds $K_n = (M, g, J)$ and $\overline{K}_n = (M, \bar{g}, J)$, $x_0 \in M$ and $\bar{x}_0 = f(x_0)$.*

Suppose that the initial condition $\bar{g}(\bar{x}_0) = kg(x_0)$ is satisfied for a $k \in \mathbf{R}$.

If the holomorphically projective tensor does not vanish at x_0, then the mapping f provides a homothety between $K_n = (M, g, J)$ and $\overline{K}_n = (M, \bar{g}, J)$ i.e. $\bar{g} = kg; k$ is const.

4 Concircular H-projective mappings

Starting from the conformal and the H-projective group of transformations, in [4] it is considered the notion of conformal geodesic mapping, generalized in [6] to conformal H-projective mapping.

We extend the study along this line and introduce the concircular holomorphically projective mappings between Kähler manifolds, getting a rigidity condition.

A diffeomorphism $f : K_n = (M, g, J) \mapsto \overline{K}_n = (M, \bar{g}, J)$ is called a concircular holomorphically projective mapping if $f = f_1 \circ f_2 \circ f_3$, where
$f_1 : K_n = (M, g, J) \mapsto {}^1K_n = (M, {}^1g, J)$ is a concircular mapping,
$f_2 : {}^1K_n = (M, {}^1g, J) \mapsto {}^2K_n = (M, {}^2g, J)$ is a HP-mapping and
$f_3 : {}^2K_n = (M, {}^2g, J) \mapsto \overline{K}_n = (M, \bar{g}, J)$ is a concircular mapping.

Proposition 4.1 *Let $K_n = (M, g, J)$ be a Kähler manifold.*

If the H-projective curvature tensor ${}^H P$ is vanishing at a fixed point x_0 of the manifold, then $W = {}^J W$ at the point x_0.

Proof. If ${}^H P^i_{jkl}(x_0) = 0$, then $S_{jl} = \frac{\rho}{n} g_{jl}$ and $W^i_{jkl} = {}^J W^i_{jkl}$ at x_0.

It is evident that the relation of being concircular holomorphically projective equivalent is symmetric and reflexive. Unfortunately, the concircular holomorphically projective mappings do not form a group because of lack of transitivity and the relation is not an equivalence relation. We found the following solution to the appropriate "equivalence problem" though, only partial.

Theorem 4.1 *Let f be a concircular holomorphically projective mapping between two Kähler manifolds $K_n = (M, g, J)$ and $\overline{K}_n = (M, \bar{g}, J)$.*

If the metrics are proportional at the point x_0 i.e. $\bar{g}(f(x_0)) = kg(x_0)$ and $(W - {}^J W)(x_0) \neq 0$, then f is a concircular mapping.

Proof. All the mappings f, f_1, f_2, f_3 are diffeomorphisms and for simplicity we denote by x_0 all the corresponding points. One has $\bar{g} = kg$, at x_0. f_1 and f_2 are concircular mappings and so $g_1 = e^{2u_1}g$, $\bar{g} = e^{2u_2}g_2$, where u_1 and u_2 are nonvanishing functions on M, verifying the concircularity condition. Therefore $g_2 = ke^{-2(u_1+u_2)}g_1$ at x_0.

Since f_1 is a concircular mapping, one has $W = W_1$, ${}^JW = {}^JW_1$. Because $(W - {}^JW)(x_0) \neq 0$, then $(W_1 - {}^JW_1)(x_0) \neq 0$ and ${}^HP_1 \neq 0$ at x_0. Indeed if ${}^HP_1(x_0) = 0$, then $W_1(x_0) = {}^JW_1(x_0)$, using Proposition 4.1.

Using theorem B, we get that f_2 is a homothety and $g_2 = cg_1$, for any point of M, where c is a constant. Hence $f = f_1 \circ f_2 \circ f_3$ is a concircular mapping. □

5 On some ${}^JW$- Riemann-Kähler flows

The Ricci flow, first introduced by Hamilton, is provided by the equation

$$\frac{\partial g_{ij}}{\partial t} = -2S_{ij},$$

where S is the Ricci tensor field. The Ricci flow, which evolves a Riemannian metric by its Ricci curvature, is a natural analogue of the heat equation for metrics. The previous approach can be adaptad to the complex case. We consider (M, g, J) a Kähler manifold. The Kähler Ricci flow is given by:

$$\frac{\partial g_{i\bar{j}}}{\partial t} = -2S_{i\bar{j}}.$$

These equations become strictly parabolic and it is easy to prove the short time existence.

The idea of Ricci flow was generalized [10] to the concept of Riemann flow, which is a PDE that evolves the metric tensor G:

$$\frac{\partial G_{ijkl}}{\partial t} = -2R_{ijkl},$$

where $G = \frac{1}{2}g \wedge g$, R is the Riemann curvature tensor associated to the metric g and " $\wedge$" is the Kulkarni-Nomizu product. For $(0,2)$-tensors a and b, their *Kulkarni-Nomizu product* $a \wedge b$ is given by

$$\begin{aligned}(a \wedge b)(X_1, X_2; X, Y) = \quad & a(X_1, X)b(X_2, Y) + a(X_2, Y)b(X_1, X) \\ & -a(X_1, Y)b(X_2, X) - a(X_2, X)b(X_1, Y).\end{aligned}$$

These extensions are natural, since some results regarding the Riemann flow resemble the case of the Ricci flow.

We generalize the notion of Riemann flow for a Riemann space and the notion of Kähler-Ricci flow on complex case to the concept of ${}^JW$-Kähler-Riemann flow on a Kähler manifold (M, g, J):

$$\frac{\partial G_{i\bar{j}k\bar{l}}}{\partial t} = -2\,{}^JW_{i\bar{j}k\bar{l}},$$

where ${}^JW$ is the J-traceless component of the conformal curvature tensor W and $G = \frac{1}{2}g \wedge g$.

Theorem 5.1 *Let (M, J, g_0) be a Kähler manifold. A class $g_t = e^{2u_t} g_0$ of concircular related metrics with g_0, given by the ${}^J W$-Kähler-Riemann type flow,*

$$\frac{\partial G_{i\bar{j}k\bar{l}}}{\partial t} = -2 \ \ {}^J W_{i\bar{j}k\bar{l}},$$

satisfies

$$G(x,t) = -2 \ \ {}^J W(g_0(x))\, t + G_0(x),$$

where ${}^J W$ is the J-traceless component of the conformal curvature tensor W.

Proof. We use the implicit solution of a Cauchy problem associated to the ${}^J W$-Kähler-Riemann flow. □

The previous theorem and the properties of concircular geometry proved by Yano lead to:

Proposition 5.1. *Let $t \in (0, \epsilon), \epsilon > 0$, $f_t : (M, J, g_0) \mapsto (M, J, g_t)$ be a concircular mapping. If (M, J, g_0) is an Einstein Kähler manifold, then (M, J, g_t) is an Einstein Kähler manifold.*

References

[1] A.M. Blaga, *Eta-Ricci solitons on para-Kenmotsu manifolds,* Balkan J. Geom. Appl., **20**, 1 (2015), 1-13.

[2] A. Čap, A. R. Gover, H. R. Macbeth, *Einstein metrics in projective geometry,* Geom. Dedicata, **168** (2014),235-244.

[3] X. X. Chen, G. Tian, *Ricci flow on Kähler Eistein manifolds*, Duke Math. J., 13, 1 (2006), 17-73.

[4] H. Chudá, J. Mikeš, *Conformally geodesic mappings satisfying certain initial condition,* Arch. Math. (Brno), **47** (2011), 389-394.

[5] H. Chudá, J. Mikeš, *On holomorphically projective mappings with certain initial conditions,* J. Appl. Math. (2011), 673-678.

[6] H. Chudá, M. Shiha, *Conformal holomorphically projective mappings satisfying a certain initial curvature condition,* Miskolc Math. Notes, **14**, 2 (2013), 569-574.

[7] S. Fubanashi, H. S. Kim, Y. M. Kim and J.S. Pak, *F-traceless component of the conformal curvature tensor on Kähler manifold,* Bull. Korean Math. Soc., **44**, 4 (2007), 795-806.

[8] R. Hamilton, *The Ricci flow on surfaces,* Contemporary Mathematics, 71 (1988) 237-261.

[9] I. Hinterleitner, *On holomorphically projective mappings of e-Kähler manifolds,* Arch. Math.(Brno), **48**, 5 (2012), 333-338.

[10] I. E. Hirică, C. Udrişte, *Basic evolution PDEs in Riemannian geometry,* Balkan J. Geom. Appl., **17**, 1 (2012), 30-40.

[11] I. E. Hirică, L. Nicolescu, *On Weyl structures,* Rend. Circolo Mat. Palermo, **II**, LIII (2004), 390-400.

[12] K. Kiyohara and P. Topalov, *On Liouville integrability of h-projectively equivalent Kähler metrics,* Proc. Amer. Math. Soc. **139** (2011), 231-242.

[13] L. Lakoma, M. Jukl, *The decomposition of tensor spaces with almost complex structure,* Rend. Circ. Mat. Palermo **2**, 72 (2004), 145-150.

[14] D. Luczyszyn and Z. Olszak, *Paraholomorphically pseudo-symmetric para-Kählerian manifolds,* J. Korean Math. Soc. **45**, 4 (2008), 953-963.

[15] J. Mikeš, M. Jukl, Juklová L., *Some results on traceless decomposition on tensors,* J. Math. Science, **174**, 5 (2011), 627-640.

[16] A. Özdeğer, *Conformal and generalized concircular mappings of Einstein-Weyl manifolds,* Acta Math. Scientia, **30B**, (5) (2010), 1739-1745.

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