Adapted metrics for Newtonian gravity models

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Abstract. We use *geodesible* vector fields (i.e. those vector fields for which there exists a Riemannian metric whose geodesics are their trajectories) in order to detail the geometrization of the Newtonian force started in [13]. Several applications for the two-bodies dynamics are included, as some perturbations of the planetaries orbits (for example, the perihelion advance) and the Pioneer 11 abnormal acceleration.

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1 Introduction

The geometrization of dynamical systems is a fundamental problem for both Mathematics and Applied Sciences. In the particular case of a phenomenon governed by one field of forces, we may consider the associated flow and look for a "good geometry" explaining the behaviour of the "moving particles".

One promising direction is the following: let M be a differentiable manifold and ξ a vector field on M; denote by $Riem(M, \xi)$ the set of the Riemannian metrics g on M, such that the trajectories of ξ are geodesics of g. In the case when $Riem(M, \xi)$ is non-void, this differential invariant may provide appropriate Riemannian metrics for modeling hidden properties of the dynamics associated to ξ.

This approach has a long history, tracing back his roots in the hints of Riemann ([15]). More recently, Gluck ([7]) and Sullivan ([17]) investigated when $Riem(M, \xi)$ is non-void (see also [10]-[13]).

In [13],we found examples of "good" metrics for the gravitational field ξ of the two-bodies problem in the classical setting. In particular, we constructed two families of such metrics, having the following remarkable properties: they are infinite (depending on two real valued parameter functions) and ξ is divergence-free with respect to any such metric. The geodesics of the metrics in the first family are Keplerian, but the geodesics of the metrics in the second family spiral "nearby" Keplerian trajectories. This "almost Keplerian" behaviour suggests possible new post-Newtonian models, accounting (for example) for perihelion advance of planetaries orbits.

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In this paper, we refine our previous study [13]. In §2 we briefly provide some known basic information about geodesic vector fields. For the particular case of the Newtonian vector field, we recall the set $Riem(M, \xi)$ (from [13]) and determine some new properties of the Riemannian metrics therein (§3). We fit the parameters in order to obtain a qualitative behaviour of the geodesics, and suggest new explanations for some well-known (but less understood) classic or modern phenomena: the perturbations of the planetaries orbits (for example, perihelion advance) (§4); the Pioneer 11 abnormal acceleration and the flyby anomaly (§5).

2 Adapted metrics for dynamical systems governed by one "force"

Let M be a n-dimensional differentiable manifold and ξ a vector field on M. We say ξ is *geodesible* if there exists a Riemannian metric g on M, such that the trajectories of ξ be geodesics of g. The set of all these metrics will be denoted $Riem(M, \xi)$ and will be called the (Riemannian) moduli space associated to ξ.

Obviously, $Riem(M, \xi)$ (if non-void) is a differentiable invariant. For a fixed $g \in Riem(M, \xi)$, the vector field ξ is called *geodesic vector field* with respect to g.

For a vector field $X \in \mathcal{X}(M)$, denote L_X and i_X the corresponding Lie derivative and interior product, respectively.

The following two characterizations were proven in [17] and [7] (see also [18], Prop.6.7 and Prop. 6.8; [14] for an updated review).

Theorem 2.1. *Suppose ξ be a nonsingular vector field on M. Then ξ is geodesible if, and only if, there exists a 1-form ω on M, such that $\omega(\xi) > 0$ and $i_\xi d\omega = 0$.*

Theorem 2.2. *Suppose ξ be a nonsingular vector field on M. Then, the following assertions are equivalent:*

(i) *there exists a Riemannian metric on M, making the trajectories of ξ geodesics and ξ of unit length;*

(ii) *there exists a 1-form ω on M, such that $\omega(\xi) = 1$ and $L_\xi \omega = 0$;*

(iii) *there exists a 1-form ω on M, such that $\omega(\xi) = 1$ and $i_\xi d\omega = 0$;*

(iv) *there exists a codimension one distribution E, complementary to* span$\{\xi\}$*, such that $[\xi, X]$ belongs to E, for any X in E.*

When *span*$\{\xi\}$ is a Riemannian foliation, Y. Carriere ([5]; [18]) completed this list with this additional equivalent assertion:

(v) *there exists a Riemannian metric, making ξ a Killing vector field* (see [16]).

Remark 2.1. (i) In [11], we proved an analogous of Theorem 2.1, in the semi-Riemannian setting.

(ii)A differential form ω on M is called *basic* for the foliation *span*$\{\xi\}$ if $L_\xi \omega = 0$ and $i_\xi \omega = 0$. (In some cases, non-trivial basic forms do not exist, cf. [4]).

Suppose ξ be a nonsingular vector field on M, such that the generated foliation admits a basic one-form ω. This condition is similar to that in Theorem 2.2, with $\omega(\xi) = 0$ instead of $\omega(\xi) = 1$.

(iii) When ξ is singular, the geodesibility is more complicated; the singular points of ξ must be degenerated geodesics. Consider the following example: let $\xi = xy\partial/\partial x$ in $\mathbb{R}^2$. Suppose, ad absurdum, there exists $g \in Riem(\mathbb{R}^2)$, such that its Levi-Civita connection ∇ satisfies $\nabla_\xi \xi = 0$. Denote by $U = \{(x, y) \in \mathbb{R} \mid xy \neq 0\}$. On U, we obtain

$$x\nabla_{\partial/\partial x}\partial/\partial x = -\partial/\partial x.$$

From the Koszul formula, we deduce

$$x\partial g_{11}/\partial x = -2g_{11},$$

where $g_{11} = g(\partial/\partial x, \partial/\partial x)$. By integrating on U the last PDE we get $g_{11}(x, y) = f(y)/x^2$, where f is a (nowhere vanishing) function of y only. So, the metric g cannot be defined on the whole $\mathbb{R}^2$, contradiction.

(iv) In [8], the geodesible singular vector fields on surfaces are characterized in extenso (see also [7]).

(v) Let ξ be a vector field on a differentiable manifold M. If ξ is parallel with respect to some Riemannian metric, it is also geodesible. For the existence of (complete) parallel vector fields, there exist strong topological obstructions (see for example [19] and references therein).

3 Adapted metrics for the 2-bodies problem: the Newtonian gravitational field

We recall and update some results from [13]; in what follows, all the assertions which are not explicitly quoted are new.

Let $M = \mathbb{R}^2 \backslash \{(0, 0)\}$ be our Universe: all particles are supposed of neglectable mass. The only source of gravitation is supposed to be a point-like massive object of (constant and positive) "mass" m, situated in the point (0,0) (so, outside the Universe, but determining all the dynamics in M).

We denote by (r, φ) the polar coordinates on M and by $\xi = -mr^{-2}\partial r$ the "Newtonian gravitational vector field" on M.

Remark 3.1. The Euclidean metric h on M has the well-known components $h_{11} = 1$, $h_{12} = 0$, $h_{22} = r^2$. The (only non-vanishing) Christoffel coefficients are $\Gamma^1_{22} = -r$, $\Gamma^2_{12} = r^{-1}$. The canonically parametrized geodesics are ("folklore" result) of the form $\gamma(s) = (r(s), \varphi(s))$, with $r^2(s) = s^2 + a^2$, $\varphi(s) = b + arctg\frac{s}{a}$, where a and b are arbitrary real constants.

Remark 3.2. (i)([13]) The Riemannian metrics $g \in Riem(M, \xi)$ are of the form:

(1) $$g = m^{-2}r^4dr^2 - 2m^{-1}r^2\beta dr d\varphi + (\beta^2 + m^2r^{-4}\gamma^2)d\varphi^2,$$

where $\beta = \beta(\varphi)$ and $\gamma = \gamma(r, \varphi)$ are differentiable functions and γ is nowhere vanishing.

(ii) For $\beta := 0$ and $\gamma := r^5 m^{-2}$, we get the particular metric

$$g_1 = m^{-2} r^4 (dr^2 + r^2 d\varphi^2),$$

which is conformally equivalent with the Euclidean metric h from the Remark 3.1.

(iii) ([13]) The coefficients of Christoffel are:

$$\Gamma^1_{11} = 2r^{-1} \quad , \quad \Gamma^2_{11} = 0 \quad , \quad \Gamma^1_{12} = -2m\beta r^{-3} + m\beta\gamma^{-1} r^{-2} \partial_r \gamma$$

$$\Gamma^2_{12} = \gamma^{-1} \partial_r \gamma - 2r^{-1} \quad , \quad \Gamma^2_{22} = \gamma^{-1} \partial_\varphi \gamma - m\beta\gamma^{-1} r^{-2} \partial_r \gamma + 2m\beta r^{-3}$$

$$\Gamma^1_{22} = m\beta\gamma^{-1} r^{-2} \partial_\phi \gamma - m\beta' r^{-2} + 2m^2\beta^2 r^{-5} +$$

$$+ 2m^4\gamma^2 r^{-9} - m^2\beta^2\gamma^{-1} r^{-4} \partial_r \gamma - m^4 \gamma r^{-8} \partial_r \gamma.$$

(iv) The Gaussian curvature is

$$K(r, \varphi) = -\frac{4\beta^2 \partial_r \gamma}{\gamma^3 r} - \frac{m^2 \partial^2_{rr} \gamma}{\gamma r^4} + \frac{6m^2 \partial_r \gamma}{\gamma r^5} - \frac{10m^2}{r^6} + \frac{2\beta^2 (\partial_r \gamma)^2}{\gamma^4}.$$

The manifold (M, g) is flat if, and only if,

$$2\beta^2 r^5 \partial_r \gamma (r \partial_r \gamma - 2\gamma) + m^2 \gamma^3 (6r \partial_r \gamma - r^2 \partial^2_{rr} \gamma - 10\gamma) = 0.$$

In the particular case when $\beta = 0$, a solution of the previous PDE is $\gamma(r, \varphi) = r^2 m^{-1} [\lambda(\varphi) r^3 + \theta(\varphi)]$, where $\lambda = \lambda(\varphi)$ and $\theta = \theta(\varphi)$ are such that γ does not vanish on M.

Another particular solution is $\gamma(r, \varphi) = r^2 m^{-1} \theta(\varphi)$, for any β.

(v) We remark that $\xi = grad\ f$, where $f(r, \varphi) = -\frac{1}{3m} r^3 + \int \beta(\varphi) d\varphi$ and the gradient is calculated with respect to g.

(vi) ([13]) The following assertions are equivalent: $(\text{vi})_1$ $div \xi = 0$; $(\text{vi})_2$ ξ is parallel; $(\text{vi})_3$ there exists a nowhere vanishing real-valued differentiable function θ such that $\gamma(r, \varphi) = r^2 m^{-1} \theta(\varphi)$.

Remark 3.3. ([13]) Suppose ξ is divergence-free with respect to $g \in Riem(M, \xi)$. Then, the following properties hold.

(i) With the notation from the previous Remark $(\text{v})_3$, we have $g_{22} = \beta^2 + \theta^2$. So, these metrics depend upon two (real valued) "parameter" functions (or "degrees of freedom") β and θ. That is

$$g = m^{-2} r^4 [dr^2 - 2m\beta r^{-2} dr d\varphi + m^2 r^{-4} (\beta^2 + \theta^2) d\varphi^2], \tag{2}$$

with $\beta = \beta(\varphi)$ and $\theta = \theta(\varphi)$.

(ii) The coefficients of Christoffel become

$$\Gamma^1_{11} = 2r^{-1} \ , \ \Gamma^2_{11} = \Gamma^1_{12} = \Gamma^2_{12} = 0 \ , \ \Gamma^2_{22} = \theta^{-1}\theta' \ , \ \Gamma^1_{22} = m\beta\theta'\theta^{-1}r^{-2} - m\beta' r^{-2},$$

where the sign ' means the derivation with respect to φ.

(iii) The curvature of g is completely provided by $R_{1212}(r, \varphi) = 0$, that is (M, g) is flat. Hence, (M, g) is locally Euclidean.

(iv) The equations of the geodesics are

$$r" + 2r^{-1}(r')^2 + mr^{-2}(\varphi')^2(\beta\theta^{-1}\theta' - \beta') = 0, \quad \theta\varphi" + \theta'(\varphi')^2 = 0.$$

Using the second equation in the first, we obtain an equivalent system

$$r^2r" + 2r(r')^2 = m(\varphi')^2(\beta\varphi'^{-2}\varphi" - \beta'), \quad (\theta(\varphi)\varphi')' = 0.$$

Denote by Θ an anti-derivative of θ and by B an anti-derivative of β; then, for arbitrary constants k_1, k_2, k_4, we have

$$r^2r' = m(\beta\varphi' + k_1), \quad \Theta(\varphi(t)) = k_2t + k_4.$$

A geodesic $c = c(t) = (r(t), \varphi(t))$ has the form

$$c(t) = ((3m(\frac{1}{2}B^2(\varphi(t)) + k_1t + k_3))^{\frac{1}{3}}, \varphi(t)), \tag{3}$$

with the condition $\Theta(\varphi(t)) = k_2t + k_4$, where k_1, k_2, k_3, k_4 are arbitrary constants.

(v) Obviously, the radial geodesics

$$c(t) = ((3m(k_1t + k_3))^{\frac{1}{3}}, \varphi(t_0))$$

are the trajectories of ξ, and, despite the Newtonian model, in (M, g) they are freely falling particles. In particular, they are parameterized by arc lenght, so their speed with respect to (M, g) is constant. Of course, in the Euclidian geometry of M, these particles are accelerated (their variable speed is increasing for incoming/crash particles and decreasing for outcoming/escape particles).

(vi) When $\beta = 0$, the pair (M, g) is a warped product Riemannian manifold.

(vii) For the particular case when

$$\beta(\varphi) = B^3 \ e \ sin\varphi \ m^{-1}(1 + ecos\varphi)^{-4}, \quad \theta(\varphi) = B^2L^{-1}(1 + e\ cos\varphi)^{-2},$$
$$\varphi(0) = 0, \quad r'(0) = 0, \quad r(0) = \tfrac{B}{1+e},$$
$$\varphi'(0) = L^2B^{-2}(1 + e)^2,$$

the geodesics are given by

$$\begin{aligned} &r = B(1 + e\ cos\varphi)^{-1} && (\text{"}Kepler's \ first \ law\text{"}) \\ &r^2\varphi' = L && (\text{"}Kepler's \ second \ law\text{"}), \end{aligned} \tag{4}$$

where we denoted $B = L^2 m^{-1}$, the "angular momentum" by L, the "mass" by m, the "eccentricity" by e.

(viii) The previous remark shows that the trajectories in Newtonian dynamics are geometric objects (geodesics) in an appropriate (Riemannian) geometry. This fact was not evident in the classical setting, where the equations of motion were direct consequence of analytical calculations; of course, those calculations were made in the (implicit) Euclidean framework of $\mathbb{R}^2$, but they were "chart dependent" and, hence, had not covariant character.(This lack of covariance led to the necessity of looking for other kind of invariance, and the Galilean group was the lucky solution).

(ix) For the particular case when $\beta(\varphi) = \varphi$, the geodesics are given by

$$2r^3(t) = 3m(\varphi^2(t) + k_1 t + k_3), \quad \Theta(\varphi(t)) = k_2 t + k_4, \tag{5}$$

where k_1, k_2, k_3 and k_4 are arbitrary constants; Θ is an anti-derivative of θ.

(x) The geodesics (5) are second order approximations of the Keplerian ones given by (4). In [13], we suggested that, for suitable parameters L ("angular momentum") and m ("mass"), the orbits (5) are elliptical spirals, which spread slowly, showing both perihelion precession-like behaviour and increasing lenght of "revolution year" (two astronomical recorded phenomena, at large time scales, which are not satisfactory explained in Newtonian and most post-Newtonian models). In §5 we shall detail this topic.

The family of metrics (2) allows multiple other such variations of the classical Newtonian model for the 2-body dynamics (by setting up the two "degrees of freedom" β and θ).

(xi) The particular cases of metrics which appeared bellow were described by ad-hoc heuristic restrictions imposed to β and θ. These restrictions are purely analytic and (up to now) we did not succeed to derive them from some variational principle or geometric hypothesis.

(xii) In the literature, there exist some similar geometrizations of the 2-bodies problem:

- J. Moser made a suitably changing of the time variable and transformed the phase flow of Kepler's problem into the geodesic flow on a surface of constant (positive or negative) curvature ([3], §1.4).

- Starting from the Maupertuis or Fermat principles, critical curves for some Hamiltonian actions may be considered as geodesics of special (conformally Euclidean) Riemannian metrics ([6],§33).

Comparing with the preceding ones, our attempt has two major advantages: firstly, it does not appeal to the formalism of Analytical Mechanics (Lagrangians, Hamiltonians, etc), working directly on the base manifold; secondly, it introduces not only a (single and lucky) "good" Riemannian metric, but a differential invariant (the moduli space), with an infinity of available metrics, waiting to be picked up for applications.

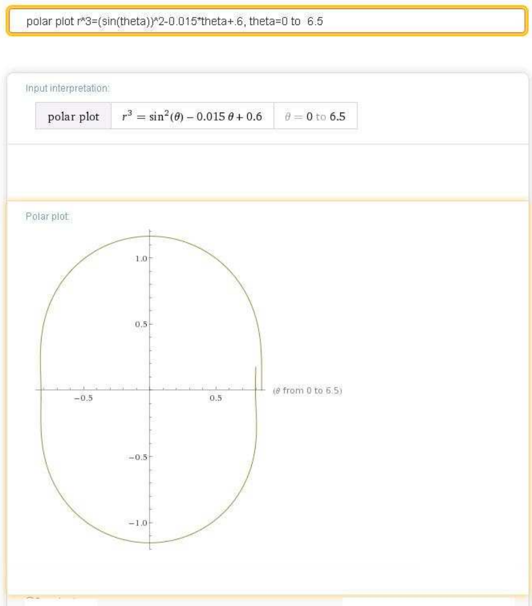

Figure 1: Orbit-like geodesic showing perihelion precession

4 Application 1: qualitative explanation for some perturbations of the orbits of planets

Classical Celestial Mechanics integrates the Newton's equation of motion in order to obtain, via the Kepler's laws, the trajectories of moving "particles" in the Solar System. In a first approximation, these trajectories are conics and describe quite accurately the dynamics of planets, asteroids, satellites and comets. The geometrical setting is Euclidean Geometry.

The problem 1. When more accurate measurements are done, it appears that significant differences arise, for example the precession of the perihelium of a planet. The success of the General Theory of Relativity in explaining this last fact (through a suitable model based on the Schwarzschid metric) shaded the interest in post-Newtonian models for explaining the same thing with Riemannian methods.

However, there exist several interesting post-Newtonian explanations for the advance of the perihelion of a planet as well (see, for example, [9]).

The problem 2. Significant periodical climate changes are produced by some variations of the Earth's orbit (for example, modification of the eccentricity).

Such variations are not satisfactory explained by the Newtonian Celestial Mechanics, nor by the General Theory of Relativity.

The model 1. Consider a 2-dimensional empty Universe, with a single, massive, static body of mass $m = 1$ (the "Sun"). This is a reduction of a 2-bodies Universe,

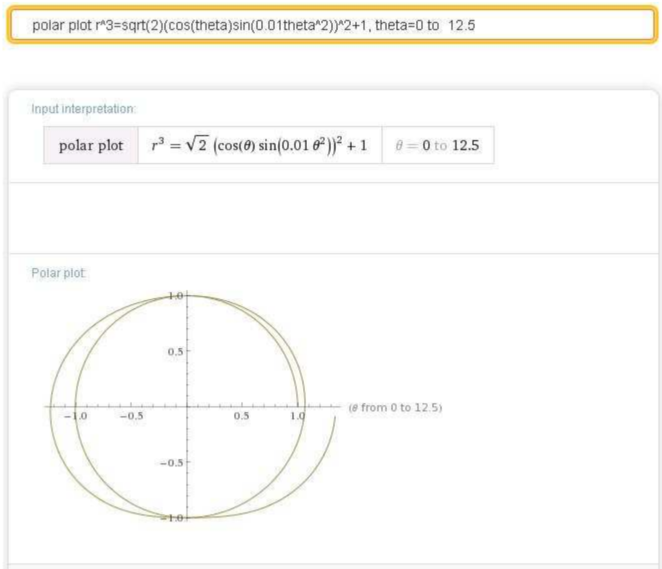

Figure 2: Orbit-like geodesic showing eccentricity perturbation

when one mass is much greater than the other. Suppose hence we are in $M := \mathbb{R}^2 \backslash \{(0,0)\}$, with polar coordinates (r,φ) and the "gravitational force" vector field $\xi = -mr^{-2}\partial_r$, as in §3. We *postulate* that the Riemannian metric $g \in Riem(M,\xi)$, governing the behaviour of this Universe, is given by a particular metric from (2):

$$g = 9r^4dr^2 - 6\sqrt{2}r^2cos\varphi drd\varphi + (2cos^2\varphi + 1)d\varphi^2.$$

Consider a particular case of a geodesic given in (3), moving on a near-elliptic trajectory (as a planet of the Solar System revolving around the Sun). The mass of the respective material particle is supposed constant and neglectable. The (explicit) equation of the geodesic, in polar coordinates, is *postulated*

$$r(\varphi) = (sin^2(\varphi) - 0.015\varphi + 0.6)^{\frac{1}{3}}.$$

Figure 1 shows the image of this geodesic, depicting a qualitative precession of the "perihelion".

The model 2. Consider the same framework as previously, with a particular Riemannian metric

$$g = 9r^4dr^2 - 6\sqrt{2}r^2\beta(\varphi)drd\varphi + (2\beta^2(\varphi) + 1)d\varphi^2,$$

where β is the anti-derivative of $B(t) = \sqrt{2}\cos t \sin(0.01t^2)$; fix a *postulated* geodesic

$$r(\varphi) = (\sqrt{2}(\cos\varphi \sin(0.01\varphi^2))^2 + 1)^{\frac{1}{3}}.$$

In Figure 2 we remark a specific behaviour of this (quasi-elliptic) geodesic, showing a modified eccentricity.

5 Application 2: qualitative explanation of the Pioneer effect and of the flyby anomaly

The problem 3. The Pioneer effect (anomaly) was, for the first time, discovered during the flight of the spacecrafts Pioneer 10 and Pioneer 11, soon after they left a flyby trajectory around Jupiter, respectively Saturn, and were put on a hyperbolic trajectory toward the border of the Solar System ([1]). Their acceleration decreased under the Sun's gravity, at a bigger rate than predicted. Afterthat, similar behaviour was detected in the data received from other spacecrafts. Several explanations were suggested, but none seems to be unanimously accepted.

The problem 4. The flyby effect (anomaly) was detected during Earth-flybys of Galileo spacecraft (and replicated afterthat with NEAR, Cassini-Huygens, Rosetta spacecrafts); it consists of a significant velocity increase. Up to now, there are no satisfactory explanations for this effect ([2]).

The model 3. Consider a 2-dimensional empty Universe, with a single massive body of mass $m = 1$. We suppose, on $M := \mathbb{R}^2 \setminus \{(0,0)\}$, the polar coordinates (r,φ) and the "gravitational force" vector field $\xi = -mr^{-2}\partial_r$, as in §3. We *postulate* that the Newtonian gravitational field ξ is divergence-free with respect to a Riemannian metric $g \in Riem(M,\xi)$, of the form (2).

Consider a Keplerian particle $\mu(t) = (r(t),t)$, $r(t) = \frac{1}{1+2cost}$ moving "outward" (as unmanned vehicle aimed toward the border of the Solar System, on a hyperbolic trajectory) with $t \in [0,1]$. This particle modelates -classically- an "ideal" spacecraft Pioneer-like, on its final trajectory.

Obviously, at the moment of time t, the velocity is $\mu'(t) = (r'(t),1)$, the "Euclidean" vector acceleration is $\mu''(t) = (r''(t),0)$, the "Euclidean" speed is $\sqrt{(r'(t))^2+1}$ and the "Euclidean" scalar acceleration is $\mid r''(t) \mid$.

Fix a metric (2), with constant β and $\theta \neq 0$. With respect to g, for $t \geq 1$, the speed is $\sqrt{r^4(t)(r'(t))^2 - 2r^2(t)\beta(t)r'(t) + \beta^2(t) + \theta^2(t)}$ and the (scalar) acceleration is $\mid r^3(t)[r(t)r''(t) - 2(r'(t))^2] \mid$.

For our particular curve μ, and for $t \in [0,1]$, this (scalar) acceleration becomes smaller than the Euclidean one. We see that, in our post-Newtonian model, the particle μ has an additional "inward" acceleration (when compared with the Euclidean one), manifesting the same behaviour as the "real" spacecrafts Pioneer.

The model 4. Fix the same (general) geometrical setting as for the model 3. We consider a particle in circular motion around the origin (as a spacecraft around Earth). The mass of such a particle is supposed constant and neglectible.This circular particle may be replaced with a (Keplerian) elliptic one, with minor changes in formalism. Denote it by $\nu(t) = (r_0,t)$ and a time moment $t_0 = 1$.

Obviously, for $t \geq 1$, the velocity is $\nu'(t) = (0,1)$ and the "Euclidean" speed is 1.

With respect to a metric (2), for $t \geq 1$, the speed is $(\beta^2(t) + \theta^2(t))$.

For $t > t_0$, $\beta^2(t) + \theta^2(t) > 1$, the particle ν has a bigger speed than previously calculated,which is in accordance with what was observed in practice.

6 Comments

Remark 6.1. In the previous two paragraphs we treated separately four different kinds of trajectories for four essentially different kinds of dynamical systems, using four different adapted Riemannian metrics. In each case, the qualitative theoretical description of some *postulated* geodesics fits quite satisfactory the qualitative observational behaviour of the respective gravitational phenomena.

(i) *The first open problem* is if numerical simulation (i.e. quantitative description) will also fit the real observational data.

(ii) *The second open problem* is if a "good" metric for the 2-bodies problem may be found, modeling, *at the same time*:

- all known trajectories of the planets (or comets, asteroids, etc.) in the simplified framework of a Sun-planet system, approximated by conics-like curves;
- all known perturbations of the orbits of the planets, in the previous framework (especially the precession of the perihelium);
- the abnormal accelerations of Pioneer 10 and Pioneer 11.
- the flyby anomaly.

This metric might provide, at some limit, all the Newtonian results known from Classical Celestial Mechanics. In our opinion, it is quite improbable such an "universal" metric exists. More probable, gravitation -in the respective post-Newtonian context- might be modelled case-by-case only (as in the General Theory of Relativity, where several models coexist, each one describing its own reality).

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