Bolyai–Lobachevsky geometrical simulation of a media acting as an ideal mirror on the particles

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Abstract. Previously it was shown that in electrodynamic context the Lobachevsky geometry can simulate an effective medium acting as an ideal mirror, oriented perpendicularly to the axes z. Such an effect exists for a scalar particle as well. In the present paper, an analogue of that phenomenon is investigated for a spin $1/2$ field. In explicit form, solutions of the Dirac equation are constructed which describe waves reflected from effective potential barrier without penetrating it. The depth of penetration into the medium is determined by characteristics of the quantum states and by the curvature radius of the Lobachevsky space; for waves with $k_1 = 0, k_2 = 0$ the effective reflecting barrier vanishes. Results are valid for Majorana fermions as well, some relevant details for this special case are specified. It is shown that for Weyl fermions, the reflecting effect does not exist. So, effects of non-Euclidean geometry can substantially depend on the type of a fermion.

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1 Introduction

Non-Euclidean geometry can be seen as a base for modeling effective medias in electrodynamic context [9]. In particular, the Lobachevsky geometry while using quasi-Cartesian coordinates effectively simulates an electrodynamic medium with the following constitutive law

$$D^i = \epsilon_0 \epsilon^{ik} E_k \; , \qquad B_i = \mu_0 \mu^{ik} H^k \; , \quad \epsilon^{ik}(x) = \mu^{ik}(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-2z} \end{pmatrix}; \tag{1.1}$$

the medium is non-homogeneous along the direction z. The flat space Maxwell equations in such a medium may be reduced in the end to a single differential equation of

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the form [2, 3]

$$\left(\frac{d^2}{dz^2} + \epsilon - U(z) \right) \varphi(z) = 0, \tag{1.2}$$

which can be associated with a Schrödinger-like equation with the potential $U(z) = (a^2 + b^2)e^{2z}$. This potential describes an effective repulsing force on the left $F_z = -2(a^2 + b^2)e^{2z}$. In the context of quantum mechanics, that equation describes the motion of a particle in potential field tending to infinity by exponential law, the particle is reflected by this potential non penetrating through it.

Thus, the Lobachevsky geometry effectively acts as an ideal mirror spreading in the space. The depth z_0 of penetration into that medium is determined by parameters of solutions and by the curvature radius of the Lobachevsky space [2], [3]. Note that at $a = k_1 = 0, b = k_2 = 0$ the barrier vanishes.

That analysis was extended [4] to the case of non-relativistic scalar particle; the main reflection features are the same. Some preliminary and non-complete study was performed in [5, 6, 7] for a relativistic Dirac particle: formal solutions in the Lobachevsky space were constructed in terms of confluent hypergeometric functions though the reflection effect was not definitely described [5, 6, 7].

In the present paper, the effects of the Lobachevsky geometry are investigated for three type of spin 1/2 particle: Weyl, Dirac, and Majorana's. It is proved the effect of reflection for Dirac and Majorana particles and absence of that phenomena for Weyl particle.

2 Majorana spinor field

Let us fix the Majorana basis by the following transformation from the spinor one [8]

$$\Psi_M = A\ \Psi\ ,\ \Gamma^a_M = A\gamma^a A^{-1}\ , \quad A = \frac{1-\gamma^2}{\sqrt{2}}\ ,\ A^{-1} = \frac{1+\gamma^2}{\sqrt{2}};$$

$$\gamma^0_M = \gamma^0\gamma^2\ , \quad \gamma^1_M = \gamma^1\gamma^2\ , \quad \gamma^2_M = \gamma^2, \quad \gamma^3_M = \gamma^3\gamma^2\ . \tag{2.1}$$

Explicitly, the matrices are given by

$$\gamma^0_M = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix},\ \gamma^1_M = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix},\gamma^2_M = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},\ \gamma^3_M = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}. \tag{2.2}$$

These matrices are purely imaginary. Therefore, in this representation the Dirac wave operator becomes explicitly real:

$$\left(i\gamma^a \frac{\partial}{\partial x^a} - m \right) \Psi_M = 0;$$

in other words there exist independent equations for real and imaginary parts of the Dirac wave function $\Psi_M = \mathrm{Re}\ \Psi + i\ \mathrm{Im}\Psi = \Psi_+ + \Psi_-$:

$$\left(i\gamma^a \frac{\partial}{\partial x^a} - m \right) \Psi_+ = 0\ , \quad \left(i\gamma^a \frac{\partial}{\partial x^a} - m \right) \Psi_- = 0\ ; \tag{2.3}$$

they describe the so-called Majorana fermions, Ψ_+ and Ψ_- with charge parity $+1$ and -1, respectively.

For the following six generators

$$\sigma^{ab} = \frac{1}{4}(\gamma^a \gamma^b - \gamma^b \gamma^a) \equiv \frac{1}{2}\gamma^a \gamma^b, \quad \sigma^{ab} = -\sigma^{ba} \; ;$$

we have

$$\sigma^{01} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \; \sigma^{02} = \frac{1}{2}\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \; \sigma^{03} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\sigma^{12} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \; \sigma^{13} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \; \sigma^{23} = \frac{1}{2}\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{2.4}$$

The Lorentzian 4-spinor transformations relative to an arbitrary basis are given by the formula [8])

$$S(k,\; k^*) = \frac{1}{2}(k_0 + k_0^*) - \frac{1}{2}(k_0 - k_0^*)\gamma^5 \; + k_1(\sigma^{01} + i\sigma^{23}) + k_1^*(\sigma^{01} - i\sigma^{23}) \; +$$
$$k_2(\sigma^{02} + i\sigma^{31}) + k_2^*(\sigma^{02} - i\sigma^{31}) \; + k_3(\sigma^{03} + i\sigma^{12}) + k_3^*(\sigma^{03} - i\sigma^{12}); \tag{2.5}$$

where the complex 4-vector parameter k_a is used. We see that in the Majorana basis, the bispinor transformations are real, so the Majorana particles are relativistic Lorentz invariant objects.

To describe the interaction of the Majorana particles with the gravitational field, it suffices to restrict the Dirac covariant equation to the Majorana case in any chosen Majorana basis (see the notations from [8]):

$$\{ \, i\gamma^{\alpha}(x) \, (\, \partial_{\alpha} \, + \, \Gamma_{\alpha}(x)) - m \} \; \Psi(x) = 0 \, ,$$
$$\gamma^{\alpha}(x) = \gamma^a \, e^{\alpha}_{(a)}(x) \, , \quad \Gamma_{\alpha}(x) = \frac{1}{2} \, \sigma^{ab} \, e^{\beta}_{(a)} \, \nabla_{\alpha} \, (\, e^{\alpha}_{(b);\beta} \,) \tag{2.6}$$

Due to the properties

$$(i \, \gamma^a_M)^* = +\gamma^a_M \, , \quad (\sigma^{ab}_M)^* = +\sigma^{ab}_M \, , \quad (i \, \gamma^5_M)^* = +\gamma^5_M \, ; \tag{2.7}$$

the covariant Dirac wave operator is real

$$[\, i\gamma^{\alpha}(x) \, (\, \partial_{\alpha} \, + \, \Gamma_{\alpha}(x) \,) \, - \, m \,]^* = [\, i\gamma^{\alpha}(x) \, (\, \partial_{\alpha} \, + \, \Gamma_{\alpha}(x) \,) \, - \, m \,] \; . \tag{2.8}$$

This means the existence of an independent equation for each Majorana component:

$$[\, i\gamma^{\alpha}(x) \, (\, \partial_{\alpha} \, + \, \Gamma_{\alpha}(x) \,) \, - \, m \,]\Psi_+ = 0 \, , \quad [\, i\gamma^{\alpha}(x) \, (\, \partial_{\alpha} \, + \, \Gamma_{\alpha}(x) \,) \, - \, m \,]\Psi_- = 0 \, . \tag{2.9}$$

3 Separating the variables

Let us consider the spin 1/2 particle on the background of Lobachevsky geometry in quasi-Cartesian coordinates (t, x, y, z are dimensionless; it is convenient to start with the Dirac case):

$$dS^2 = dt^2 - e^{-2z}(dx^2 + dy^2) - dz^2 \; , \quad \sqrt{-g} = e^{-2z} \; , \quad z \in (-\infty, +\infty). \tag{3.1}$$

The most simple form of covariant Dirac equation in any orthogonal coordinates is [8]

$$\left[i\gamma^{a} \left(e^{\alpha}_{(a)} \frac{\partial}{\partial x^{\alpha}} + \frac{1}{2} \left(\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^{\alpha}} \sqrt{-g} e^{\alpha}_{(a)} \right) \right) - m \right] \Psi(x) = 0 \; ; \tag{3.2}$$

in the diagonal tetrad $e^{\beta}_{(a)} = \text{diag}\,(1, e^{z}, e^{z}, 1)$, eq. (3.2) takes the form

$$\left[i\gamma^{0} \frac{\partial}{\partial t} + i\gamma^{1} e^{z} \frac{\partial}{\partial x} + i\gamma^{2} e^{z} \frac{\partial}{\partial y} + i\gamma^{3} \left(\frac{\partial}{\partial z} - 1 \right) - m \right] \Psi = 0 \; . \tag{3.3}$$

There exist three operators: $i\partial_t, \; -i\partial_x, \; -i\partial_y$, which commute with the wave operator in (3.3); so the solutions may be searched in the form

$$\Psi^{\epsilon, k_1, k_2} = e^{-i\epsilon t} \; e^{ik_1 x} \; e^{ik_2 y} \; \begin{pmatrix} f_1(z) \\ f_2(z) \\ f_3(z) \\ f_4(z) \end{pmatrix} . \tag{3.4}$$

With the use of the spinor representation for Dirac matrices, we get 4 equations for $f_i(z), \; i = 1, 2, 3, 4$[1]:

$$\begin{gathered} -i\epsilon F_3 - iae^{z} F_4 - be^{z} F_4 - \frac{d}{dz} F_3 + im \; F_1 = 0 \; , \\ -i\epsilon F_4 - iae^{z} F_3 + be^{z} F_3 + \frac{d}{dz} F_4 + im \; F_2 = 0 \; , \\ -i\epsilon F_1 + iae^{z} F_2 + be^{z} F_2 + \frac{d}{dz} F_1 + imF_3 = 0 \; , \\ -i\epsilon F_2 + iae^{z} F_1 - be^{z} F_1 - \frac{d}{dz} F_2 + imF_4 = 0 \; . \end{gathered} \tag{3.5}$$

There exists an additional commuting operator – the generalized helicity operator

$$\Sigma = \frac{1}{2} \; (e^{z} \gamma^{2} \gamma^{3} \frac{\partial}{\partial x} + e^{z} \gamma^{3} \gamma^{1} \frac{\partial}{\partial y} + \gamma^{1} \gamma^{2} \frac{\partial}{\partial z}) \; . \tag{3.6}$$

From the eigenvalue equation $\Sigma \; \Psi = p \; \Psi$, we get

$$\begin{gathered} ae^{z} F_2 - ibe^{z} F_2 - i \frac{d}{dz} F_1 = pF_1 \; , \quad ae^{z} F_1 + ibe^{z} F_1 + i \frac{d}{dz} F_2 = pF_2 \; , \\ ae^{z} F_4 - ibe^{z} F_4 - i \frac{d}{dz} F_3 = pF_3 \; , \quad ae^{z} F_3 + ibe^{z} F_3 + i \frac{d}{dz} F_4 = pF_4 \; . \end{gathered} \tag{3.7}$$

Considering the equations (3.5) and (3.7) jointly, we derive the algebraic equations for F_i:

$$\begin{gathered} -i\epsilon F_3 - ipF_3 + imF_1 = 0 \; , \quad -i\epsilon F_4 - ipF_4 + imF_2 = 0 \; , \\ -i\epsilon F_1 + ipF_1 + imF_3 = 0 \; , \quad -i\epsilon F_2 + ipF_2 + imF_4 = 0 \; . \end{gathered} \tag{3.8}$$

[1]We perform a change of notations, $k_1 = a, k_2 = b$; as well, it is convenient to separate the simple multiplier: $f_i = e^{z} F_i$.

Further, we get two values for p, and the corresponding restrictions on F_i:

$$p = \pm\sqrt{\epsilon^2 - m^2}\,, \qquad F_3 = \frac{\epsilon - p}{m}\,F_1\,, \qquad F_4 = \frac{\epsilon - p}{m}\,F_2\,. \tag{3.9}$$

Taking into account (3.9) instead of the four equations (3.5), we get only two ones:

$$\left(\frac{d}{dz} - ip\right) F_1 + ie^z(a - ib)F_2 = 0\,, \qquad \left(\frac{d}{dz} + ip\right) F_2 - ie^z(a + ib)F_1 = 0\,; \tag{3.10}$$

the corresponding wave functions are

$$\Psi^{\epsilon,a,b,\lambda} = e^{-i\epsilon t}\,e^{iax}\,e^{iby}e^{z}\begin{pmatrix} F_1(z)\\ F_2(z)\\ \lambda F_1(z)\\ \lambda F_2(z)\end{pmatrix}, \quad \lambda = \frac{\epsilon - p}{m},\; p = \pm\sqrt{\epsilon^2 - m^2}. \tag{3.11}$$

Let us detail the transition to Weyl neutrinos. In accordance with the spinor structure of the Dirac wave function, we have:

$$\psi(x) = \begin{pmatrix}\xi(x)\\ \eta(x)\end{pmatrix}, \quad \xi(x) = \begin{pmatrix}\xi^1(x)\\ \xi^2(x)\end{pmatrix}, \quad \eta(x) = \begin{pmatrix}\eta_{\dot 1}(x)\\ \eta_{\dot 2}(x)\end{pmatrix}, \tag{3.12}$$

and we obtain the corresponding substitutions for different Weyl 2-spinor, ξ (anti-neutrino) and η (neutrino):

$$\xi = e^{-i\epsilon t}\,e^{iax}\,e^{iby}e^{z}\,\begin{pmatrix}F_1(z)\\ F_2(z)\end{pmatrix}, \quad \eta = e^{-i\epsilon t}\,e^{iax}\,e^{iby}e^{z}\,\begin{pmatrix}F_3(z)\\ F_4(z)\end{pmatrix}. \tag{3.13}$$

Accordingly, we have two independent subsystems:

$$-i\epsilon F_3 - iae^z F_4 - be^z F_4 - \frac{d}{dz}F_3 = 0,\;\; -i\epsilon F_4 - iae^z F_3 + be^z F_3 + \frac{d}{dz}F_4 = 0\,; \tag{3.14}$$

$$-i\epsilon F_1 + iae^z F_2 + be^z F_2 + \frac{d}{dz}F_1 = 0,\;\; -i\epsilon F_2 + iae^z F_1 - be^z F_1 - \frac{d}{dz}F_2 = 0\,. \tag{3.15}$$

Helicity operator is diagonalized on these Weyl's subsystems in accordance with the following relations:

$$-i\epsilon F_3 - ipF_3 = 0\,, \;\; -i\epsilon F_4 - ipF_4F_2 = 0 \quad \Longrightarrow \quad p = -1\,; \tag{3.16}$$

$$-i\epsilon F_1 + ipF_1 = 0\,, \;\; -i\epsilon F_2 + ipF_2 = 0 \quad \Longrightarrow \quad p = +1\,; \tag{3.17}$$

neutrino and anti-neutrino are eigenfunctions of the helicity operator with opposite corresponding eigenvalues.

Now, let us consider the transition to Majorana particles. Decomposition of the Dirac waves into the sum of Majorana waves $\Psi = \Psi_+ + \Psi_-$ in spinor basis is given by the formulas

$$\Psi^c = \gamma^2\Psi^* = \begin{pmatrix}0 & -\sigma^2\\ \sigma^2 & 0\end{pmatrix}\begin{pmatrix}\xi^*\\ \eta^*\end{pmatrix} = \begin{pmatrix}-\sigma^2\eta^*\\ \sigma^2\xi^*\end{pmatrix}, \quad \Psi = \Psi_+ + \Psi_- = \frac{\Psi + \Psi^c}{2} + \frac{\Psi - \Psi^c}{2}$$

$$\Psi_+ = \begin{pmatrix}\xi_+ = (\xi - \sigma_2\eta^*)/2\\ \eta_+ = (\eta + \sigma_2\xi^*)/2\end{pmatrix}\,, \qquad \Psi_- = \begin{pmatrix}\xi_- = (\xi + \sigma_2\eta^*)/2\\ \eta_- = (\eta - \sigma_2\xi^*)/2\end{pmatrix}. \tag{3.18}$$

With the shortening notation

$$\xi = \varphi \begin{pmatrix} F_1(z) \\ F_2(z) \end{pmatrix}, \quad \eta = \varphi \begin{pmatrix} \lambda F_1(z) \\ \lambda F_2(z) \end{pmatrix}, \quad \xi^* = \varphi^* \begin{pmatrix} F_1^*(z) \\ F_2^*(z) \end{pmatrix}, \quad \eta^* = \varphi^* \begin{pmatrix} \lambda F_1^*(z) \\ \lambda F_2^*(z) \end{pmatrix},$$

where $\varphi = e^{-i\epsilon t} e^{iax} e^{iby} e^{z}$, $\varphi^* = (e^{-i\epsilon t} e^{iax} e^{iby} e^{z})^*$, we get the needed decompositions:

$$\Psi_+ = \begin{pmatrix} \varphi F_1 + i\varphi^* \lambda F_2^* \\ \varphi F_2 - i\varphi^* \lambda F_1^* \\ \varphi\lambda F_1 - i\varphi^* F_2^* \\ \varphi\lambda F_2 + i\varphi F_1^* \end{pmatrix}, \quad \Psi_- = \begin{pmatrix} \varphi F_1 - i\varphi^* \lambda F_2^* \\ \varphi F_2 + i\varphi^* \lambda F_1^* \\ \varphi\lambda F_1 + i\varphi^* F_2^* \\ \varphi\lambda F_2 - i\varphi^* F_1^* \end{pmatrix}.$$

These formulas for Majorana components refer to the spinor basis. The Majorana nature of these solution becomes most evident after translating the formulas to the Majorana basis by the rule

$$\Psi^M_{\pm} = \frac{1-\gamma^2}{\sqrt{2}} \Psi_{\pm} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -i \\ 0 & 1 & i & 0 \\ 0 & i & 1 & 0 \\ -i & 0 & 0 & 1 \end{pmatrix} \Psi_{\pm}\,. \tag{3.19}$$

In this way, we get

$$\Psi_+ = \begin{pmatrix} (\varphi F_1 + i\varphi^* \lambda F_2^*) - i(\varphi\lambda F_2 + i\varphi F_1^*) \\ (\varphi F_2 - i\varphi^* \lambda F_1^*) + i(\varphi\lambda F_1 - i\varphi^* F_2^*) \\ i(\varphi F_2 - i\varphi^* \lambda F_1^*) + (\varphi\lambda F_1 - i\varphi^* F_2^*) \\ -i(\varphi F_1 + i\varphi^* \lambda F_2^*) + (\varphi\lambda F_2 + i\varphi F_1^*) \end{pmatrix} = \begin{pmatrix} \text{Re}\ \varphi F_1 + \lambda\ \text{Im}\ \varphi F_2 \\ \text{Re}\ \varphi F_2 - \lambda\ \text{Im}\ \varphi F_1 \\ \lambda\ \text{Re}\ \varphi F_1 - \text{Im}\ \varphi F_2 \\ \lambda\ \text{Re}\ \varphi F_2 + \text{Im}\ \varphi F_1 \end{pmatrix},$$

$$\Psi_- = \begin{pmatrix} (\varphi F_1 - i\varphi^* \lambda F_2^*) - i(\varphi\lambda F_2 - i\varphi^* F_1^*) \\ (\varphi F_2 + i\varphi^* \lambda F_1^*) + i(\varphi\lambda F_1 + i\varphi^* F_2^*) \\ i(\varphi F_2 + i\varphi^* \lambda F_1^*) + (\varphi\lambda F_1 + i\varphi^* F_2^*) \\ -i(\varphi F_1 - i\varphi^* \lambda F_2^*) + (\varphi\lambda F_2 - i\varphi^* F_1^*) \end{pmatrix} = i \begin{pmatrix} \text{Im}\ \varphi F_1 - \lambda\ \text{Re}\ \varphi F_2 \\ \text{Im}\ \varphi F_2 + \lambda\ \text{Re}\ \varphi F_1 \\ \lambda\ \text{Im}\ \varphi F_1 + \text{Re}\ \varphi F_2 \\ \lambda\ \text{Im}\ \varphi F_2 - \text{Re}\ \varphi F_1 \end{pmatrix}. \tag{3.20}$$

So, it suffices to find solutions of the Dirac equation, and after that we can restrict ourselves to Majorana particles in accordance with the formulas (3.20).

4 Constructing and analyzing the Dirac solutions

Let us turn back to eqs. (3.10) and transform them in terms of the variable $Z = e^z$, $Z \in (0, +\infty)$:

$$\left(\frac{d}{dZ} - \frac{ip}{Z}\right) F_1 + i(a - ib) F_2 = 0\,, \quad \left(\frac{d}{dZ} + \frac{ip}{Z}\right) F_2 - i(a + ib) F_1 = 0\,. \tag{4.1}$$

These give two second order equations

$$\left(\frac{d^2}{dZ^2} + \frac{p^2 + ip}{Z^2} - a^2 - b^2\right) F_1 = 0\,, \quad \left(\frac{d^2}{dZ^2} + \frac{p^2 - ip}{Z^2} - a^2 - b^2\right) F_2 = 0\,. \tag{4.2}$$

We note the symmetry among them with respect to complex conjugation; for each solution of (4.1), its conjugate will be a solution as well:

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \qquad \begin{pmatrix} F_2^* \\ F_1^* \end{pmatrix}. \tag{4.3}$$

The equations (4.2) both have one regular point $Z = 0$ and one irregular one of rank 2 in $Z = \infty$ (at $a^2 + b^2 \neq 0$); this means that we have a confluent hypergeometric equation. Because the equations in (4.2) mutually relate by complex conjugation, it suffices to detail only one of them. For $F_1(Z)$, we use the following substitution: $F_1(Z) = Z^A e^{BZ} \bar{F}_1(Z)$, which yields

$$\bar{F}_1'' + (\frac{2A}{Z} + 2B)\bar{F}_1' + \frac{2AB}{Z}\bar{F}_1 + (\frac{A(A-1)}{Z^2} + \frac{p^2+ip}{Z^2})\bar{F}_1 + [B^2 - (a^2+b^2)]\bar{F}_1 = 0 \; .$$

Let us fix the parameters A, B as $A = +ip,\ 1-ip,\ B = \pm\sqrt{a^2+b^2}$; then we get

$$Z\bar{F}_1'' + (2A + 2BZ)\,\bar{F}_1' + 2AB\bar{F}_1 = 0 \; .$$

Without loss of generality, we consider the values $A = +ip,\ B = -\sqrt{a^2+b^2}$. Translating the last equation in terms of y: $2BZ = -y,\ y = +2\sqrt{a^2+b^2}e^z$, we obtain

$$y\frac{d^2}{dy^2}\bar{F}_1 + (2A - y)\frac{d}{dy}\bar{F}_1 - A\bar{F}_1 = 0 \; ,$$

which is a confluent hypergeometric equation

$$\Phi'' + (\gamma - y)\Phi' - \alpha\Phi = 0, \quad \alpha = A = ip, \quad \gamma = 2A = 2ip \; .$$

For two linearly independent solutions, we may take the following ones [1]:

$$\bar{F}_1^{(1)}(y) = \Phi(\alpha, \gamma; y) = \Phi(ip, 2ip; y) \; ,$$

$$\bar{F}_1^{(2)}(y) = y^{1-\gamma}\Phi(\alpha - \gamma + 1, 2 - \gamma; y) = y^{1-2ip}\Phi(1 - ip, 2 - 2ip; y) \; , \tag{4.4}$$

which provide us with two complete functions $F_1(Z) = Z^A e^{BZ}\bar{F}_1$:

$$F_1^{(1)} = y^{ip}e^{-y/2}\; \Phi(ip, 2ip; y) \; , \quad F_1^{(2)} = y^{1-ip}e^{-y/2}\; \Phi(1 - ip, 2 - 2ip; y) \; . \tag{4.5}$$

Applying the above mentioned symmetry, we obtain similar results for F_2:

$$F_2^{(1)} = y^{1+ip}e^{-y/2}\; \Phi(1 + ip, 2 + 2ip; y) \; , \qquad F_2^{(2)} = y^{-ip}e^{-y/2}\; \Phi(-ip, -2ip; y) \; . \tag{4.6}$$

It should be stressed that the pairwise coupling of the 4 functions from $\left\{F_1^{(1)}, F_1^{(2)};\ \ F_2^{(1)}, F_2^{(2)}\right\}$ can be reached only by using equations of first order (4.1).

We first state the answers and afterwards prove them. We shall further have two different alternatives depending on the sign of A $(+A, -A)$:

$$I^+. \qquad F_1^{+(1)} = e^{-y/2}y^A\;\Phi(A, 2A, y) = f \; ,$$
$$F_2^{+(1)} = Le^{-y/2}y^{1+A}\;\Phi(1 + A, 2 + 2A, y) = g \; ,$$

$$II^+. \qquad F_1^{+(2)} = L^*e^{-y/2}y^{1-A}\Phi(1 - A, 2 - 2A, y) = g^* \; ,$$
$$F_2^{+(2)} = e^{-y/2}y^{-A}\Phi(-A, -2A, y) = f^* \; ; \tag{4.7}$$

$A \Longrightarrow -A$

$$I^-. \qquad F_1^{-(1)} = e^{-y/2}y^{-A}\;\Phi(-A, -2A, y) = f^* \; ,$$
$$F_2^{-(1)} = L^*e^{-y/2}y^{1-A}\;\Phi(1 - A, 2 - 2A, y) = g^* \; ,$$

$$II^-. \qquad F_1^{-(2)} = Le^{-y/2}y^{1+A}\Phi(1 + A, 2 + 2A, y) = g \; ,$$
$$F_2^{-(2)} = e^{-y/2}y^A\Phi(A, 2A, y) = f \; . \tag{4.8}$$

We further determine the numerical coefficient L. The functions F_1 and F_2 obey the first order system

$$\left(\frac{d}{dy} - \frac{A}{y}\right) F_1 + \frac{e^{i\alpha}}{2} F_2 = 0 \, , \quad \left(\frac{d}{dy} + \frac{A}{y}\right) F_2 + \frac{e^{-i\alpha}}{2} F_1 = 0 \, ,$$

where

$$e^{i\alpha} = i\frac{a - ib}{\sqrt{a^2 + b^2}}, \quad e^{-i\alpha} = -i\frac{a + ib}{\sqrt{a^2 + b^2}} \, .$$

We shall prove that the needed pair is

$$F_1^{+(1)} = e^{-y/2} y^A \, \Phi(A, 2A, y) = f \, , \qquad F_2^{+(1)} = L e^{-y/2} y^{1+A} \, \Phi(1 + A, 2 + 2A, y) = g \, ;$$

$$F_2^{+(2)} = e^{-y/2} y^{-A} \Phi(-A, -2A, y) = f^*, \quad F_1^{+(2)} = L^* e^{-y/2} y^{1-A} \Phi(1 - A, 2 - 2A, y) = g^* \, .$$

In fact, it suffices to consider only the first pair. Substituting the two functions into the first equation, we get

$$\left(\frac{d}{dy} - \frac{A}{y}\right) e^{-y/2} y^A \, \Phi(A, 2A, y) + \frac{e^{i\alpha}}{2} L e^{-y/2} y^{1+A} \, \Phi(1 + A, 2 + 2A, y) = 0$$

and further,

$$-\Phi(A, 2A) + \Phi(A + 1, 2A + 1) + e^{i\alpha} L y \Phi(1 + A, 2 + 2A) = 0 \, .$$

By studying the first terms of the series, we readily get the identity

$$-\Phi(A, 2A) + \Phi(A + 1, 2A + 1) = \frac{x}{2(2A + 1)} \, \Phi(A + 1, 2A + 2) \, ,$$

and we obtain the needed relation

$$(4.9) \quad \frac{1}{2(2A + 1)} + e^{i\alpha} L = 0 \quad \Longrightarrow \quad L = -\frac{e^{-i\alpha}}{2(2A + 1)} = \frac{i}{2} \frac{1}{2A + 1} \frac{a + ib}{\sqrt{a^2 + b^2}} \, .$$

We shall describe now the asymptotic behavior of $f(z)$ and $g(z)$ for $z \to -\infty (y \to 0)$:

$$\begin{aligned} f \sim y^A &= \left(2\sqrt{a^2 + b^2}\right)^{ip} e^{ipz} \, , \\ f^* \sim y^{-A} &= \left(2\sqrt{a^2 + b^2}\right)^{-ip} e^{-ipz} \, ; \end{aligned} \qquad (4.10)$$

$$\begin{aligned} g \sim L y^{1+A} &= L \left(2\sqrt{a^2 + b^2}\right)^{1+ip} e^{(1+ip)z} \to 0 \, , \\ g^* \sim y^{-A} &= L^* \left(2\sqrt{a^2 + b^2}\right)^{1-ip} e^{(1-ip)z} \to 0 \, ; \end{aligned} \qquad (4.11)$$

Now, with the use of the known asymptotic formula

$$\Phi(\alpha, \gamma, x) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)} \, e^x (x)^{\alpha - \gamma} \, , \quad \text{Re} \, x \to +\infty,$$

we obtain the behavior of the functions for $z \to +\infty$ $(y =\to +\infty)$:

$$f \sim e^{-y/2} y^{ip} \, \frac{\Gamma(2ip)}{\Gamma(ip)} \, e^{y} y^{-ip} \to \infty \,, \quad f^{*} \to \infty \,;$$
$$g \sim L e^{-y/2} y^{1-ip} \, \frac{\Gamma(2ip+2)}{\Gamma(ip+1)} \, e^{y} y^{1-ip} \to \infty \,, \quad g^{*} \to \infty \,. \tag{4.12}$$

The last two relations mean that the constructed solutions f, g do not have the expected behavior in the region $z \to +\infty$, needed for interpreting them as being referred to the reflection effect.

In the above relations (4.7)–(4.8), we notice the evident symmetry (4.3). In fact, the solutions of the type $(-)$ are conjugate to those of the type $(+)$. The difference between the types $(+)$ and $(-)$ is associated with the two polarization states of the Dirac particle (helicity operator). By considering the function F_1 as the *main* one, we will construct the needed solutions for this main function, and then we shall find their counterparts F_2.

5 Reflection effect for the Dirac field

We have used above a definite pair of linearly independent solutions of the confluent hypergeometric equation (with two values for A: $+A, -A$)

$$Y^{+(1)} = \Phi(A, 2A, y) \,, \quad Y^{+(2)} = y^{1-2A} \Phi(1-A, 2-2A, y) \,;$$
$$Y^{-(1)} = \Phi(-A, -2A, y) \,, \quad Y^{-(2)} = y^{1+2A} \Phi(1+A, 2+2A, y) \,. \tag{5.1}$$

To construct solutions with the needed asymptotic behavior, we employ other two solutions [1] (again with different A: $+A, -A$):

$$Y^{+(5)} = \Psi(A, 2A, y) \,, \quad Y^{+(7)} = e^{y} \Psi(A, 2A, -y) \,;$$
$$Y^{-(5)} = \Psi(-A, -2A, y) \,, \quad Y^{-(7)} = e^{y} \Psi(-A, -2A, -y) \,. \tag{5.2}$$

These sets of solutions (5.1)–(5.2) relate to each other by the Kummer formulas [1]:

$$Y^{+(5)} = \frac{\Gamma(1-2A)}{\Gamma(1-A)} Y^{+(1)} + \frac{\Gamma(2A-1)}{\Gamma(A)} Y^{+(2)},$$
$$Y^{+(7)} = \frac{\Gamma(1-2A)}{\Gamma(1-A)} Y^{+(1)} - \frac{\Gamma(2A-1)}{\Gamma(A)} Y^{+(2)};$$
$$Y^{-(5)} = \frac{\Gamma(1+2A)}{\Gamma(1+A)} Y^{-(1)} + \frac{\Gamma(-2A-1)}{\Gamma(-A)} Y^{-(2)},$$
$$Y^{-(7)} = \frac{\Gamma(1+2A)}{\Gamma(1+A)} \, Y^{-(1)} - \frac{\Gamma(-2A-1)}{\Gamma(-A)} Y^{-(2)};$$

after multiplying them with $y^{A} e^{-y/2}$ (and respectively with $y^{-A} e^{-y/2}$), we get

$$F_1^{+(5)} = \frac{\Gamma(1-2A)}{\Gamma(1-A)} f + \frac{\Gamma(2A-1)}{\Gamma(A)} \, \frac{1}{L^{*}} g^{*}, \; F_1^{+(7)} = \frac{\Gamma(1-2A)}{\Gamma(1-A)} f - \frac{\Gamma(2A-1)}{\Gamma(A)} \, \frac{1}{L^{*}} g^{*}; \tag{5.3}$$

$$F_1^{-(5)} = \frac{\Gamma(1+2A)}{\Gamma(1+A)} f^* + \frac{\Gamma(-2A-1)}{\Gamma(-A)} \frac{1}{L} g, \; F_1^{-(7)} = \frac{\Gamma(1+2A)}{\Gamma(1+A)} f^* - \frac{\Gamma(-2A-1)}{\Gamma(-A)} \frac{1}{L} g; \tag{5.4}$$

The functions $F_1^{+(5)}$ and $F_1^{+(7)}$ behave for $z \longrightarrow -\infty$ as follows:

$$F_1^{+(5)} = \frac{\Gamma(1-2A)}{\Gamma(1-A)} \left(2\sqrt{a^2+b^2}\right)^{+ip} e^{+ipz},$$
$$F_1^{+(7)} = \frac{\Gamma(1-2A)}{\Gamma(1-A)} \left(2\sqrt{a^2+b^2}\right)^{+ip} e^{+ipz}\,, \tag{5.5}$$

while the functions $F_1^{-(5)}$ and $F_1^{-(7)}$ behave for $z \longrightarrow -\infty$ as

$$F_1^{-(5)} = \frac{\Gamma(1+2A)}{\Gamma(1+A)} \left(2\sqrt{a^2+b^2}\right)^{-ip} e^{-ipz},$$
$$F_1^{-(7)} = \frac{\Gamma(1+2A)}{\Gamma(1+A)} \left(2\sqrt{a^2+b^2}\right)^{-ip} e^{-ipz}. \tag{5.6}$$

Now let us find the asymptotic behavior of $F_1^{\pm(5)} z$ at $z \to +\infty$. Applying the known formulas [1]: $Y_5 = \Psi(A, 2A, y) \sim y^{-A}$, we get (for $y \to +\infty,\ z \to +\infty$)

$$F_1^{+(5)} = y^A e^{-y/2}\, y^{-A} \sim e^{-y/2} \sim \exp\left(-\sqrt{a^2+b^2} e^z\right) \longrightarrow 0\,,$$
$$F_1^{-(5)} = y^{-A} e^{-y/2}\, y^{+A} \sim e^{-y/2} \sim \exp\left(-\sqrt{a^2+b^2} e^z\right) \longrightarrow 0\,. \tag{5.7}$$

Analogously, applying the formula $Y_7 = e^y \Psi(A, 2A, -y) \sim e^y y^{-A}$, we infer for $y \to +\infty$ $(z \to +\infty)$ that:

$$F_1^{+(7)} \sim y^A e^{-y/2}\, e^y y^{-A} \sim e^{+y/2} \sim \exp\left(+\sqrt{a^2+b^2} e^z\right) \longrightarrow \infty\,,$$
$$F_1^{-(7)} \sim y^{-A} e^{-y/2}\, e^y y^{+A} \sim e^{+y/2} \sim \exp\left(+\sqrt{a^2+b^2} e^z\right) \longrightarrow \infty\,. \tag{5.8}$$

The most interesting are the solutions of the type $\pm(5)$, since they tend to zero for $z \to$, whereas they behave as flat waves for $z \to -\infty$.

Let us define the new combination of the solutions $F_1^{+(5)}$ and $F_1^{-(5)}$:

$$H_1 = \left(2\sqrt{a^2+b^2}\right)^{-ip} F_1^{+(5)} + \left(2\sqrt{a^2+b^2}\right)^{+ip} F_1^{-(5)}\,, \quad H^* = H\,;$$
$$G_1 = \left(2\sqrt{a^2+b^2}\right)^{-ip} F_1^{+(5)} - \left(2\sqrt{a^2+b^2}\right)^{+ip} F_1^{-(5)}\,, \quad G^* = -G\,. \tag{5.9}$$

These behave as follows

$$H_1(z \to -\infty) \sim \frac{\Gamma(1-2A)}{\Gamma(1-A)}\, e^{+ipz} + \frac{\Gamma(1+2A)}{\Gamma(1+A)}\, e^{-ipz}\,, \quad H_1(z \to +\infty) \sim 0\,;$$
$$G_1(z \to -\infty) \sim \frac{\Gamma(1-2A)}{\Gamma(1-A)}\, e^{+ipz} - \frac{\Gamma(1+2A)}{\Gamma(1+A)}\, e^{-ipz}\,, \quad G_1(z \to +\infty) \sim 0\,. \tag{5.10}$$

For such solutions we can define the reflection coefficient

$$\psi \sim M_- e^{-ipz} \pm M_+ e^{+ipz}, \qquad R = \left|\frac{M_-}{M_+}\right|^2 = \left|\frac{\Gamma(1+2A)}{\Gamma(1-2A)} \frac{\Gamma(1-A)}{\Gamma(1+A)}\right|^2 = 1. \tag{5.11}$$

We remind that $A = +ip,\ A^* = -A$. With the notation

$$\frac{\Gamma(1-2A)}{\Gamma(1-A)} = \rho + i\sigma, \qquad \frac{\Gamma(1+2A)}{\Gamma(1+A)} = \rho - i\sigma, \tag{5.12}$$

the asymptotic behavior of two standing waves is given by the formulas

$$\begin{aligned} H_1(z \to -\infty) &= 2(\rho \cos pz - \sigma \sin pz)\ , \\ G_1(z \to -\infty) &= 2i(\sigma \cos pz + \rho \sin pz)\ , \end{aligned} \tag{5.13}$$

where the first is real, and the second is imaginary. The various choices of complex (and conjugate) coefficients when constructing the functions F, G in (5.9) influence only the total amplitude of the standing waves and their phase shifts.

Note that the direct interpretation of the effect in terms of 'barrier– reflection' is difficult, because in (4.2) we note the complex potentials:

$$\left(\frac{d^2}{dZ^2} + \frac{p^2+ip}{Z^2} - (a^2+b^2)\right) F_1 = 0\ , \quad \left(\frac{d^2}{dZ^2} + \frac{p^2-ip}{Z^2} - (a^2+b^2)\right) F_2 = 0\ . \tag{5.14}$$

The structure of these equations assumes the relation $F_2 = F_1^*$. With this fact in mind, we can derive for the functions with the structure

$$H = cF_1 + c^* F_1^*,\ G = cF_1 - c^* F_1^*$$

subsequent equations which will contain only real-valued potentials. Indeed, we readily get:

$$\begin{aligned} \left(\frac{d^2}{dZ^2} + \frac{p^2}{Z^2} - (a^2+b^2)\right) H + \frac{ip}{Z^2} G = 0\ , \\ \left(\frac{d^2}{dZ^2} + \frac{p^2}{Z^2} - (a^2+b^2)\right) G + \frac{ip}{Z^2} H = 0\ . \end{aligned} \tag{5.15}$$

In terms of the coordinate $z = \ln Z$ they read[2]:

$$\begin{aligned} \left(\frac{d^2}{dz^2} + p^2 - \frac{1}{4} - (a^2+b^2)e^{2z}\right) H + ipG = 0\ , \\ \left(\frac{d^2}{dz^2} + p^2 - \frac{1}{4} - (a^2+b^2)e^{2z}\right) G + ipH = 0\ . \end{aligned} \tag{5.16}$$

We easily find the critical point z_0, so that for $z > z_0$ the function should fall dawn to zero:

$$p^2 - 1/4 = (k_1^2 + k_2^2)\ e^{2z_0} \quad \Longrightarrow \quad z_0 = \ln \sqrt{\frac{p^2 - 1/4}{a^2+b^2}}\ . \tag{5.17}$$

In the neighborhood of this point z_0, the equations become simpler:

$$\frac{d^2}{dz^2} H + ipG = 0\ , \quad \frac{d^2}{dz^2} G + ipH = 0\ . \tag{5.18}$$

[2] In order to eliminate the term corresponding to the first derivative, we separate the special multiplier: $H \Longrightarrow e^{z/2} H,\ G \Longrightarrow e^{z/2} G$.

Their solutions may have only exponential form $H = Me^{\nu(z-z_0)}$, $G = Ne^{\nu(z-z_0)}$; we further obtain the algebraic equations with the four solutions

$$\begin{pmatrix} \nu^2 & ip \\ ip & \nu^2 \end{pmatrix} \begin{pmatrix} M \\ N \end{pmatrix} = 0 \quad \Longrightarrow \quad \nu = -\frac{1 \pm i}{\sqrt{2}}\sqrt{p}, \; +\frac{1 \pm i}{\sqrt{2}}\sqrt{p} \, . \tag{5.19}$$

Two first ones with negative real parts are what we need. Note that in usual units, the critical point z_0 is given by

$$z_0 = \rho \, \ln \sqrt{\frac{(E^2 - M^2c^4)/c^2\hbar^2 - 1/4\rho^2}{(K_1^2 + K_2^2)}} \; ; \tag{5.20}$$

where K_1, K_2 are the wave numbers; here ρ stands for the curvature radius of the Lobachevsky space. Note that when K_1, K_2 approach zero, the depth of the penetrating z_0 approaches infinity.

6 Additional calculations

Having constructed the needed main functions – the solutions of equations for F_1, let us find the form of the relevant functions – solutions of the equation for F_2. To this end, let us turn back to the solutions:

$$H_1 = \left(2\sqrt{a^2+b^2}\right)^{-ip} F_1^{+(5)} + \left(2\sqrt{a^2+b^2}\right)^{+ip} F_1^{-(5)} = CF_1^{+(5)} + C^* F_1^{-(5)} \, ,$$
$$G_1 = \left(2\sqrt{a^2+b^2}\right)^{-ip} F_1^{+(5)} - \left(2\sqrt{a^2+b^2}\right)^{+ip} F_1^{-(5)} = CF_1^{+(5)} - C^* F_1^{-(5)} \, . \tag{6.1}$$

By substituting the expressions for $F_1^{\pm(5)}$, we obtain

$$H_1 = C\left(\frac{\Gamma(1-2A)}{\Gamma(1-A)} f + \frac{\Gamma(2A-1)}{\Gamma(A)} \frac{1}{L^*} g^*\right) + C^*\left(\frac{\Gamma(1+2A)}{\Gamma(1+A)} f^* + \frac{\Gamma(-2A-1)}{\Gamma(-A)} \frac{1}{L} g\right),$$
$$G_1 = C\left(\frac{\Gamma(1-2A)}{\Gamma(1-A)} f + \frac{\Gamma(2A-1)}{\Gamma(A)} \frac{1}{L^*} g^*\right) - C^*\left(\frac{\Gamma(1+2A)}{\Gamma(1+A)} f^* + \frac{\Gamma(-2A-1)}{\Gamma(-A)} \frac{1}{L} g\right)$$

(6.2)

Now, for these functions H_1, G_1 we may find relevant H_2, G_2. Since we deal with a linear task, we have:

$$H_1 \Longrightarrow H_2, \qquad \left(\frac{d}{dz} - ip\right) H_1 + ie^z(a - ib)H_2 = 0 \, ;$$
$$G_1 \Longrightarrow G_2, \qquad \left(\frac{d}{dz} - ip\right) G_1 + ie^z(a - ib)G_2 = 0 \, ,$$

$$F_1^{+(1)} = f \quad \Longrightarrow \quad F_2^{+(1)} = g \, , \quad F_1^{+(2)} = g^* \quad \Longrightarrow \quad F_2^{+(2)} = f^* \, ,$$
$$F_1^{-(1)} = f^* \quad \Longrightarrow \quad F_2^{-(1)} = g^* \, , \quad F_1^{-(2)} = g \quad \Longrightarrow \quad F_2^{-(2)} = f \, .$$

Hence, we get

$$H_2 = C\left(\frac{\Gamma(1-2A)}{\Gamma(1-A)}g + \frac{\Gamma(2A-1)}{\Gamma(A)}\frac{1}{L^*}f^*\right) + C^*\left(\frac{\Gamma(1+2A)}{\Gamma(1+A)}g^* + \frac{\Gamma(-2A-1)}{\Gamma(-A)}\frac{1}{L}f\right),$$

$$G_2 = C\left(\frac{\Gamma(1-2A)}{\Gamma(1-A)}g + \frac{\Gamma(2A-1)}{\Gamma(A)}\frac{1}{L^*}f^*\right) - C^*\left(\frac{\Gamma(1+2A)}{\Gamma(1+A)}g^* + \frac{\Gamma(-2A-1)}{\Gamma(-A)}\frac{1}{L}f\right).$$

In fact, the transition from (6.2) to (**??**) is performed by the change $f \Longleftrightarrow g$. Assuming the asymptotic behavior of f, g for $z \to -\infty$):

$$\text{(6.3)}\quad f \sim y^A = \left(2\sqrt{a^2+b^2}\right)^{ip} e^{ipz}\ , \quad f^* \sim y^{-A} = \left(2\sqrt{a^2+b^2}\right)^{-ip} e^{-ipz}\ ;$$

$$g \sim Ly^{1+A} = L\left(2\sqrt{a^2+b^2}\right)^{1+ip} e^{(1+ip)z} \to 0\ ,$$

$$\text{(6.4)}\qquad g^* \sim y^{-A} = L^*\left(2\sqrt{a^2+b^2}\right)^{1-ip} e^{(1-ip)z} \to 0\ ;$$

we establish the behavior of H_2, G_2:

$$\text{(6.5)}\qquad H_2(G_2) = \frac{C^2}{L^*}\frac{\Gamma(2A-1)}{\Gamma(A)}\ e^{-ipz} + (-)\frac{C^{*2}}{L}\frac{\Gamma(-2A-1)}{\Gamma(-A)}\ e^{+ipz}\ .$$

Here, we remark the existence of standing waves of two types:

$$\text{(6.6)}\quad H_2(z\to-\infty) = 2(\rho'\cos pz - \sigma'\sin pz)\ ,\ \ G_2(z\to-\infty) = 2i(\sigma'\cos pz + \rho'\sin pz)\ .$$

7 The Majorana case

To get the results for the Majorana particles, we start with the formulas (3.20)

$$\text{(7.1)}\qquad \Psi_+ = \begin{pmatrix} \text{Re}\ \varphi F_1 + \lambda\ \text{Im}\ \varphi F_2 \\ \text{Re}\ \varphi F_2 - \lambda\ \text{Im}\ \varphi F_1 \\ \lambda\ \text{Re}\ \varphi F_1 - \text{Im}\ \varphi F_2 \\ \lambda\ \text{Re}\ \varphi F_2 + \text{Im}\ \varphi F_1 \end{pmatrix}, \quad \Psi_- = i\begin{pmatrix} \text{Im}\ \varphi F_1 - \lambda\ \text{Re}\ \varphi F_2 \\ \text{Im}\ \varphi F_2 + \lambda\ \text{Re}\ \varphi F_1 \\ \lambda\ \text{Im}\ \varphi F_1 + \text{Re}\ \varphi F_2 \\ \lambda\ \text{Im}\ \varphi F_2 - \text{Re}\ \varphi F_1 \end{pmatrix},$$

in which we should consider the two different solutions (H_1, H_2) and (G_1, G_2):

$$\text{(7.2)}\qquad \begin{aligned} F_1 = H_1 &= C\left(\tfrac{\Gamma(1-2A)}{\Gamma(1-A)}f + \tfrac{\Gamma(2A-1)}{\Gamma(A)}\tfrac{1}{L^*}g^*\right) + C^*\left(\tfrac{\Gamma(1+2A)}{\Gamma(1+A)}f^* + \tfrac{\Gamma(-2A-1)}{\Gamma(-A)}\tfrac{1}{L}g\right),\\ F_2 = H_2 &= C\left(\tfrac{\Gamma(1-2A)}{\Gamma(1-A)}g + \tfrac{\Gamma(2A-1)}{\Gamma(A)}\tfrac{1}{L^*}f^*\right) + C^*\left(\tfrac{\Gamma(1+2A)}{\Gamma(1+A)}g^* + \tfrac{\Gamma(-2A-1)}{\Gamma(-A)}\tfrac{1}{L}f\right); \end{aligned}$$

$$\text{(7.3)}\qquad \begin{aligned} F_1 = G_1 &= C\left(\tfrac{\Gamma(1-2A)}{\Gamma(1-A)}f + \tfrac{\Gamma(2A-1)}{\Gamma(A)}\tfrac{1}{L^*}g^*\right) - C^*\left(\tfrac{\Gamma(1+2A)}{\Gamma(1+A)}f^* + \tfrac{\Gamma(-2A-1)}{\Gamma(-A)}\tfrac{1}{L}g\right),\\ F_2 = G_2 &= C\left(\tfrac{\Gamma(1-2A)}{\Gamma(1-A)}g + \tfrac{\Gamma(2A-1)}{\Gamma(A)}\tfrac{1}{L^*}f^*\right) - C^*\left(\tfrac{\Gamma(1+2A)}{\Gamma(1+A)}g^* + \tfrac{\Gamma(-2A-1)}{\Gamma(-A)}\tfrac{1}{L}f\right). \end{aligned}$$

We will present these two solutions in short form:

$$(7.4)\quad \begin{cases} F_1 = H_1 = (\alpha f + \alpha^* f^*) + (\gamma^* g^* + \gamma g) = 2\text{Re}(\alpha f + \gamma g)\ , & F_1^* = F_1\ , \\ F_2 = H_2 = (\alpha g + \alpha^* g^*) + (\gamma^* f^* + \gamma f) = 2\text{Re}(\alpha g + \gamma f)\ , & F_2^* = F_2\ ; \end{cases}$$

$$(7.5)\quad \begin{cases} F_1 = G_1 = (\alpha f - \alpha^* f^*) + (\gamma^* g^* - \gamma g) = 2i\ \text{Im}(\alpha f - \gamma g)\ , & F_1^* = -F_1\ , \\ F_2 = G_2 = (\alpha g - \alpha^* g^*) + (\gamma^* f^* - \gamma f) = 2i\ \text{Im}(\alpha g - \gamma f)\ , & F_2^* = -F_2\ . \end{cases}$$

Considering (7.4), we find the expressions of the Majorana solutions $\Psi_\pm$ as related to (H_1, H_2):

$$\Psi_+ = \begin{pmatrix} \text{Re}\ \varphi\times\text{Re}(\alpha f+\gamma g)+\lambda\ \text{Im}\ \varphi\times\text{Re}(\alpha g+\gamma f) \\ \text{Re}\ \varphi\times\text{Re}(\alpha g+\gamma f)-\lambda\ \text{Im}\ \varphi\times\text{Re}(\alpha f+\gamma g) \\ \lambda\ \text{Re}\ \varphi\times\text{Re}(\alpha f+\gamma g)-\text{Im}\ \varphi\times\text{Re}(\alpha g+\gamma f) \\ \lambda\ \text{Re}\ \varphi\times\text{Re}(\alpha g+\gamma f)+\text{Im}\ \varphi\times\text{Re}\ \ \varphi(\alpha f+\gamma g) \end{pmatrix},$$

$$(7.6)\qquad \Psi_- = i\begin{pmatrix} \text{Im}\ \varphi\times\text{Re}(\alpha f+\gamma g)-\lambda\ \text{Re}\ \varphi\times\text{Re}(\alpha g+\gamma f) \\ \text{Im}\ \varphi\times\text{Re}(\alpha g+\gamma f)+\lambda\ \text{Re}\varphi\times\text{Re}(\alpha f+\gamma g) \\ \lambda\ \text{Im}\ \varphi\times\text{Re}(\alpha f+\gamma g)+\text{Re}\ \varphi\times\text{Re}(\alpha g+\gamma f) \\ \lambda\ \text{Im}\ \varphi\times\text{Re}(\alpha g+\gamma f)-\text{Re}\ \varphi\times\text{Re}(\alpha f+\gamma g) \end{pmatrix}.$$

In the same manner, with the use of (7.5) we find expressions for Majorana solutions $\Psi_\pm$ as related to (G_1, G_2):

$$\Psi_+ = \begin{pmatrix} \text{Re}\ i\varphi\times\text{Im}(\alpha f-\gamma g)+\lambda\ \text{Im}\ i\varphi\times\text{Im}(\alpha g-\gamma f) \\ \text{Re}\ i\varphi\times\text{Im}(\alpha g-\gamma f)-\lambda\ \text{Im}\ i\varphi\times\text{Im}(\alpha f-\gamma g) \\ \lambda\ \text{Re}\ i\varphi\times\text{Im}(\alpha f-\gamma g)-\text{Im}\ i\varphi\times\text{Im}(\alpha g-\gamma f) \\ \lambda\ \text{Re}\ i\varphi\times\text{Im}(\alpha g-\gamma f)+\text{Im}\ i\varphi\times\text{Im}(\alpha f-\gamma g), \end{pmatrix},$$

$$(7.7)\qquad \Psi_- = i\begin{pmatrix} \text{Im}\ i\varphi\times\text{Im}(\alpha f-\gamma g)-\lambda\ \text{Re}\ i\varphi\times\text{Im}(\alpha g-\gamma f) \\ \text{Im}\ i\varphi\times\text{Im}(\alpha g-\gamma f)+\lambda\ \text{Re}\ i\varphi\times\text{Im}(\alpha f-\gamma g) \\ \lambda\ \text{Im}\ i\varphi\times\text{Im}(\alpha f-\gamma g)+\text{Re}\ i\varphi\times\text{Im}(\alpha g-\gamma f) \\ \lambda\ \text{Im}\ i\varphi\times\text{Im}(\alpha g-\gamma f)-\text{Re}\ i\varphi\times\text{Im}(\alpha f-\gamma g) \end{pmatrix}.$$

To get the asymptotic at $z \to -\infty$, we shoul take into consideratuin that non-is only from function $f(z)$:

$$g(z), g^*(z) - -- > 0\ , \quad f(z) - -- > e^{-ipz}\ .$$

So, the asymptotics for Majorana solutions related to (H_1, H_2) are

$$\Psi_+ = \begin{pmatrix} \text{Re}\ \varphi\times\text{Re}\ \alpha e^{-ipz}+\lambda\ \text{Im}\ \varphi\times\text{Re}\ \gamma e^{-ipz} \\ \text{Re}\ \varphi\times\text{Re}\ \gamma e^{-ipz}-\lambda\ \text{Im}\ \varphi\times\text{Re}\alpha e^{-ipz} \\ \lambda\ \text{Re}\ \varphi\times\text{Re}\ \alpha e^{-ipz}-\text{Im}\ \varphi\times\text{Re}\ \gamma e^{-ipz} \\ \lambda\ \text{Re}\ \times\text{Re}\ \gamma e^{-ipz}+\text{Im}\ \varphi\times\text{Re}\ \alpha e^{-ipz}, \end{pmatrix},$$

$$(7.8)\qquad \Psi_- = i\begin{pmatrix} \text{Im}\ \varphi\times\text{Re}\ \alpha e^{-ipz}-\lambda\ \text{Re}\ \varphi\times\text{Re}\gamma e^{-ipz} \\ \text{Im}\ \varphi\times\text{Re}\ \gamma e^{-ipz}+\lambda\ \text{Re}\varphi\times\text{Re}\ \alpha e^{-ipz} \\ \lambda\ \text{Im}\ \varphi\times\text{Re}\ \alpha e^{-ipz}+\text{Re}\ \varphi\times\text{Re}\ \gamma e^{-ipz} \\ \lambda\ \text{Im}\ \varphi\times\text{Re}\ \gamma e^{-ipz}-\text{Re}\ \varphi\times\text{Re}\ \alpha e^{-ipz} \end{pmatrix};$$

the asymptotics for Majorana solutions related to (G_1, G_2) are

$$\Psi_+ = \begin{pmatrix} \text{Re}\ i\varphi\times\text{Im}\ \alpha e^{-ipz}-\lambda\ \text{Im}\ i\varphi\times\text{Im}\gamma e^{-ipz} \\ -\text{Re}\ i\varphi\times\text{Im}\gamma e^{-ipz}-\lambda\ \text{Im}\ i\varphi\times\text{Im}\ \alpha e^{-ipz} \\ \lambda\ \text{Re}\ i\varphi\times\text{Im}\ \alpha e^{-ipz}+\text{Im}\ i\varphi\times\text{Im}\gamma e^{-ipz} \\ -\lambda\ \text{Re}\ i\varphi\times\text{Im}\gamma e^{-ipz}+\text{Im}\ i\varphi\times\text{Im}\ \alpha e^{-ipz} \end{pmatrix},$$

$$(7.9)\qquad \Psi_- = i\begin{pmatrix} \text{Im}\ i\varphi\times\text{Im}\ \alpha e^{-ipz}+\lambda\ \text{Re}\ i\varphi\times\text{Im}\gamma e^{-ipz} \\ -\text{Im}\ i\varphi\times\text{Im}\gamma e^{-ipz}+\lambda\ \text{Re}\ i\varphi\times\text{Im}\ \alpha e^{-ipz} \\ \lambda\ \text{Im}\ i\varphi\times\text{Im}\ \alpha e^{-ipz}-\text{Re}\ i\varphi\times\text{Im}\gamma e^{-ipz} \\ -\lambda\ \text{Im}\ i\varphi\times\text{Im}\gamma e^{-ipz}-\text{Re}\ i\varphi\times\text{Im}\ \alpha e^{-ipz} \end{pmatrix}.$$

We are able now to write down the general structure of elementary blocks which enter the formulas (7.8)–(7.9):

$$\varphi = e^{i(-\epsilon t + ax + by)} = e^{i\Delta}\ , \quad \text{Re}\,\varphi = \cos\Delta\ , \quad \text{Im}\ \ \varphi = \sin\Delta\ ,$$

$$i\varphi, \quad \text{Re}\, i\varphi = \cos(\Delta + \pi/2)\ , \quad \text{Im}\ \ i\varphi = \sin(\Delta + \pi/2)\ ,$$

$$\text{Re}\,\alpha e^{-ipz} \sim ...\cos(pz + ...)\ , \quad \text{Im}\ \alpha e^{-ipz} \sim ...\sin(pz + ...)\ ,$$

$$\text{Re}\,\gamma e^{-ipz} \sim ...\cos(pz + ...)\ , \quad \text{Im}\ \gamma e^{-ipz} \sim ...\sin(pz + ...)\ .,$$

All terms have a similar general structure:

$$2\sin(\Delta + ..)\cos(pz + ...) = \sin(\Delta + pz + ...) + \sin(\Delta - pz + ...)\ ,$$

$$2\cos(\Delta + ..)\cos(pz + ...) = \cos(\Delta + pz + ...) + \cos(\Delta - pz + ...)\ ,$$

$$2\sin(\Delta + ..)\sin(pz + ...) = \cos(\Delta - pz + ...) - \cos(\Delta + pz + ...)\ ,$$

$$2\cos(\Delta + ..)\sin(pz + ...) = \sin(\Delta + pz + ...) - \sin(\Delta - pz + ...)\ .$$

Thus, for a Majorana particle, for fixed (x, y) in $\Delta(t, x, y)$ we have standing waves (superpositions of two running waves in the variable z). In other words, for Majorana particle, we get the effect of complete reflection on effective barrier generated be Lobachevsky geometry. Also, it should be noted that Majorana component-solutions for the solutions H_1, H_1 and G_1, G_2 represent standing waves (being real or imaginary) in the whole region of the variable z (not only for $z \to -\infty$).

8 Weyl particles

Finally, we shall discuss the effects of Lobachevsky geometry on Weyl 2-component fields. Recall the equations for the Weyl anti-neutrino

(8.1) $$\left(\frac{d}{dz} - i\epsilon\right) F_1 + ie^z(a - ib)F_2 = 0\ , \quad \left(\frac{d}{dz} + i\epsilon\right) F_2 - ie^z(a + ib)F_1 = 0\ ;$$

and neutrino

(8.2) $$\left(\frac{d}{dz} + i\epsilon\right) F_3 + ie^z(a - ib)F_4 = 0\ , \quad \left(\frac{d}{dz} - i\epsilon\right) F_4 - ie^z(a + ib)F_3 = 0\ .$$

Both subsystems are symmetric under complex conjugation: they have as solutions the pairs

(8.3) $$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \quad \begin{pmatrix} F_2^* \\ F_1^* \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} F_3 \\ F_4 \end{pmatrix}, \quad \begin{pmatrix} F_4^* \\ F_3^* \end{pmatrix},$$

respectively. We may have used the known results for the system (3.10),

(8.4) $$\left(\frac{d}{dz} - ip\right) F_1 + ie^z(a - ib)F_2 = 0\ , \left(\frac{d}{dz} + ip\right) F_2 - ie^z(a + ib)F_1 = 0\ .$$

Recall that $p = \pm\sqrt{\epsilon^2 - m^2}$, so at $m = 0$ we have $p = -\epsilon,\ +\epsilon$. Therefore, when $p = -\epsilon$, eqs. (8.4) coincide with (8.2), and refer to neutrino; at $p = +\epsilon$, (8.4) give eqs. (8.1), and refer to anti-neutrino. However, the main issue is that in the Weyl case, it is forbidden to combine the solutions of (8.2) and (8.1). Therefore, only a part of results obtained for the Dirac particle can be preserved in the Weyl case.

For definiteness, we consider below the anti-neutrino (that is $p = +\epsilon$; to get the neutrino case, it suffices to make the formal change $p \Longrightarrow -p = -\epsilon$). We have the two second order equations

$$(8.5) \quad \left(\frac{d^2}{dZ^2} + \frac{p^2 + ip}{Z^2} - a^2 - b^2 \right) F_1 = 0 \,, \quad \left(\frac{d^2}{dZ^2} + \frac{p^2 - ip}{Z^2} - a^2 - b^2 \right) F_2 = 0$$

which have the following asymptotic behavior:

$$z \to -\infty, \quad F_1 \sim e^{ipz}, \; e^{(1-ip)z} \to 0, \quad F_2 \sim e^{-ipz}, \; e^{(1+ip)z} \to 0, \quad ;$$

$$z \to +\infty, \quad F_1 \sim e^{\pm\sqrt{a^2+b^2} e^z}, \quad F_2 \sim e^{\pm\sqrt{a^2+b^2} e^z} \; .$$

We see that when we cannot use solutions with opposite values of p, it is impossible to construct solutions which refer to the reflecting process (since this concerns both F_1 and F_2). In other words, the Weyl fields cardinally differ from Maxwell, Dirac or Majorana cases: for Weyl fermions, the reflecting effect vanishes. A general conclusion may be drawn, that the effects of non-Euclidean geometry can substantially depend on the type of fermion.

9 Conclusions

Previously it was shown that, in electrodynamic context, the Lobachevsky geometry can simulate an effective medium acting as an ideal mirror, perpendicularly oriented towards the axes. In the present paper, an analogue of that effect is investigated for spin 1/2 field. In explicit form, the solutions of the Dirac equation are constructed, and shown to describe waves in the space reflected from the effective potential barrier without penetrating it. The depth of penetration into the medium is determined by characteristics of the quantum states and by the curvature radius of the Lobachevsky space; for waves with $k_1 = 0, k_2 = 0$, the effective reflecting barrier vanishes. The results are valid for Majorana fermions as well, and some relevant details are specified. It is shown that for Weyl fermions, the reflecting effect vanishes. The drawn general conclusion is that the effects of non-Euclidean geometry can substantially depend on the type of fermion.

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