Spectral Cartan properties in Randers-type spaces

Jelena Stojanov, Vladimir Balan

Abstract. The classical algebraic eigenproblem is extended for a third-order directionally dependent tensor field over a linearly deformed Euclidean space endowed with a Finsler structure of Randers type. The $Z-$ and the $H-$ eigenvalues of the associated Cartan tensor are discussed. It is emphasized that the $Z-$eigendata produce global information, while the $H-$eigendata exhibit a strong local character.

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Key words: Eigenvalues; $Z-$eigenproblem; $H-$eigenproblem; Finsler structure; Randers structure; Cartan tensor.

1 Introduction

Higher order tensor spectral analysis [13, 18] is a natural extension from endomorphisms to multilinear (tensor) algebra [15, 20, 25] and addresses topics as the extension of the singular value decomposition, lower rank approximation, and dimensionality reduction [23].

The present work deals with the spectral theory for a collection of directionally dependent third order tensors in non-Euclidean space. The main tools to be used are the multilinear spectral algebra and tensor field analysis on manifolds. The $Z-$ and $H-$eigenproblems will be stated for the Cartan tensor field over an anisotropic differentiable manifold, and the related issues will be addressed means of spectral theory for higher order symmetric tensors [13, 14].

The paper structure is centered on the following topics: geometrical structures and tensors, higher order tensor spectral analysis and its adjustments for non-Euclidean manifolds, and finally an illustrative case - the eigenproblems for the Cartan tensor of the oncologic Garner-Finsler model of locally Minkowski type.

2 Preliminaries

2.1 Geometrical structures and tensors

An $n-$dimensional Euclidean space M can locally be approximated by the real vector space supplied with the standard inner product defined by the δ-metric structure

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$(T_xM, \delta_{ij}) = (\mathbb{R}^n, \delta_{ij})$. The Euclidean metric is a constant second order covariant tensor field over the space $M = \mathbb{R}^n$,

$$\delta : M \to T^*M \otimes T^*M, \quad \delta(x) = (\delta_{ij})_{i,j=\overline{1,n}},$$

which identifies the covariant or contravariant character the geometrical objects, via

$$v_a = \delta_{ab}v^b = v^a,\ a, b = \overline{1,n},\ \ (v^1, \dots, v^n) \in T_xM,\ \ (v_1, \dots, v_n) \in T_x^*M.$$

An $n-$dimensional differentiable manifold (M, g) is called a Riemannian space if the metric structure is determined by a symmetric second order covariant tensor field

$$g : M \to T^*M \otimes T^*M, \quad g(x) = (g_{ij}(x))_{i,j=\overline{1,n}}\,,$$

such that the associated quadratic form $g(x) : T_xM \to \mathbb{R}$, $v \mapsto g(x)(v, v)$ is regular and positive definite. Each tangent vector space $(T_xM, g_{ij}(x))$ has the inner product $\langle v, u\rangle_g = g_{ij}(x)v^iu^j$.

An anisotropic differentiable manifold is generalization of the Riemannian one by means of supplying each tangent vector space not with a single inner product, but with a smooth collection

$$\{\langle\cdot,\cdot\rangle_g \,|\, g(x, y), y \in T_xM\}. \tag{2.1}$$

The origin of the inner product collection declares the type of anisotropy. The Finsler space will be introduced in the following as an illustrative example of anisotropic space.

A Finsler space (M, F) is an n-dimensional differentiable manifold M with a fundamental Finsler function i.e., a mapping $F : TM \to [0, \infty)$, satisfying the following conditions:

- F is continuous and smooth on the slit tangent space $TM\backslash\{0\}$;
- F is positively 1-homogeneous in its second argument, $F(x, \lambda y) = \lambda F(x, y)$, $\lambda > 0$;
- the fundamental metric tensor field $g = g_{ij}dx^i \otimes dx^j$ having the components $g_{ij} : TM \setminus \{0\} \to \mathbb{R}$, $i, j \in \overline{1,n}$,

  $$g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}, \tag{2.2}$$

  is symmetric, regular outside the null section of TM and continuous at 0, and positive definite.

The fundamental Finsler function endows each tangent vector space $T_xM \cong \mathbb{R}^n$ with an inner product collection (2.1) that smoothly depends on direction, naturally provided by the (2.2).

The components g_{ij} of the fundamental metric tensor field (2.2) and of the corresponding dual tensor field defined by $g^{is}g_{sj} = \delta^i_j$, will be used to change the nature of geometrical objects, i.e., to lower and to raise indices of tensors, for constructing geometric objects specific to the Finsler structure.

The metric tensor $g(x,y)$ is a 0-homogeneous in its directional argument, totally symmetric and covariant d-tensor, related to the Finsler norm via $F(x,y) = g_{ij}(x,y)y^i y^j$. A fixed tangent vector $y \in \widetilde{T_x M}$ is called *flagpole*.

The dependence of the metric tensor on the directional argument is exhibited by the Cartan tensor field, which is globally defined on the slit tangent bundle

$$C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k} = \frac{1}{4}\frac{\partial^3 F^2}{\partial y^i \partial y^j \partial y^k}. \tag{2.3}$$

The Cartan tensor is a covariant totally symmetric d-tensor, positively homogeneous of order -1. It further endows each tangent space $T_x M$, $x \in M$ with the family of 3-linear operators $\mathbf{C} = \{C_y \,|\, y \in T_x M \setminus \{0\}\}$, i.e., for a fixed flagpole $y \in T_x M \setminus \{0\}$, the mapping $C_y : (T_x M)^3 \to \mathbb{R}$ is defined as

$$C_y : (u,v,z) \mapsto C_y(u,v,z) = C_{ijk}(x,y)u^i v^j z^k. \tag{2.4}$$

The detailed theory of Finsler and other anisotropic spaces can be found in [11, 12, 16, 17].

A linear deformation of the Riemannian norm extends the Riemannian space to a Finslerian one of special Randers type. The Finsler space has a metric of Randers type if the fundamental function is

$$F(x,y) = \sqrt{g_{ij}(x)y^i y^j} + b_i(x)y^i =: \alpha(x,y) + \beta(x,y),$$

where $\|y\| = \alpha(x,y)$ is a norm function in the tangent space $T_x M$, and $\beta(x,y) = b_i(x)y^i$ is a linear functional in $T_x M$. The Randers-Finsler norm(2.1) in $T_x M$ provides a regular and positive metric tensor field if the α-norm of the one-form $b(x) = b_i(x)dx^i$ is strictly less then 1.

In the simplest case, when a Euclidean norm is linearly deformed by $b(x) = b$, the obtained Finsler structure is of Minkowski type,

$$F(x,y) = \sqrt{y^T y} + b_i y^i, \tag{2.5}$$

and the Finsler norm is point-independent. The details for the metric and the Cartan tensors in a 2-dimensional Randers case can be found in [6].

In the present work, the Cartan tensor will be subject to the commonly developed spectral tools initiated by Liqun Qi.

It will be of the interest to us make a clear distinction between the notions of tensor, and tensor field. The tensor A is related to a single vector space, while the tensor field specifies varying of tensor in space in terms of the position variable x.

The tensor A of order m is a scalar-valued multilinear mapping with m vectorial arguments, which is independent of coordinate representation. The tensor is described by the components $A_{i_1 i_2 \cdots i_m}$ that form an m-dimensional array, fixed by a chosen coordinate system.

If the tensor is considered at a point x of a manifold M, it acts on the tangent vector space $T_x M$. The tensor field is a smooth assignment of tensors to the points of the manifold, $x \mapsto A(x)$, and further $x \mapsto A_{i_1 i_2 \cdots i_m}(x)$ in a local coordinate chart. The analysis of a tensor field observes the variation of its tensor characteristics over the manifold.

A tensor field defined over the tangent space TM of a manifold M, whose local representation depends directly on the base manifold id called d-tensor field[1]. At each point $x \in M$ there is a collection of tensors $A(x, y)$, $y \in T_xM$, smooth with respect to flagpoles. The characteristics of tensors are used in analysis of multidimensional data arrays, which makes them also be named tensors.

2.2 Higher order tensor spectral analysis

The extension of eigenvalue problem is developed in various manners. An algebraic approach to the generalization of classical spectral theory was initiated in [13, 14], while the variational approach can be found in, e.g., [18]. All the existing definitions apply to tensors, considered as a multilinear maps in Euclidean vector spaces. The computational methods are rather complicated, and are considered in [19].

The present work follows the algebraic approach, where the stated $Z-$eigenproblem and $H-$eigenproblem differ by homogeneity. Both eigenproblems relate to a given m-th order totally symmetric covariant tensor A with its components $A = (A_{ii_2\cdots i_m})$, acting as a multilinear map on the flat n-dimensional real vector space $\mathbb{R}^n$,

$$A : \mathbb{R}^n \otimes \ldots \otimes \mathbb{R}^n \to \mathbb{R}, \quad (v_1, v_2, \ldots, v_m) \mapsto A_{ii_2\cdots i_m} v_1^i v_2^{i_2} \cdots v_m^{i_m}.$$

This action of the tensor on the vector v is compared with the scaled vector or with its componentwise appropriate power,

$$A_{ii_2\cdots i_m} v^{i_2} \cdots v^{i_m} = \lambda v^i, \quad i \in \{1, m-1\}.$$

The dual objects, covariant and contravariant ones, are equated, which means that the Euclidean metric structure is imposed, $\langle u, v\rangle_\delta = \delta_{ij} u^i v^j$.

The exact definitions of the two eigenproblems consider a scalar - vector pair, $\lambda \in \mathbb{C}$ and $v = (v^1, \ldots, v^n) \in \mathbb{C}^n$.

Definition 2.1. If λ and v are real solutions of the following $Z-$eigenproblem

$$\text{(2.6)} \qquad \begin{cases} A_{ii_2\cdots i_m} v^{i_2} \cdots v^{i_m} = \lambda\, v^i \\ v^T\, v = 1, \end{cases}$$

then λ is a $Z-$eigenvalue and v is a corresponding $Z-$eigenvector of the tensor A, briefly (λ, v) is a $Z-$eigenpair. If the solution vector of the problem is not a real vector, but $v \in \mathbb{C}^n$, then the objects are labelled as $E-$eigenvalue and $E-$eigenvector, respectively.

Definition 2.2. If λ and v are real solutions of the following $H-$eigenproblem

$$\text{(2.7)} \qquad A_{ii_2\cdots i_m} v^{i_2} \cdots v^{i_m} = \lambda\, (v^i)^{m-1},$$

then λ is an $H-$eigenvalue and v is a corresponding $H-$eigenvector of the tensor A, i.e., (λ, v) is an $H-$eigenpair. Similarly, if v is not a real solution, then (λ, v) is an $N-$eigenpair.

[1]Details on (distinguished) d-tensors can be found in [11, 12]

Main results on the four types of eigendata can be found in [13, 14]. The $Z-$eigenvalues always exist and they are invariants of the tensor A. However, they do not have eigenspaces, but corresponding single $Z-$eigenvectors. The existence of $H-$eigenvalues is ensured in the case of an even order of the tensor. The homogeneity of (2.7) provides an eigenspace to each $H-$eigenvalue. The $Z-$spectra $\sigma_Z(A)$ and the $H-$spectra $\sigma_H(A)$ are defined as collections of corresponding eigenvalues, and the produced spectral radii $\rho_Z(A)$ and $\rho_H(A)$ which are invariant as well.

The spectral characteristics of higher order tensors are extremely useful for their decomposition and lower rank approximation apart from their $m-$action on the vector space. Therefore the Big Data sets, arranged as a multidimensional arrays, can be subjected to the higher order tensor spectral analysis of simplest type, i.e., with respect to the Euclidean δ metric, see [21, 22, 24].

3 Higher order tensor spectral analysis in non-Euclidean spaces

The third- and the fourth-order constant tensor fields, in a 3- or 4-dimensional Finsler space of particular $4-$th root type may be regarded as linear operators in an Euclidean vector space in [2, 3, 4]. Their spectral analysis is accomplished by the use of (2.6) and (2.7). In other words, both the eigenproblems may be stated and solved with respect to the δ metric,

$$\text{(3.1)} \qquad \text{Z}: \quad \begin{cases} C_{ijk}v^j v^k = \lambda\,\delta_{ih}v^h \\ \langle v, v\rangle_\delta = 1 \end{cases} \qquad \text{and} \qquad \text{H}: \quad C_{ijk}v^j v^k = \lambda\,(\delta_{ih}v^h)^2.$$

A comprehensive observation for the eigenproblem over a manifold, is that the metric the manifold changes, while confined to distinct tangent spaces. Under such auspices, the eigenproblem relates to the geometrical properties of the space, and provide geometrically meaningful information.

The adjustment of the spectral tensor approach for real non-Euclidean manifolds is proposed in this section with emphasis on the real solutions of the eigenproblems, since the geometric relevance of the complex case over real manifolds an open problem. The generalization considers the real anisotropic differentiable manifolds as environment for the eigenproblems of d-tensor fields.

Let A be a d-tensor field of order m, over an n-dimensional real anisotropic differentiable manifold (M, g) with regular metric. A tensor $A(x, y)$, for $x \in M$ and a flagpole $y \in T_xM$ acts on the tangent space T_xM endowed with the scalar product $\langle u, v\rangle_g = g_{ij}(x, y)u^i v^j$, $u, v \in T_xM$, at the flagpole y.

In accordance with the algebraic approach to multilinear mappings, the proposed generalization of the eigenproblem adjusts the lefthendsides of the spectral equations, and agrees on the tensorial character of both sides.

Definition 3.1. If $\lambda(x, y)$ and $z(x, y)$ are real solutions of the following anisotropic Z-eigenproblem over the manifold

$$\text{(3.2)} \qquad \begin{cases} A_{ii_2\cdots i_m}(x, y)g^{ia}(x, y)z^{i_2}\cdots z^{i_m} = \lambda\, z^a \\ \langle z, z\rangle_{g(x,y)} = 1 \end{cases}$$

then we call λ a Z-eigenvalue and z its corresponding Z-eigenvector for the tensor $A(x, y)$.

In brief, (λ, z) is a Z-eigenpair of the tensor $A(x, y)$ at the point x and flagpole y. If the vector z is not real, then the two spectral objects are called E-eigenvalue and E-eigenvector, respectively.

Definition 3.2. If $\lambda(x, y)$ and $z(x, y)$ are real solutions of the following (generally anisotropic) H-eigenproblem

$$A_{ii_2\cdots i_m}(x, y)g^{ia}(x, y)z^{i_2}\cdots z^{i_m} = \lambda\,(z^a)^{m-1}, \tag{3.3}$$

then λ is called H-eigenvalue and z corresponding H-eigenvector of the real tensor $A(x, y)$. We say that (λ, z) is an H-eigenpair of A at x for fixed y.

In the case when z is not a real solution, but a complex one, then (λ, z) is called an N-eigenpair.

In both definitions we look for nontrivial vector solutions. In any case, $(\lambda, z = 0)$ is an eigenpair for each possible value $\lambda \in \mathbb{C}$.

The previous eigenproblems are stated with respect to a regular metric tensor. This fact can be emphasized by denoting the spectral data as Z_g- and H_g, respectively. A particular eigenproblem occurs when g is the Euclidean metric δ. In such case, the used notations are Z_δ- and H_δ respectively. This leads to the case (3.1). The eigenproblems generally depend on the considered flagpole and the corresponding eigendata do, as well.

The Z_g-eigenproblem (3.2) is invariant with respect to the local representation, hence if solutions exist, they are global, and $\lambda(x, y)$ is a scalar field and $z(x, y)$ is a vector field over the slit tangent space, and provide the spectral radius scalar field.

The H_g-eigenproblem is not invariant with respect to the local representation. For $m \geq 2$, the righthandside of the spectral equation has strictly local character. Since the equation (3.3) is homogeneous in z, if a nontrivial solution exists, then it belongs to a generally larger space of solutions.

4 Spectral data of the Cartan tensor

We shall further focus on the eigenproblem associated to the geometrical model of Garner's dynamical system of cancer cell population, studied in [6]. The configuration space of this system is an open domain in the first quadrant of the $\mathbb{R}^2$, $x = (x^1, x^2) \in M \subset \mathbb{R}^2$, endowed with a Finsler metric structure of Minkowski type, $F(x, y) = F(y)$, which is fit to support the characteristics of the emerging dynamical system. This Finsler norm extends the Euclidean one $\alpha(y) := \|y\|_\delta = \sqrt{(y^1)^2 + (y^2)^2}$, $y = (y^1, y^2) \in T_xM$ in a Randers manner,

$$F_R(y) = \alpha(y) + 0.63y^1 - 0.27y^2. \tag{4.1}$$

The related metric tensor determined by the Randers norm (4.1) is symmetric, positive definite and regular. Its flag-dependent components obtained by means of (2.2) are

$$g_{11} = \frac{1.26(y^1)^3 + 1.89y^1(y^2)^2 + 1.397\alpha^3(y) - 0.27(y^2)^3}{\alpha^3(y)},$$
$$g_{12} = \frac{0.63(y^2)^3 - 0.27(y^1)^3 - 0.17\alpha^3(y)}{\alpha^3(y)},$$
$$g_{22} = \frac{0.63(y^1)^3 - 0.81(y^1)^2y^1 + 1.073\alpha^3(y) - 0.54(y^2)^3}{\alpha^3(y)}$$

and determine at the flagpole $(x,y) \in \widetilde{T_xM}$ an inner product (2.1) a metric norm $\|z\|_{g(y)} = \sqrt{g_{11}(z^1)^2 + 2g_{12}z^1z^2 + g_{22}(z^2)^2}$. As well, the determinant of the metric tensor field $G(x,y) = g_{11}g_{22} - (g_{12})^2$ is a positive scalar field over $\widetilde{T_xM}$.

The components of the totally symmetric flag-dependent Cartan tensor (2.3) are

$$C_{111} = (y^2)^3K, \quad C_{112} = -(y^2)^2y^1K, \quad C_{122} = (y^1)^2y^2K, \quad C_{222} = -(y^1)^3K,$$

where $K = 0.135\left(7\,y^2 + 3\,y^1\right)/\alpha^5(y)$. The mapping defined by (2.4),

$$C_{(x,y)} : T_xM \to T_xM, \quad z \mapsto C_{ijk}(x,y)g^{ia}(x,y)z^jz^k$$

is trilinear, its trace is 3-homogeneous, and in applications, the related mapping $z \mapsto C_{ijk}(x,y)\delta^{ia}(x,y)z^jz^k$ is observed, for the extrinsic perception of the eigenproblems.

Since the structure is of Minkowski type, the Z-eigenproblems reduce,

$$\mathrm{Z}_\delta : \begin{cases} C_{ijk}(y)\delta^{ia}z^jz^k = \lambda\, z^a \\ \langle z,z\rangle_{g(y)} = 1, \end{cases} \qquad \mathrm{Z}_g : \begin{cases} C_{ijk}(y)g^{ia}(y)z^jz^k = \lambda\, z^a \\ \langle z,z\rangle_{g(y)} = 1. \end{cases}$$

Assuming the flag fixed, the anisotropic H-eigenproblem for the Cartan tensor has the form

$$\mathrm{H}_\delta : C_{ijk}(y)\delta^{ia}z^jz^k = \lambda\,(z^a)^2 \qquad \mathrm{H}_g : C_{ijk}(y)g^{ia}(y)z^jz^k = \lambda\,(z^a)^2.$$

Both H-eigenproblems have strongly local character and their solutions exist only within a local coordinate framework. As well, the H-eigendata have no natural geometric character.

The explicit expressions of the Cartan eigenproblems esentially depend on the metric structure and on the flagpole components.

Proposition 4.1. *The eigendata of the anisotropic Z_δ-eigenproblem are (λ, z) the solutions of the following system*

$$\begin{cases} \dfrac{3y^2(0.63y^2 + 0.27y^1)(-y^2z^1 + y^1z^2)^2}{2\alpha^5} = \lambda\, z^1 \\ \dfrac{-3y^1(0.63y^2 + 0.27y^1)(-y^2z^1 + y^1z^2)^2}{2\alpha^5} = \lambda\, z^2 \\ g_{11}(z^1)^2 + g_{12}z^1z^2 + g_{22}(z^2)^2 = 1. \end{cases}$$

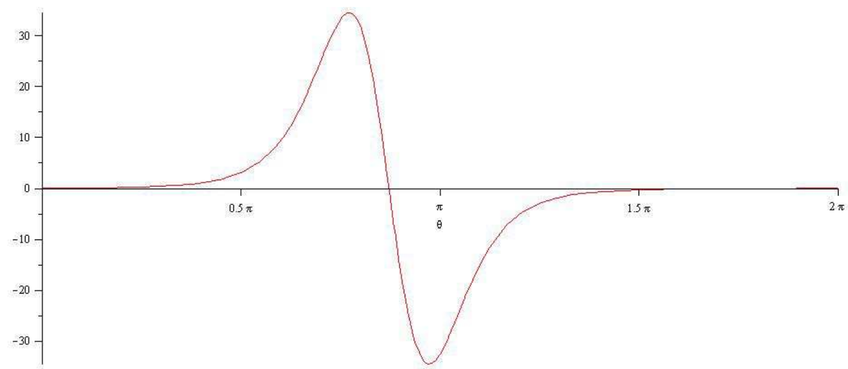

Figure 1: Z_δ-eigenvalue function $\lambda_\delta = \lambda_\delta(y(\theta))$

Proposition 4.2. *The eigendata of the anisotropic Z_g-eigenproblem are the solutions (λ, z) of the system*

$$\begin{cases} K\left[\left((y^2)^3 g_{22} + (y^2)^2 y^1 g_{12}\right)(z^1)^2 - 2\left((y^2)^2 y^1 g_{22} + (y^1)^2 y^2 g_{12}\right) z^1 z^2 \right. \\ \qquad \left. + \left((y^1)^2 y^2 g_{22} + (y^1)^3 g_{12}\right)(z^2)^2\right] = \lambda\, G\, z^1, \\ K\left[-\left((y^2)^3 g_{12} + (y^2)^2 y^1 g_{11}\right)(z^1)^2 + 2\left((y^2)^2 y^1 g_{12} + (y^1)^2 y^2 g_{11}\right) z^1 z^2 \right. \\ \qquad \left. - \left((y^1)^2 y^2 g_{12} + (y^1)^3 g_{11}\right)(z^2)^2\right] = \lambda\, G\, z^2, \\ g_{11}(z^1)^2 + 2g_{12} z^1 z^2 + g_{22}(z^2)^2 = 1. \end{cases}$$

It was proved ([13, 14]) that the Z-eigenproblem generally admits four eigenvalues. The zero Z-eigenvalue exhibits multiplicity 2 and allows two mutually opposite eigenvectors. The other two Z-eigenvalues are real or purely imaginary opposite numbers, with the corresponding eigenvectors similarly related. The spectra of the both Z-eigenproblems are

$$\sigma_{Z_\delta}(y) = \{0, 0, \lambda_\delta(y), -\lambda_\delta(y)\}, \quad \sigma_{Z_g}(y) = \{0, 0, \lambda_g(y), -\lambda_g(y)\},$$

and the spectral radii are $\rho_{Z_\delta}(y) = |\lambda_\delta(y)|$ and $\rho_{Z_g}(y) = |\lambda_g(y)|$. The nonzero Z_δ-eigenvalue smoothly depends on the direction determined by the polar angle of $y = y(\theta)$ (see Figure 1), and hence $\lambda_\delta(y)$ becomes a scalar field over the sphere tangent bundle of the manifold.

The nonzero Z_g-eigenvalues are purely imaginary for a subregion of flagpoles, and their modules are presented by the dark curve in Figure 2. The function $\lambda_g(y)$ is not a proper scalar field over the normalized flagpole sphere space.

Proposition 4.3. *a) The eigendata (λ, z) of the anisotropic H_δ-eigenproblem are the solutions of the following system*

$$\begin{cases} \dfrac{3y^2(0.63y^2 + 0.27y^1)(-y^2 z^1 + y^1 z^2)^2}{2\alpha^5} = \lambda\,(z^1)^2 \\ \dfrac{-3y^1(0.63y^2 + 0.27y^1)(-y^2 z^1 + y^1 z^2)^2}{2\alpha^5} = \lambda\,(z^2)^2, \end{cases}$$

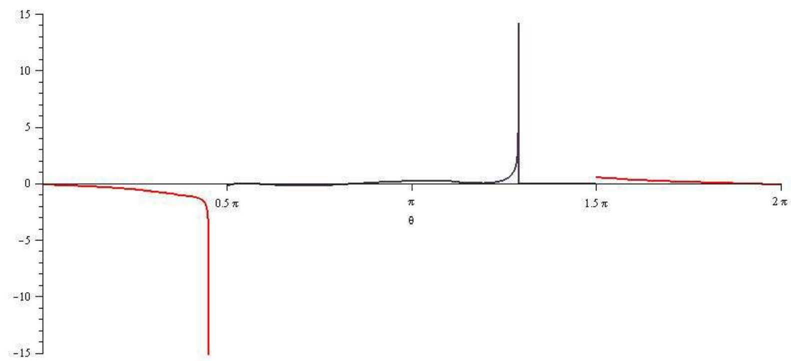

Figure 2: Real Z_g-eigenvalues are shown by the red curve

b) The eigendata (λ, z) of the anisotropic H_g-eigenproblem are the solutions of the following system

$$\begin{cases} K\left[\left((y^2)^3 g_{22} + (y^2)^2 y^1 g_{12}\right)(z^1)^2 - 2\left((y^2)^2 y^1 g_{22} + (y^1)^2 y^2 g_{12}\right) z^1 z^2 \right. \\ \qquad \left. + \left((y^1)^2 y^2 g_{22} + (y^1)^3 g_{12}\right)(z^2)^2\right] = \lambda\, G\,(z^1)^2 \\ K\left[-\left((y^2)^3 g_{12} + (y^2)^2 y^1 g_{11}\right)(z^1)^2 + 2\left((y^2)^2 y^1 g_{12} + (y^1)^2 y^2 g_{11}\right) z^1 z^2 \right. \\ \qquad \left. - \left((y^1)^2 y^2 g_{12} + (y^1)^3 g_{11}\right)(z^2)^2\right] = \lambda\, G\,(z^2)^2 \end{cases}$$

The study of H-eigendata in [13] was based on generalized algebraic concepts. Since the H-eigenvalues are solutions of the characteristic polynomial, their number does not exceed four.

4.1 Numeric illustration

In order to solve the Cartan eigenproblem, a possible approach might use the Maple software. The procedure of finding the eigendata in the non-Euclidean case consists of the following: for a given flagpole, the components of the metric and of the Cartan tensor are determined and provide the eigenproblem, and its eigenpairs are further obtained $(\lambda(y), z(y))$.

To visualize the dependence of the eigenpairs on flagpoles, we discretize the sphere domain by using flagpoles defined by indicatrix harmonics

$$I_N = \{y = y(\theta) \,|\, \theta = \tfrac{h}{N} 2\pi, h = \overline{1, N}\},$$
$$(y^1, y^2) = \left(\tfrac{\cos\theta}{\|(\cos\theta, \sin\theta)\|_g}, \tfrac{\sin\theta}{\|(\cos\theta, \sin\theta)\|_g}\right).$$

The number of samples $N = 64$ is fit for the used graphical illustrative representations.

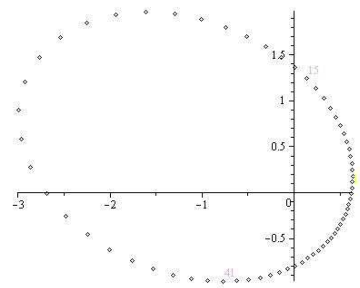

Figure 3: Discretized Indicatrix of the Randers space

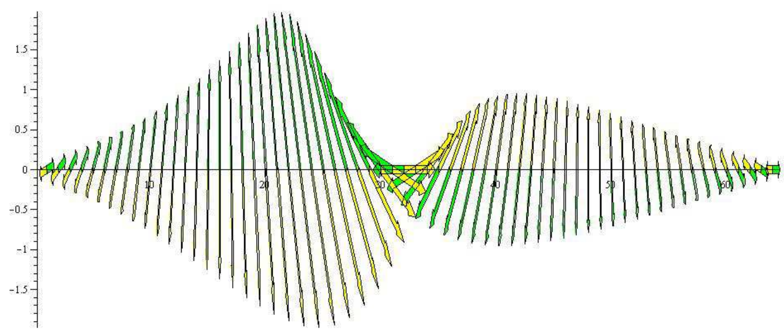

Figure 4: Zero-related Z_δ-Eigenvectors, two opposite for each flagpole

The indicatrix contains the Randers-unit flagpoles, as observed in Figure 3.

A loop iterates the prescribed flagpoles, and each iteration produces the metric and the Cartan data, and finds the solutions of the spectral equations,

$$\{\ \lambda(y^1, y^2);\ z(y^1, y^2) \equiv (z^1(y^1, y^2), z^2(y^1, y^2))\ \}.$$

We further represent the dependence of the eigenpairs on the flagpoles for the samples $h = \overline{1, 64}$.

For the null Z_δ- and Z_g- eigenvalue, the corresponding eigenvectors form vector fields, and in the Z_g case a whole flagpole subregion provides purely imaginary eigenvectors (see Figures 4 and 5).

The nonzero Z_δ-eigendata are fields (scalar and vector ones) over the slit sphere space; the dependence on flagpoles is illustrated in Figure 6.

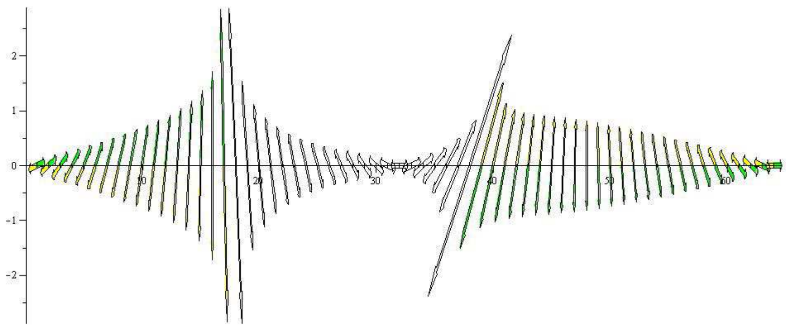

Figure 5: Not proper nonzero-related Z_g-eigenvector fields. The gray vectors are purely imaginary.

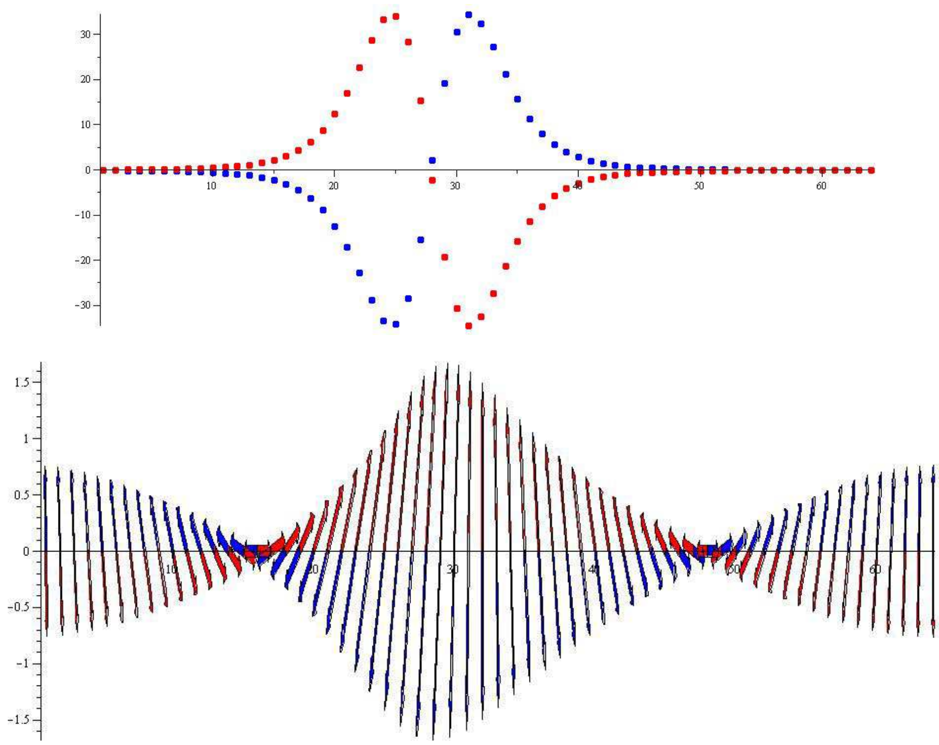

Figure 6: Two opposite nontrivial Z_δ-Eigenvalues and corresponding eigenvectors of nontrivial Z_δ-eigenvalues

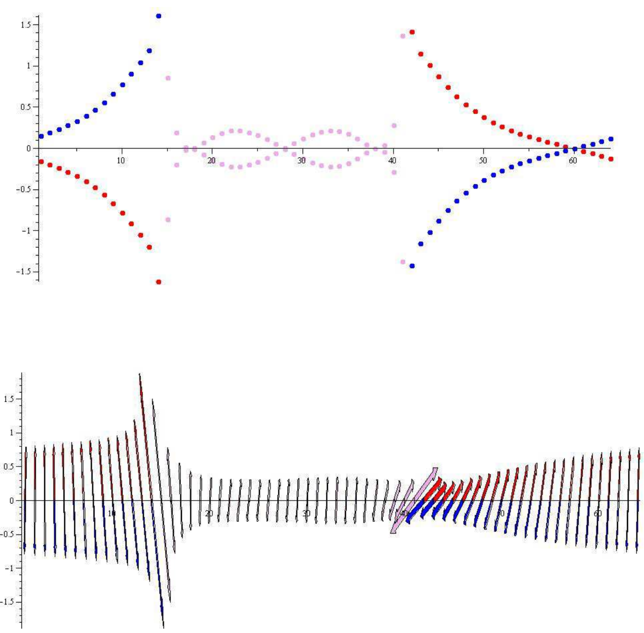

Figure 7: Nonproper Z_g-eigendata field. The "Imaginary zone": $h \in \{15, 16, \ldots, 41\}$.

The nonzero Z_g-eigendata are not real for all the flagpoles, but taking their real correspondents are geometrically meaningful (see Figure 7).

As well, the spectral radii in both the Z_g- and Z_δ cases are scalar fields.

As for the H-eigendata, a *technical constrain* is used to optimize the numeric solving. Namely, the condition of unit length vector is included into the H-eigenproblem, and the solution H-eigenvector determines a basis of the corresponding one dimensional H-eigenspace. It is shown that the solutions of these adjusted systems (H_g and H_δ) are of the form $(0, z_0), (0, -z_0), (\lambda, z_\lambda), (\lambda, -z_\lambda)$.

The H-eigendata are generally complex; the spectral radii of the H_g and H_δ eigenproblems were graphically represented in Figure 8.

5 Conclusions

The classical algebraic eigenproblem is considered over a differentiable manifold and its extension is described. Further, the anisotropic Z- and H-eigenproblems for the third-order directionally dependent Cartan tensor field over the Finsler structure of Randers type, are stated. The Z- and the H-eigenvalues of the Cartan tensor always exist as complex numbers, and their flag-dependence is numerically determined and illustrated. It was shown that the Z_δ and Z_g eigendata smoothly vary on the direction of the flagpole, and that the H_δ and H_g eigendata are generally complex, lacking a proper geometrical interpretation.

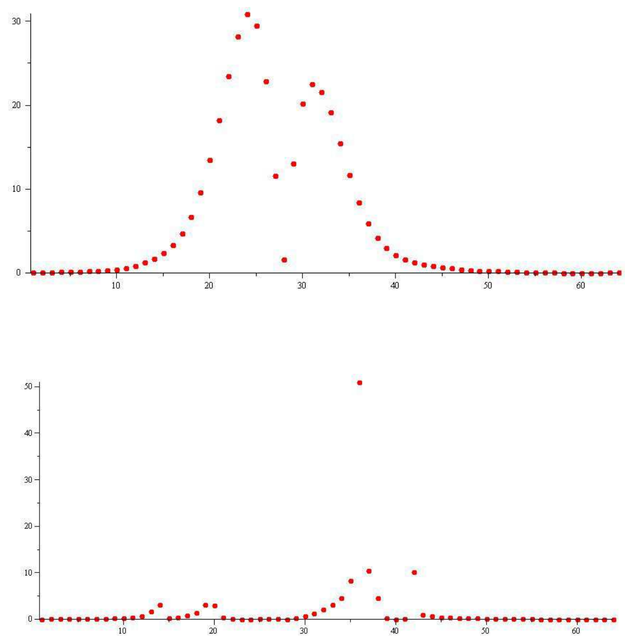

Figure 8: H_δ- and H_g- spectral radii of the Cartan tensor

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