Pencils on a curve giving independent conditions to a very ample special line bundle

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Abstract. We classify the triples (C, L, D), with C a smooth curve, L a very ample special line bundle on C and D an effective divisor of C with $\dim |D| > 0$ and D imposing exactly $\deg(D) - \dim |D| - 1$ independent conditions to $|L|$, i.e. the maximal possible number of independent conditions to the complete special linear system $|L|$.

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1 Introduction

Let C be a smooth projective curve of genus $g \geq 3$ defined over an algebraically closed field with characteristic 0. Let L be a very ample line bundle on C. For all integers $b > a \geq 0$ let $V_b^a(L)$ denote the set of the effective divisors D of C with $\deg(D) = b$ and $h^0(L(-D)) \geq h^0(L) - a$. Each set $V_b^a(L)$ is a closed subset of the set $C^{(b)}$ of all degree b effective divisors of C.

Now assume $h^1(L) > 0$ and that D is any effective divisor with $k := h^0(\mathcal{O}_C(D)) \geq 2$. The geometric form of Riemann-Roch gives $h^0(K_C(-D)) = h^0(K_C) - \deg(D) + k - 1$. Since $K_C \otimes L^\vee$ is effective, we get $h^0(L(-D)) \geq h^0(L) - \deg(D) + k - 1$. Hence for divisors D with $h^0(\mathcal{O}_C(D)) = k$ and L a special line bundle we consider D " exceptional " for L only if $h^0(L(-D)) \geq h^0(L) - \deg(D) + k$. Note that $h^0(L(-D')) = h^0(L(-D))$ for all $D' \in |\mathcal{O}_C(D)|$. In this note we prove the following result.

Theorem 1.1. *Let C be a smooth curve of genus $g \geq 5$ and L a very ample line bundle on C with $h^0(L) \geq 4$ and $h^1(L) > 0$. Write $L = K_C(-B)$ with B effective. Fix an integer $k \geq 2$. Let $D \subset C$ be an effective divisor with $h^0(\mathcal{O}_C(D)) = k$ and $d := \deg(D) \leq h^0(L) - 2$. We have $D \in V_d^{d-k+1}(L)$.*

(a) Assume $h^1(L) \geq 2$ and set $t := h^1(L) - 1$. We have $D \in V_d^{d-k} \setminus V_d^{d-k-1}(L)$ if and only if there are effective divisors B_1, B_2 and a base point free pencil A such that $B = B_1 + tA$, $D = B_2 + (k-1)A$ and $h^0(\mathcal{O}_C(B_1 + B_2 + (t + k - 1)A)) = t + k$.

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(b) Assume $h^1(L) = 1$. We have $D \in V_d^{d-k}(L)$ if and only if $h^0(\mathcal{O}_C(D+B)) > k$.

Part (b) of Theorem 1.1 is an obvious consequence of duality. The only interesting statement is part (a), which is related to the study of the dependency loci of complete linear systems (see [10, Conjecture at page 353], [4], [7] for general linear systems on general curves, [6] for the tangent space of the associated functor). This year we came to it from [2].

To get at least one L as in part (a) we need the existence of a base point free linear system $|A|$ with $\dim |A| = 1$, $h^0(\mathcal{O}_C((t+k-1)A)) = t+k$. It never exists on a curve with gonality x if $(t+k-1)x \geq g+t+k$. See Example 2.1 for the range of integers g, k, t, $\deg(B_1)$, $\deg(B_2)$ which occur when we may take as C a general x-gonal curve. If the gonality of C is not very low with respect to the genus we first fix a base point free pencil $|A|$ and then a very ample L and $D = B_2 + |(k-1)A|$ as in part (a) of Theorem 1.1; for the case $k = 2$ and $t = 1$ we only need that $\dim |2A| = 3$. This condition is satisfied if C is a smooth plane curve of degree $z \geq 3$ and A is a degree $z-1$ pencil computing the gonality of C, while for this pair (C, A) we have $\dim |(t+k-1)A| > t+k$ if $t+k \geq 4$.

There are linearly normal curves $C \subset \mathbb{P}^r$ for which $h^0(\mathcal{O}_C(D)) = 2$ for all linearly dependent divisors D with $h^0(\mathcal{O}_C(1)(-D)) \geq 2$ (e.g., take curves on a smooth quadric surface of $\mathbb{P}^3$). In this case there are irreducible components of $V_d^a(L)$ whose general element D satisfies $h^0(\mathcal{O}_C(D)) \geq 2$.

Take (C, L, D) as in Theorem 1.1 with $k = h^0(\mathcal{O}_C(D)) \geq 2$. Set $d := \deg(D)$. Since $h^0(L(-D')) = h^0(L(-D))$ for every $D' \in |\mathcal{O}_C(D)|$ and a general $S \subset C$ with $\sharp(S) = k-1$ is contained in some $D' \in |\mathcal{O}_C(D)|$, we have $D \in V_d^{k-1}(L)$. Since L is very ample, we have $D \in V_d^k(L)$ (use [11, Corollary 5.2]).

2 General curves with fixed gonality and the proof of Theorem 1.1

Proof of Theorem 1.1: Since $h^0(\mathcal{O}_C(D)) = k \leq g-3$, the geometric form of Riemann-Roch gives $h^0(K_C) - h^0(K_C(-D)) = d+1-k$. Since L is special, we get $h^0(L) - h^0(L(-D)) \leq d+1-k$, i.e. $D \in V_d^{d+1-k}(L)$. Duality gives that $D \in V_d^{d-k}(L) \setminus V_d^{d-k-1}(L)$ if and only if $h^0(\mathcal{O}_C(D+B)) = k$. Part (b) follows at once.

Now assume $h^1(L) \geq 2$, i.e. $h^0(\mathcal{O}_C(B)) = t+1 \geq 2$. Write $|B| = B_1 + |F|$ and $|D| = B_2 + |G|$ with $|F|$ and $|G|$ base point free, $\dim |F| = \dim |B|$ and $\dim |G| = \dim |D|$. We have two addition maps $u : |B| \times |D| \times |B+D|$ and $v : |F| \times |G| \to |F+G|$. All addition maps of linear systems of effective divisors are finite. Since $h^0(\mathcal{O}_C(D+B)) = k$, we have $\dim |B| + \dim |D| = \dim |B+D|$. Hence u is surjective. Thus v is surjective and $|B+D| = B_1 + B_2 + |F+G|$. Since v is surjective, there is a base point free linear system $|A|$ with $\dim |A| = 1$, $\mathcal{O}_C(F) = \mathcal{O}_C(tA)$, $\mathcal{O}_C(D) = \mathcal{O}_C(kA)$, $S^t(H^0(A)) = H^0(F)$ and $S^{k-1}(H^0(A)) = H^0(G)$ ([11, Corollary 5.2]). Since u and v are surjective, we also get that $S^{t+k-1}(H^0(A)) = H^0((t+k-1)A)$ and that $B_1 + B_2$ is the base locus of $|B+D|$. □

Example 2.1. Fix integers $x \geq 4$, $t \geq 1$, $k \geq 2$ and $g \geq (t+k-1)(x-1)$; if $x = 4$ assume $g \geq 4 + (t+k-1)(x-1)$. Let C be a general x-gonal curve. Call $|A|$ its unique g^1_x with A effective ([1, Theorem 2.6]). Since $g \ge (t+k-1)(x-1)$, we have $h^0(\mathcal{O}_C((t+k-1)A) = t+k$ ([3], [10, Proposition 2.1.1]). Set $F := tA$, $G := kA$ and $L := K_C(-tA)$. By duality L is very ample if and only if $h^0(\mathcal{O}_C((t-1)A+Z)) = t$ for every zero-dimensional scheme $Z \subset C$ with $\deg(Z) = 2$. Now fix effective divisors B_1, B_2. We have $h^0(\mathcal{O}_C((t+k-1)A+B_1+B_2) = t+k$ if either $2\deg(B_1)+2\deg(B_2)+(t+k-1)(x-1) \le g$ and $h^0(\mathcal{O}_C(B_1+B_2-A)) = 0$ ([8, Proposition 1.1]) or B_1+B_2 is general and $\deg(B_1)+\deg(B_2)+(t+k-1)x \le t+k+g-1$. In these cases we also have $h^0(\mathcal{O}_C(B_2+(k-1)A)) = k$ and $h^0(\mathcal{O}_C(B_1+(t-1)A)) = t$. The line bundle $L(-B_1)$ is very ample if and only if $h^0(\mathcal{O}_C(B_1+(t-1)A+Z)) = t$ for every degree 2 effective divisor $Z \subset C$. If $t = 1$ it is sufficient to assume that $\deg(B_1)+2 < x$. For any t by [8, Proposition 1.1] it is sufficient to assume $2\deg(B_1)+4+(t-1)(x-1) \le g$ and $\deg(B_1) \le x-3$ ([8, Proposition 1.1]). If B_1 is general, to have the very ampleness it is sufficient first to have $h^0(\mathcal{O}_C((t-1)A+Z)) = t$ (true by [8, Proposition 1.1], because $4+(t-1)(x-1) \le g$ and $h^0(\mathcal{O}_C(Z-A)) = 0$) and then to take a general $B_1 \subset C$ with $\sharp(B_1)+2+(t-1)x \le t+g-1$. Any line bundle L as above are called *type 2* in [8].

Now suppose that we want to use instead of A another effective divisor A' with $\dim|A'| = 1$ and $|A'| \ne |A|$. By [1, Theorem 2.6] we have $\deg(A') \ge (g+2)/2$ and hence we always have $h^0(\mathcal{O}_C(3A')) > 4$, while $h^0(\mathcal{O}_C(2A')) > 3$, except in the case g even and $\deg(A') = (g+2)/2$. Take A' with $\deg(A') = (g+2)/2$ (it exists by [10, Corollary 2.3.2]). If $h^0(\mathcal{O}_C(2A')) = 3$, then $h^1(\mathcal{O}_C(2A')) = 0$ and hence $h^1(\mathcal{O}_C(B+D)) = 0$. Hence $h^0(L(-D)) = 0$ and so A' does not give an example.

In some ranges for g, x, $\deg(L)$ and $h^1(L)$ any very ample L is of the form $K_C(-B_1-(t-1)A)$ with $t = h^1(L)$ and $|B_1+(t-1)A| = B_1+|(t-1)A|$ (see [10] and see [9] for a complete list when $x = 4$).

Another case in which we may identify all pencils A (or at least their degrees, or at least the ones whose degree is the gonality of the curve) is for the smooth curves C with a plane model which is either a smooth plane curve, or a plane curves with " a small number of singularities " ([5], [12], [13]) or contained in a Hirzebruch surface. In each case one can fix A and then, for some k, t see which B_1 and B_2 one may add. On a Hirzebruch surface F_e it is easy to check when the condition $\dim|(k+t-1)A| = t+k$ is satisfied if $|A|$ is induced by the ruling of F_e. In the next example we work out some of the details.

Example 2.2. Let $X := F_e$, $e \ge 0$, be the Hirzebruch surface with a section with self-interesection $-e$. We have $\mathrm{Pic}(X) \cong \mathbb{Z}^2$ and we may take as free generator of $\mathrm{Pic}(X)$ two smooth rational curves h, f with $f^2 = 0$, $f \cdot h = 1$ and $h^2 = -e$. Let $Y \in |xh+yf|$ be an integral divisor, $x > 0$, $y \ge \max\{1, xe\}$ and let $u : C \to Y$ be the normalization of Y. Let g be the genus of C. Let A be any effective divisor of $|u^*(\mathcal{O}_Y(f))|$. Fix integers $b_1 \ge 0$, $b_2 \ge 0$ and $B_i \in C^{(b_i)}$, $i = 1, 2$, with the restriction that $u^{-1}(\mathrm{Sing}(Y))$ contains no point of the support of B_1+B_2. Set $U_i := u(B_i)$. Since we only need of gonality > 3, we assume $x \ge 4$. There is a zero-dimensional scheme $W \subset X$ with $W_{\mathrm{red}} = \mathrm{Sing}(Y)$, $\deg(W) = p_a(Y)-g$. Since $\omega_X \cong \mathcal{O}_X(-2h-(2+e)f)$, the adjunction formula gives $\omega_Y \cong \mathcal{O}_Y((x-2)h+(y-2-e)f)$ and hence $p_a(Y) = 1+(2xy-ex^2-2y-2x+ex)/2$. Since X is a smooth rational surface, $|\omega_C|$ is induced by the complete linear system $|\mathcal{I}_W((x-2)h+(y-2-e)f)|$. We get $\dim|A| = 1$ if and only if $h^1(\mathcal{I}_W((x-2)h+(y-3-e)) = 0$. Since Y is irreducible we have

$x \geq ey$. We get that $h^1(\mathcal{I}_W((x-2)h+(y-3-e)) = 0$ if (but not only if) either $e \geq 2$ and $\deg(W) \leq x-1$ or $e = 1$ and $\deg(W) \leq \min\{x-1, y-3\}$ or $e = 0$ and $\deg(W) \leq \min\{x-1, y-1\}$. We also get that $\dim |B_1 + B_2 + (t+k-1)A| = t+k$ if $h^1(\mathcal{I}_{W\cup U_1\cup U_2}((x-2)h+(y-1-e-t-k)) = 0$. If $e \geq 1+t+k$ it is sufficient to assume that $\deg(W) + \deg(B_1) + \deg(B_2) \leq x-1$. If $e = 0$ it is sufficient to assume that $\deg(W)+\deg(B_1)+\deg(B_2) \leq \min\{x-1, y-t-k\}$. If $e = 1$, then C has a plane singular birational model Y' with $|A|$ induced by the pencil of lines of $\mathbb{P}^2$ containing one of the singular points of Y'.

Example 2.3. Let C be a general curve of genus $g \geq 5$ and L a very ample line bundle on C with $h^1(L) \geq 2$. Assume that $D \in V_d^{d-k}(L) \setminus V_d^{d-k-1}(L)$. Since C has gonality $\lfloor (g+2)/2 \rfloor$, we saw in the introduction that we have g even, $t = 1$, $k = 2$, $\deg(A) = (g+2)/2$ and $B_1 = B_2 = \emptyset$. As in Example 2.1 we exclude this case in the following way. If $\dim |2A| = 3$, then $h^1(\mathcal{O}_C(2A)) = 0$ and so $h^1(\mathcal{O}_C(B+D)) = 0$. Thus $h^0(L(-D)) = 0$.

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