$\Psi-$uniform stability of solutions of a nonlinear Lyapunov matrix differential equation

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Abstract. We obtain sufficient $\Psi-$uniform stability conditions for the trivial solution of a nonlinear Lyapunov matrix differential equation with the integral term at the right-hand side.

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1 Introduction

The Lyapunov matrix differential equations occur in many branches of control theory such as optimal control and stability analysis.

Recent works for $\Psi-$stability, $\Psi-$uniform stability, $\Psi-$boundedness, $\Psi-$instability, controllability, dichotomy and conditioning for Lyapunov matrix differential equations have been given in many papers. See [3], [4], [5], [6], [7], [9], [12], [13], [14] and the references therein.

In this paper are presented some new sufficient conditions for $\Psi-$uniform stability of the trivial solution to the nonlinear Lyapunov matrix differential equation

$$Z' = A(t)Z + ZB(t) + \int_0^t F(t,s,Z(s))ds. \tag{1.1}$$

These conditions can be expressed in the terms of a fundamental matrices of the matrix differential equations

$$X' = A(t)X \tag{1.2}$$

$$Y' = YB(t) \tag{1.3}$$

and on the function F.

Here, Ψ is a matrix function whose introduction permits to obtaining a mixed asymptotic behavior for the components of solutions.

The main tool used in this paper is the technique of Kronecker product of matrices (combined with the variation of constants formula), which has been successfully applied in various fields of matrix theory, group theory and particle physics. See, for example, the cited papers and the references cited therein.

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2 Preliminaries

In this section we present some basic notations, definitions, hypotheses and results which are useful later on.

Let $\mathbb{R}^d$ be the Euclidean $d-$ dimensional space. For $x = (x_1, x_2, ..., x_d)^T \in \mathbb{R}^d$, let $||x|| = \max\{| x_1|, | x_2|, ..., | x_d|\}$ be the norm of x (here, ^{T} denotes transpose). Let $\mathbb{M}_{d\times d}$ be the linear space of all real $d \times d$ matrices. For $A = (a_{ij}) \in \mathbb{M}_{d\times d}$, we define the norm $| A |$ by formula $| A | = \sup\limits_{||x||\leq 1} ||Ax||$. It is well-known that $| A | = \max\limits_{1\leq i\leq d}\{\sum\limits_{j=1}^{d} | a_{ij}|\}$.

By a solution of the equation (1.1) we mean a continuous differentiable $d \times d$ matrix function satisfying the equation (1.1) for all $t \in \mathbb{R}_+ = [0, \infty)$

In equation (1.1), we assume that A and B are continuous $d\times d$ matrices on $\mathbb{R}_+$ and $F : D \longrightarrow \mathbb{M}_{d\times d}$, $D = \{(t, s, Z) \mid 0 \leq s \leq t < +\infty, Z \in \mathbb{M}_{d\times d}\}$, is continuous function such that $F(t, s, O_d) = O_d$. It is well-known that these conditions ensure the local existence of a solution of (1.1) passing through any given point $(t_0, Z_0) \in \mathbb{R}_+ \times \mathbb{M}_{d\times d}$, but it does not guarantee that the solution is unique or that it can be continued for large values of $t \in \mathbb{R}_+$.

Let $\Psi_i : \mathbb{R}_+ \longrightarrow (0, \infty)$, $i = 1, 2, ..., d$, be continuous functions and

$$\Psi = \text{diag } [\Psi_1, \Psi_2, \cdots \Psi_d].$$

In this paper, we will admit the following hypothesis:

(H) For all $t_0 \in \mathbb{R}_+$, $Z_0 \in \mathbb{M}_{d\times d}$ and $\rho > 0$, if $| \Psi(t_0)Z_0 | < \rho$, then any solution $Z(t)$ of (1.1) which satisfies the equality $Z(t_0) = Z_0$ exists on $\mathbb{R}_+$ and satisfies the inequality $| \Psi(t)Z(t) | \leq \rho$ for all $t \in [0, t_0]$.

This is a natural hypothesis in studying $\Psi-$ (uniform) stability of trivial solution of (1.1). See, for example, [10], [8].

Definition 2.1. ([3]) The trivial solution of the matrix differential equation $Z' = F(t, Z)$ (where $Z \in \mathbb{M}_{d\times d}$ and F is a continuous $d \times d$ matrix function) is said to be $\Psi-$ stable over $\mathbb{R}_+$ if for each $\varepsilon > 0$ and each $t_0 \in \mathbb{R}_+$ there is a a corresponding $\delta = \delta(\varepsilon, t_0) > 0$ such that any solution $Z(t)$ of the equation which satisfies the inequality $| \Psi(t_0)Z(t_0) | < \delta$, exists and satisfies the inequality $| \Psi(t)Z(t) | < \varepsilon$ for all $t \geq t_0$.
The trivial solution of equation $Z' = F(t, Z)$ is said to be $\Psi-$ *uniformly* stable over $\mathbb{R}_+$ if it is $\Psi-$ stable over $\mathbb{R}_+$ and the above δ is independent of t_0.

Remark 2.2. 1. The Definition extends the definition of (*uniform*) stability from (vector) differential equations to matrix differential equations.
2. For $\Psi = I_d$, one obtain the notion of classical (*uniform*) stability (see [2]).
3. It is easy to see that if Ψ and Ψ^{-1} are bounded on $\mathbb{R}_+$, then the $\Psi-$ (*uniform*) stability is equivalent with the classical (*uniform*) stability.

Definition 2.3. ([1]) Let $A = (a_{ij}) \in M_{m\times n}$ and $B = (b_{ij}) \in M_{p\times q}$. The Kronecker product of A and B, written $A \otimes B$, is defined to be the partitioned matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}.$$

Obviously, $A \otimes B \in \mathbb{M}_{mp\times nq}$.

The important rules of calculation of the Kronecker product are given in [1], [11], Chapter 2 and Lemma 1, [3].

Definition 2.4. ([11]) The application $\mathcal{V}ec : \mathbb{M}_{m\times n} \longrightarrow R^{mn}$, defined by

$$\mathcal{V}ec(A) = (a_{11}, a_{21}, \cdots, a_{m1}, a_{12}, a_{22}, \cdots, a_{m2}, \cdots, a_{1n}, a_{2n}, \cdots, a_{mn})^T,$$

where $A = (a_{ij}) \in \mathbb{M}_{m\times n}$, is called *the vectorization operator.*

For important properties and rules of calculation of the $\mathcal{V}ec$ operator, see Lemmas 2, 3, 4, [3]. For "*corresponding Kronecker product system associated with (1.1)",* see Lemma 5, [3].

The Lemmas 6 and 8, [3], play an important role in the proofs of main results of present paper.

3 Main results

Theorem 3.1. *Suppose that:*
(1). the hypothesis (H) is fulfilled;
(2). there exists a constant $K > 0$ *such that the fundamental matrices* $X(t)$ *and* $Y(t)$ *for equations (1.2) and (1.3) respectively satisfy the inequality*

$$\left|\left(Y^T(t)(Y^T)^{-1}(s)\right) \otimes \left(\Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s)\right)\right| \le K, \ \textit{for } t \ge s \ge 0;$$

(3). the matrix function $F(t,s,Z)$ *satisfies the inequality*

$$\mid \Psi(t)F(t,s,Z) \mid \le f(t,s) \mid \Psi(s)Z \mid,$$

for $(t,s,Z) \in D$, *where* $f(t,s)$ *is a continuous nonnegative function for* $t \ge s \ge 0$ *such that*

$$\exists M > 0 : \int_0^t \int_0^s f(s,u)duds \le M, \ \textit{for all } t \in \mathbb{R}_+.$$

Then, the trivial solution of (1.1) is $\Psi-$ uniformly *stable over* $\mathbb{R}_+$.

Proof. We will use the Definition of $\Psi-$ uniform stability.
For a given $\varepsilon > 0$, we choose $\delta = \delta(\varepsilon) = [2dK(1+dM)\exp(dKM)]^{-1}\varepsilon$. Let $Z(t)$ be any solution of equation (1.1) which satisfies the inequality $\mid \Psi(t_0)Z(t_0) \mid < \delta$ for some $t_0 \in \mathbb{R}_+$.

From the hypothesis (1), the solution $Z(t)$ exists on $\mathbb{R}_+$ and satisfies the inequality $\mid \Psi(t)Z(t) \mid \le \delta$ for all $t \in [0, t_0]$.

From Lemma 5, [3], the vector function $z(t) = \mathcal{V}ec(Z(t))$ is a solution of the "*corresponding Kronecker product system associated with (1.1)*", i.e., of the system

$$z' = \left(I_d \otimes A(t) + B^T(t) \otimes I_d\right) z + \int_0^t f(t, s, z(s))ds, \tag{3.1}$$

where $f(t, s, z) = \mathcal{V}ec\left(F(t, s, Z)\right)$, on the same interval $\mathbb{R}_+$.

From Lemma 8, [3], we know that the matrix $U(t) = Y^T(t) \otimes X(t)$ is a fundamental matrix for the linear homogeneous system associated with the system (3.1), i.e., for the differential system

$$z' = \left(I_d \otimes A(t) + B^T(t) \otimes I_d\right) z. \tag{3.2}$$

Therefore, by the formula of variation of constants ([2, Ch. II, s. 2 (8)]),

$$z(t) = U(t)U^{-1}(t_0)z_0 + \int_{t_0}^t U(t)U^{-1}(s) \int_0^s f(s, u, z(u))duds,$$

for $t \geq t_0$, where $z_0 = \mathcal{V}ec(Z(t_0))$. Because (see Lemma 1, [3]) we have

$$\begin{aligned} U(t)U^{-1}(s) &= \left(Y^T(t) \otimes X(t)\right) \cdot \left(Y^T(s) \otimes X(s)\right)^{-1} \\ &= \left(Y^T(t) \left(Y^T\right)^{-1}(s)\right) \otimes \left(X(t)X^{-1}(s)\right), t \geq s, \end{aligned}$$

for $t \geq t_0$, we have that

$$\begin{aligned} z(t) = &\left[\left(Y^T(t) \left(Y^T\right)^{-1}(t_0)\right) \otimes \left(X(t)X^{-1}(t_0)\right)\right] z_0 \\ &+ \textstyle\int_{t_0}^t \left[\left(Y^T(t) \left(Y^T\right)^{-1}(s)\right) \otimes \left(X(t)X^{-1}(s)\right)\right] \int_0^s f(s, u, z(u))duds, \end{aligned}$$

We note that $w(t) = \left(I_d \otimes \Psi(t)\right) z(t)$, $t \geq t_0$ and

$$\Phi(t, s) = \left(Y^T(t) \left(Y^T\right)^{-1}(s)\right) \otimes \left(\Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s)\right), \text{ for all } t \geq s \geq 0.$$

As a result,

$$w(t) = \Phi(t, t_0)w(t_0) + \int_{t_0}^t \Phi(t, s) \int_0^s \left(I_d \otimes \Psi(s)\right) f(s, u, z(u))duds, \text{ for all } t \geq t_0,$$

and then,

$$||w(t)|| \leq | \Phi(t, t_0) | \, ||w(t_0)|| + \int_{t_0}^t | \Phi(t, s) | \int_0^s \| \left(I_d \otimes \Psi(s)\right) f(s, u, z(u)) \| \, duds, \tag{3.3}$$

for all $t \geq t_0$. From hypotheses and Lemma 6, [3], for all $s \geq u \geq 0$, we have

$$\begin{aligned} &\| \left(I_d \otimes \Psi(s)\right) f(s, u, z(u)) \| = \| \left(I_d \otimes \Psi(s)\right) \mathcal{V}ec(F(s, u, Z(u))) \| \\ &\leq | \Psi(s)F(s, u, Z(u)) | \leq f(s, u) \, | \Psi(u)Z(u) | \\ &\leq d f(s, u) \, \| \left(I_d \otimes \Psi(u)\right) \mathcal{V}ec(Z(u)) \| = d f(s, u)||w(u)||_{R^{d^2}}. \end{aligned}$$

Therefore, from (3.3) and the above results, for $t \geq t_0$,

$$\begin{aligned} ||w(t)|| &\leq K||w(t_0)|| + Kd\textstyle\int_{t_0}^{t}\int_0^s f(s,u)||w(u)||duds \\ &= K||w(t_0)|| + Kd\textstyle\int_{t_0}^{t}\int_0^{t_0} f(s,u)||w(u)||duds \\ &+Kd\textstyle\int_{t_0}^{t}\int_{t_0}^{s} f(s,u)||w(u)||duds \\ &\leq K||w(t_0)|| + Kd\delta\textstyle\int_{t_0}^{t}\int_0^{t_0} f(s,u)duds + Kd\int_{t_0}^{t}\int_{t_0}^{s} f(s,u)||w(u)||duds, \end{aligned}$$

and then

$$||w(t)|| \leq K\delta(1+dM) + Kd\int_{t_0}^{t}\int_{t_0}^{s} f(s,u)||w(u)||duds, \tag{3.4}$$

for all $t \geq t_0$. It is easy to see that the function

$$W(t) = \int_{t_0}^{t}\int_{t_0}^{s} f(s,u)||w(u)||duds,\ t \geq t_0,$$

is continuously differentiable and increasing on $[t_0,\infty)$. For $t \in [t_0,\infty)$, we have

$$\begin{aligned} W'(t) &= \textstyle\int_{t_0}^{t} f(t,u)||w(u)||du \\ &\leq \textstyle\int_{t_0}^{t} f(t,u)\left[K\delta(1+dM) + KdW(u)\right]du \\ &= K\delta(1+dM)\textstyle\int_{t_0}^{t} f(t,u)du + Kd\int_{t_0}^{t} f(t,u)W(u)du \end{aligned}$$

and then,

$$\begin{aligned} &\left[W(t)\exp\left(-Kd\textstyle\int_{t_0}^{t}\int_{t_0}^{s} f(s,u)duds\right)\right]' \\ &\quad= \left[W'(t) - KdW(t)\textstyle\int_{t_0}^{t} f(t,u)du\right]\cdot\exp\left(-Kd\textstyle\int_{t_0}^{t}\int_{t_0}^{s} f(s,u)duds\right) \\ &\quad\leq \left[K\delta(1+dM)\textstyle\int_{t_0}^{t} f(t,u)du + Kd\int_{t_0}^{t} f(t,u)\left(W(u)-W(t)\right)du\right]\cdot \\ &\qquad\cdot\exp\left(-Kd\textstyle\int_{t_0}^{t}\int_{t_0}^{s} f(s,u)duds\right) \\ &\quad\leq \left[K\delta(1+dM)\textstyle\int_{t_0}^{t} f(t,u)du\right]\cdot\exp\left(-Kd\textstyle\int_{t_0}^{t}\int_{t_0}^{s} f(s,u)duds\right) \\ &\quad= \left[-\delta(1+dM)d^{-1}\exp\left(-Kd\textstyle\int_{t_0}^{t}\int_{t_0}^{s} f(s,u)duds\right)\right]',\quad t \in [t_0,\infty). \end{aligned}$$

By a simple integration from t_0 to t, we get

$$\begin{aligned} &W(t)\exp\left(-Kd\textstyle\int_{t_0}^{t}\int_{t_0}^{s} f(s,u)duds\right) \\ &\qquad\leq -\delta(1+dM)d^{-1}\left[\exp\left(-Kd\textstyle\int_{t_0}^{t}\int_{t_0}^{s} f(s,u)duds\right) - 1\right], \end{aligned}$$

and then

$$W(t) \leq \delta(1+dM)d^{-1}\left[\exp\left(Kd\int_{t_0}^{t}\int_{t_0}^{s} f(s,u)duds\right) - 1\right],\ \text{for } t \in [t_0,\infty).$$

From this, (3.4) and hypothesis (3), we deduce that

$$\begin{aligned}||w(t)|| &\leq K\delta(1+dM)+KdW(t)\\ &\leq K\delta(1+dM)+Kd\cdot\delta(1+dM)d^{-1}\left[\exp\left(Kd\int_{t_0}^t\int_{t_0}^s f(s,u)duds\right)-1\right]\\ &\leq K\delta(1+dM)\exp\left(Kd\int_{t_0}^t\int_{t_0}^s f(s,u)duds\right)\leq K\delta(1+dM)\exp(dKM),\end{aligned}$$

for $t\in[t_0,\infty)$. From this and Lemma 6, [3], we deduce that

$$\mid\Psi(t)Z(t)\mid\leq d\delta K(1+dM)\exp(dKM)<\varepsilon,\ \text{for } t\in[t_0,\infty).$$

From Definition, it follows that the trivial solution of (1.1) is $\Psi-$ uniformly stable over $\mathbb{R}_+$. □

Theorem 3.2. *Suppose that:*

(1). the hypothesis (H) is fulfilled;

(2). there exists a constant $K>0$ *such that the fundamental matrix* $X(t)$ *for equation (1.2) satisfies the inequality*

$$\left|\Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s)\right|\leq K,\ \textit{for } t\geq s\geq 0;$$

(3). there exists $m=\inf_{t\in\mathbb{R}_+}\mid B(t)\mid>0$;

(4). the matrix function $F(t,s,Z)$ *satisfies the inequality*

$$\mid\Psi(t)F(t,s,Z)\mid\leq f(t,s)\mid\Psi(t)Z\mid,$$

for $(t,s,Z)\in D$, *where* $f(t,s)$ *and its partial derivatives* $f_t(t,s)$ *exists and are continuous functions for* $0\leq s\leq t<\infty$ *such that* $f(t,s)\geq 0$ *and* $f_t(t,s)\leq 0$;

(5). the functions $b(t)=\max\{K,\frac{K}{m}\}\mid B(t)\mid$ *and* $f(t,s)$ *satisfy the conditions*

$$\exists M>0:\quad\int_0^t\int_0^s f(s,u)duds\leq M,\ \textit{for all } t\in\mathbb{R}_+;$$

$$\exists P>0:\quad\int_0^t b(s)e^{\int_0^s(b(u)+f(u,u))du}ds\leq P,\ \textit{for all } t\in\mathbb{R}_+;$$

Then, the trivial solution of (1.1) is $\Psi-$ uniformly *stable over* $\mathbb{R}_+$.

Proof. We will use the Definition of $\Psi-$ uniform stability. For a given $\varepsilon>0$, we choose $\delta=\delta(\varepsilon)=\left[2K(1+M)(1+P)\right]^{-1}\varepsilon$. Let $Z(t)$ be any solution of equation (1.1) which satisfies the inequality $\mid\Psi(t_0)Z(t_0)\mid<\delta$ for some $t_0\in\mathbb{R}_+$.

From the hypothesis (1), the solution $Z(t)$ exists on $\mathbb{R}_+$ and satisfies the inequality $\mid\Psi(t)Z(t)\mid\leq\delta$ for all $t\in[0,t_0]$.

Obviously, the function $Z(t)$ is a solution of equation

$$Z'=A(t)Z+\left[ZB(t)+\int_0^t F(t,s,Z(s))ds\right],\ t\geq t_0.$$

Therefore, by the formula of variation of constants ([2, Ch. II, s. 2 (8))],

$$Z(t) = X(t)X^{-1}(t_0)Z(t_0) + \int_{t_0}^{t} X(t)X^{-1}(s)\left[Z(s)B(s) + \int_0^s F(s,u,Z(u))du\right]ds, \tag{3.5}$$

for $t \geq t_0$. From (3.5), multiplying by $\Psi(t)$ and noting $w(t) =\mid \Psi(t)Z(t) \mid$ for $t \geq 0$ and $\Phi(t,s) = \Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s)$, for all $t \geq s \geq 0$, we obtain

$$\begin{aligned} w(t) &\leq \mid \Phi(t,t_0) \mid w(t_0) \\ &+ \textstyle\int_{t_0}^{t} \mid \Phi(t,s) \mid \left[\mid B(s) \mid w(s) + \int_0^s \mid \Psi(s)F(s,u,Z(u)) \mid du\right] ds, \text{ for } t \geq t_0. \end{aligned} \tag{3.6}$$

From hypothesis (4),

$$\mid \Psi(s)F(s,u,Z(u)) \mid\leq f(s,u) \mid \Psi(u)Z(u) \mid, \text{ for } s \geq u \geq 0.$$

As a result, the above inequality (3.6) becomes

$$w(t) \leq Kw(t_0) + K\int_{t_0}^{t}\left[\mid B(s) \mid w(s) + \int_0^s f(s,u)w(u)du\right]ds, \text{ for } t \geq t_0. \tag{3.7}$$

From hypothesis (3), we have $\mid B(s) \mid\geq m > 0$, for all t$\in \mathbb{R}_+$. Since $w(t) =\mid \Psi(t)Z(t) \mid\leq \delta$ for all $t \in [0,t_0]$, the last term from inequality (3.7) can be written as

$$\begin{aligned} &K\textstyle\int_{t_0}^{t}\left(\int_0^s f(s,u)w(u)du\right)ds \\ &= K\textstyle\int_{t_0}^{t}\left(\int_0^{t_0} f(s,u)w(u)du\right)ds + K\int_{t_0}^{t}\left(\int_{t_0}^{s} f(s,u)w(u)du\right)ds \\ &\leq \delta K\textstyle\int_{t_0}^{t}\int_0^{t_0} f(s,u)duds + K\int_{t_0}^{t}\frac{|B(s)|}{m}\int_{t_0}^{s} f(s,u)w(u)duds \\ &\leq \delta KM + \textstyle\frac{K}{m}\int_{t_0}^{t} \mid B(s) \mid \int_{t_0}^{s} f(s,u)w(u)duds, \text{ for } t \geq t_0. \end{aligned}$$

Now, from (3.7) and the above results, the function $w(t)$ satisfies the inequality

$$w(t) \leq \delta K(1+M) + \int_{t_0}^{t} b(s)w(s)ds + \int_{t_0}^{t} b(s)\left(\int_{t_0}^{s} f(s,u)w(u)du\right)ds, \tag{3.8}$$

for $t \geq t_0$. From Theorem 2.1, [15], it follows that

$$w(t) \leq \delta K(1+M)\left[1 + \int_{t_0}^{t} b(s)e^{\int_{t_0}^{s}(b(u)+f(u,u))du}ds\right], \text{ for } t \geq t_0. \tag{3.9}$$

From (3.9), Lemma 6, [3], and hypothesis (5), it follows that

$$\mid \Psi(t)Z(t) \mid< \varepsilon, \text{ for } t \geq t_0.$$

From the Definition, the trivial solution of (1.1) is $\Psi-$ uniformly stable over $\mathbb{R}_+$. □

Theorem 3.3. *Suppose that:*

1. the hypothesis (H) is fulfilled;

2. there exists a constant $K > 0$ *such that the fundamental matrix* $Y(t)$ *for equation (1.3) satisfies the inequality*

$$\mid \left(Y^T(t)\left(Y^T\right)^{-1}(s)\right) \otimes \left(\Psi(t)\Psi^{-1}(s)\right) \mid\leq K, \text{ for } t \geq s \geq 0;$$

(3). there exists $m = \inf_{t \in \mathbb{R}_+} \mid \Psi(t)A(t)\Psi^{-1}(t) \mid > 0;$

4. the matrix function $F(t, s, Z)$ *satisfies the inequality*

$$\mid \Psi(t)F(t, s, Z) \mid\leq f(t, s) \mid \Psi(t)Z \mid,$$

for $(t, s, Z) \in D$, *where* $f(t, s)$ *and its partial derivatives* $f_t(t, s)$ *exists and are continuous functions for* $0 \leq s \leq t < \infty$ *such that* $f(t, s) \geq 0$ *and* $f_t(t, s) \leq 0$;

(5). the functions $a(t) = \max\{K, \frac{dK}{m}\} \mid \Psi(t)A(t)\Psi^{-1}(t) \mid$ *and* $f(t, s)$ *satisfy the conditions*

$$\exists M > 0 : \quad \int_0^t \int_0^s f(s, u)duds \leq M, \text{ for all } t \in \mathbb{R}_+;$$

$$\exists P > 0 : \quad \int_0^t a(s)e^{\int_0^s (a(u)+f(u,u))du}ds \leq P, \text{ for all } t \in \mathbb{R}_+;$$

Then, the trivial solution of (1.1) is $\Psi-$ uniformly *stable over* $\mathbb{R}_+$.

Proof. We will use the Definition of $\Psi-$ uniform stability. For a given $\varepsilon > 0$, we choose

$$\delta = \delta(\varepsilon) = [2dK(1 + dM)(1 + P)]^{-1}\,\varepsilon.$$

Let $Z(t)$ be any solution of equation (1.1) which satisfies the inequality $\mid \Psi(t_0)Z(t_0) \mid< \delta$ for some $t_0 \in \mathbb{R}_+$. From the hypothesis (1), the solution $Z(t)$ exists on $\mathbb{R}_+$ and satisfies the inequality $\mid \Psi(t)Z(t) \mid\leq \delta$ for all $t \in [0, t_0]$. From Lemma 5, [3], the vector function $z(t) = \mathcal{V}ec(Z(t))$ is a solution of the "*corresponding Kronecker product system associated with (1.1)*", i.e., of the system

$$z' = \left(I_d \otimes A(t) + B^T(t) \otimes I_d\right) z + \int_0^t f(t, s, z(s))ds, \tag{3.10}$$

where $f(t, s, z) = \mathcal{V}ec\,(F(t, s, Z))$, on the same interval $\mathbb{R}_+$. This system can be written in the form

$$z' = \left(B^T(t) \otimes I_d\right) z + \left[(I_d \otimes A(t))\, z + \int_0^t f(t, s, z(s))ds\right], \ t \geq t_0.$$

From Lemma 8, [3], we know that the matrix $U(t) = Y^T(t) \otimes I_d$ is a fundamental matrix for the linear homogeneous system

$$z' = \left(B^T(t) \otimes I_d\right) z \tag{3.11}$$

Therefore, by the formula of variation of constants ([2, Ch. II, s. 2 (8))],

$$z(t) = U(t)U^{-1}(t_0)z_0 + \int_{t_0}^t U(t)U^{-1}(s)\left[(I_d \otimes A(s))z(s) + \int_0^s f(s, u, z(u))du\right] ds,$$

for $t \geq t_0$, where $z_0 = \mathcal{V}ec(Z(t_0))$. From Lemma 1, [3], we have

$$U(t)U^{-1}(s) = \left(Y^T(t) \otimes I_d\right) \cdot \left(Y^T(s) \otimes I_d\right)^{-1} = \left(Y^T(t)\left(Y^T\right)^{-1}(s)\right) \otimes I_d,\ t \geq s \geq 0.$$

Then, for $t \geq t_0$, we infer

(3.12)

$$z(t) = \left[\left(Y^T(t)\left(Y^T\right)^{-1}(t_0)\right) \otimes I_d\right] z_0$$
$$+ \textstyle\int_{t_0}^t \left[\left(Y^T(t)\left(Y^T\right)^{-1}(s)\right) \otimes I_d\right] \cdot \left[\left(I_d \otimes A(s)\right) z(s) + \int_0^s f(s,u,z(u))du\right] ds.$$

Define the vector function $w(t)$ by setting $w(t) = \left(I_d \otimes \Psi(t)\right) z(t),\ t \geq 0$ and the matrix function $\Phi(t,s) = \left(Y^T(t)\left(Y^T\right)^{-1}(s)\right) \otimes \left(\Psi(t)\Psi^{-1}(s)\right)$, for $t \geq s \geq 0$. From (3.12), we have (see Lemma 1, [3])

$$w(t) = \Phi(t,t_0)w(t_0)$$
$$+ \textstyle\int_{t_0}^t \Phi(t,s)\left[\left(I_d \otimes \left(\Psi(s)A(s)\Psi^{-1}(s)\right)\right) w(s) + \int_0^s \left(I_d \otimes \Psi(s)\right) f(s,u,z(u))du\right] ds,$$

for $t \geq t_0$. As a result, for $t \geq t_0$,

(3.13)

$$||w(t)|| \leq | \Phi(t,t_0) | \, ||w(t_0)||$$
$$+ \textstyle\int_{t_0}^t | \Phi(t,s) | \left[| \Psi(s)A(s)\Psi^{-1}(s) | \, ||w(s)|| + \int_0^s \| \left(I_d \otimes \Psi(s)\right) f(s,u,z(u)) \| \, du\right] ds.$$

From hypotheses and Lemma 6, [3],

$$\begin{aligned} &\| \left(I_d \otimes \Psi(s)\right) f(s,u,z(u)) \| = \| \left(I_d \otimes \Psi(s)\right) \mathcal{V}ec(F(s,u,Z(u))) \| \\ &\leq | \Psi(s)F(s,u,Z(u)) | \leq f(s,u) \, | \Psi(u)Z(u) | \leq \\ &\leq df(s,u) \, \| \left(I_d \otimes \Psi(u)\right) \mathcal{V}ec(Z(u)) \| = df(s,u)||w(u)||_{R^{d^2}}, \end{aligned} \tag{3.14}$$

for $s \geq t_0,\ s \geq u \geq 0$. From (3.13), (3.14) and hypotheses, the continuous function $||w(t)||$ satisfies the inequality

$$||w(t)|| \leq K||w(t_0)|| + K\int_{t_0}^t \left[\tilde{a}(s)||w(s)|| + \int_0^s df(s,u)||w(u)||du\right] ds,\ t \geq t_0, \tag{3.15}$$

where $\tilde{a}(s) = | \Psi(s)A(s)\Psi^{-1}(s) |$, $s \in \mathbb{R}_+$. Because

$$||w(u)|| = ||(I_d \otimes \Psi(u))\mathcal{V}ec(Z(u))|| \leq | \Psi(u)Z(u) | \leq \delta \text{ for } u \in [0,t_0],$$

the last term from the above inequality can be written as

$$\begin{aligned} &K \textstyle\int_{t_0}^t \left(\int_0^s df(s,u)||w(u)||du\right) ds \\ &= K \textstyle\int_{t_0}^t \left(\int_0^{t_0} df(s,u)||w(u)||du\right) ds + K \int_{t_0}^t \left(\int_{t_0}^s df(s,u)||w(u)||du\right) ds \\ &\leq d\delta K \textstyle\int_{t_0}^t \int_0^{t_0} f(s,u)duds + dK \int_{t_0}^t \int_{t_0}^s f(s,u)||w(u)||duds \\ &\leq d\delta KM + dK \textstyle\int_{t_0}^t \int_{t_0}^s f(s,u)||w(u)||duds. \end{aligned}$$

From (3.15), hypotheses and the above results, $||w(t)||$ satisfies the inequality

$$||w(t)|| \leq \delta K(1+dM) + \int_{t_0}^{t} a(s)||w(s)||ds + \int_{t_0}^{t} a(s)\left(\int_{t_0}^{s} f(s,u)||w(u)||du\right) ds, \tag{3.16}$$

for $t \geq t_0$. From Theorem 2.1, [15], it follows that

$$||w(t)|| \leq \delta K(1+dM)\left[1 + \int_{t_0}^{t} a(s)e^{\int_{t_0}^{s}(a(u)+f(u,u))du} ds\right], \text{ for } t \geq t_0. \tag{3.17}$$

From hypothesis (5), it follows that

$$||w(t)|| \leq \delta K(1+dM)(1+P), \text{ for } t \geq t_0. \tag{3.18}$$

From Lemma 6, [3], we get

$$\mid \Psi(t)Z(t) \mid \leq d||w(t)|| \leq d\delta K(1+dM)(1+P) < \varepsilon, \text{ for } t \geq t_0.$$

From the Definition, the trivial solution of (1.1) is $\Psi-$ uniformly stable over $\mathbb{R}_+$. □

Remark 3.1. The above Theorems generalize the results from Theorem 3.1, [10] and Theorem 4.5, [8], from differential systems to Lyapunov matrix differential equations.

Remark 3.2. The above Theorems have very useful corollaries in the particular cases when $f(t,s) = h(t)g(s)$ or $f(t,s) = k(t-s)$.

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