

New connections on the fiber-bundle of generalized statistical manifolds

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Abstract. In this paper we construct a family of α -connections on a fiber-bundle of a generalized statistical manifold. We prove that the exponential and mixture connections are curvature-free and we investigate the associated parallel transport.

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Key words: Riemannian manifold; generalized statistical manifold; fiber-bundle; α -connection; parallel transport.

1 Introduction

Information geometry investigates the differential-geometric structure of statistical models and has many applications in statistical inference or machine learning for example (see [4]). Since the seminal work of Rao[14] where Fisher information is viewed as a Riemannian metric on a probability distributions space, statistical manifolds have been widely studied. The Fisher information metric on statistical manifold is related to the Kullback-Leibler divergence which is a measure of dissimilarity between two probability distributions. Considering a family of α -divergences, Amari[2] proposed a family of α -connections on statistical manifolds. To elucidate the structures and properties of estimating functions, Amari and Kumon[3] constructed a family of α -connections on Hilbert bundle of statistical manifold, endowed with the Fisher information metric. Vigelis et al.[17] introduced a new metric and α -connections using u_0 -mappings (or φ -functions), which generalize Fisher information metric and Amari's α -connections. The obtained geometric structure is called a generalized statistical manifold. Recently, de Andrade et al.[5] investigated the mixture and exponential arcs on generalized statistical manifold.

In this paper, we extend the results of Vigelis et al.[17] by defining a new family of α -connections on Hilbert bundle of generalized statistical manifold. We prove that the curvatures of our proposed (1)-connection and (-1)-connection vanish everywhere. Moreover, we give the α -parallel transport associated with α -connection for $\alpha = \pm 1$.

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The rest of the paper is organized as follows. In section 2, we review the relevant concepts related to generalized statistical manifold. In section 3 we introduce our new α -connection on Hilbert bundle of generalized statistical manifold and prove the mains results.

2 Generalized statistical manifolds and α -connections

In this section, we recall some useful definitions and properties related to generalized statistic manifolds (see[17, 10, 16]). Let (χ, Σ, μ) be a measure space and $P_\mu = \{p \in L^0 : p > 0, \int_\chi p(x; \theta) d\mu(x) = 1\}$ where L^0 denotes the set of all real-valued, measurable functions on χ .

Definition 2.1. (see[17]) Let $u_0 : \chi \rightarrow (0, \infty)$ be a measurable function. A function $\phi : \mathbb{R} \rightarrow (0, \infty)$ is said to be a u_0 -mapping if :

- ϕ is convex,
- $\lim_{x \rightarrow -\infty} \phi(x) = 0$ and $\lim_{x \rightarrow \infty} \phi(x) = \infty$,
- for all measurable function $c : \chi \rightarrow \mathbb{R}$ satisfying $\int_\chi \phi(c(x)) d\mu(x) = 1$, we have $\int_\chi \phi(c(x) + \lambda u_0(x)) d\mu(x) < \infty$, for all $\lambda > 0$.

Example 2.2. The function ϕ defined by

$$\phi(x) = \exp(ax + b), a \in (0; \infty), b \in \mathbb{R}, \quad \forall x \in \mathbb{R}$$

is a 1_χ -mapping.

Example 2.3. (see[16]) The Kaniadakis' κ -exponential $exp_\kappa : \mathbb{R} \rightarrow (0, \infty)$ defined by

- for $\kappa = 0$, exp_κ is the usual exponential map,
- for $\kappa \in [-1, 0] \cup]0, 1]$, $exp_\kappa(x) = (\kappa x + \sqrt{1 + \kappa^2 x^2})^{1/\kappa}$

is a u_0 -mapping where u_0 satisfies $\int_\chi exp_\kappa(u_0) d\mu < +\infty$.

Definition 2.4. (see [17]) Let ϕ be a smooth u_0 -mapping. A generalized statistical manifold is a family of probability distributions

$$M = \{p(\cdot; \theta) : \theta \in \Theta\} \subset P_\mu$$

such that:

1. Θ is an open and connected set in \mathbb{R}^n .
2. Each $p(\cdot, \theta)$ is given in terms of $\theta \in \Theta$ by a one to one mapping.
3. Every function $p(x; \cdot)$ is smooth for all x and the operations of integration with respect to μ and differentiation with respect to θ^i (i.e. $\partial/\partial\theta^i$) are always commutative.
4. The support of $p(\cdot, \theta)$ does not depend on θ for all $\theta \in \Theta$.

5. The matrix $g = (g_{ij})$, which is defined by

$$g_{ij} = -\mathbb{E}'_{\theta} \left[\frac{\partial^2 f_{\theta}}{\partial \theta^i \partial \theta^j} \right]$$

is positive definite at each $\theta \in \Theta$, where $f_{\theta}(\cdot) = \phi^{-1}(p(\cdot; \theta))$ and

$$(2.1) \quad \mathbb{E}'_{\theta} [\cdot] = \frac{\int_{\mathcal{X}} (\cdot) \phi^{(1)}(f_{\theta}) d\mu}{\int_{\mathcal{X}} u_0 \phi^{(1)}(f_{\theta}) d\mu}.$$

g_{ij} is invariant under reparametrization. When ϕ is the usual exponential function and $u_0 = 1$, g is the Fisher information matrix.

Lemma 2.1. [17] For $i, j \in \{1, 2, \dots\}$ and $\theta \in M$,

$$\mathbb{E}'_{\theta} \left[\frac{\partial f_{\theta}}{\partial \theta^i} \right] = 0 \quad \text{and} \quad g_{ij} = \mathbb{E}''_{\theta} \left[\frac{\partial f_{\theta}}{\partial \theta^i} \frac{\partial f_{\theta}}{\partial \theta^j} \right],$$

where

$$(2.2) \quad \mathbb{E}''_{\theta} [\cdot] = \frac{\int_{\mathcal{X}} (\cdot) \phi^{(2)}(f_{\theta}) d\mu}{\int_{\mathcal{X}} u_0 \phi^{(1)}(f_{\theta}) d\mu}.$$

Let $\partial_i = \partial/\partial\theta^i$ be the tangent vector of the i -th coordinate curve θ^i at the point θ . Then, n such tangent vectors ∂_i , $i = 1, \dots, n$, span the tangent space $\mathbb{T}_{\theta}M$ at the point θ of the manifold M . Any tangent vector $A \in \mathbb{T}_{\theta}M$ is a linear combination of the basis vectors ∂_i ,

$$A = A^i \partial_i,$$

where A^i are the components of vector A and Einstein's summation convention is assumed throughout the paper. The tangent space $\mathbb{T}_{\theta}M$ is a linearized version of a small neighborhood of θ in M and an infinitesimal vector $d\theta = d\theta^i \partial_i$ denotes the vector connecting two neighboring points θ and $\theta + d\theta$ or two neighboring distributions $p(\cdot, \theta)$ and $p(\cdot, \theta + d\theta)$. Let us introduce a metric in the tangent space $\mathbb{T}_{\theta}M$. It can be done by defining the inner product of two basis vectors ∂_i and ∂_j . Usually the vector $\partial_i \in \mathbb{T}_{\theta}M$ is represented by a function $\partial_i f_{\theta}$ and the metric is defined by

$$(2.3) \quad g_{ij}(\theta) = g(\partial_i, \partial_j) = \langle \partial_i f_{\theta}, \partial_j f_{\theta} \rangle = \mathbb{E}''_{\theta} [\partial_i f_{\theta} \partial_j f_{\theta}], \forall \theta \in M$$

The α -covariant derivative (see [17]) of the basis vector ∂_j in the direction ∂_i is

$$(2.4) \quad \langle \nabla_{\partial_i}^{(\alpha)} \partial_j, \partial_k \rangle = \Gamma_{ij,k}^{(\alpha)},$$

where

$$\Gamma_{ij,k}^{(\alpha)} = \frac{1 + \alpha}{2} \Gamma_{ij,k}^{(1)} + \frac{1 - \alpha}{2} \Gamma_{ij,k}^{(-1)},$$

$$\Gamma_{ij,k}^{(1)} = \mathbb{E}''_{\theta} \left[\frac{\partial^2 f_{\theta}}{\partial \theta^i \partial \theta^j} \frac{\partial f_{\theta}}{\partial \theta^k} \right] - \mathbb{E}'_{\theta} \left[\frac{\partial^2 f_{\theta}}{\partial \theta^i \partial \theta^j} \right] \mathbb{E}''_{\theta} \left[u_0 \frac{\partial f_{\theta}}{\partial \theta^k} \right],$$

$$\begin{aligned} \Gamma_{ij,k}^{(-1)} &= \mathbb{E}_\theta'' \left[\frac{\partial^2 f_\theta}{\partial \theta^i \partial \theta^j} \frac{\partial f_\theta}{\partial \theta^k} \right] + \mathbb{E}_\theta''' \left[\frac{\partial f_\theta}{\partial \theta^i} \frac{\partial f_\theta}{\partial \theta^j} \frac{\partial f_\theta}{\partial \theta^k} \right] \\ &\quad - \mathbb{E}_\theta'' \left[\frac{\partial f_\theta}{\partial \theta^j} \frac{\partial f_\theta}{\partial \theta^k} \right] \mathbb{E}_\theta'' \left[u_0 \frac{\partial f_\theta}{\partial \theta^i} \right] - \mathbb{E}_\theta'' \left[\frac{\partial f_\theta}{\partial \theta^i} \frac{\partial f_\theta}{\partial \theta^k} \right] \mathbb{E}_\theta'' \left[u_0 \frac{\partial f_\theta}{\partial \theta^j} \right] \end{aligned}$$

and

$$\mathbb{E}_\theta''' [\cdot] = \frac{\int_\chi (\cdot) \phi^{(3)}(f_\theta) d\mu}{\int_\chi u_0 \phi^{(1)}(f_\theta) d\mu}.$$

3 The Hilbert bundle of a generalized statistical manifold

3.1 Hilbert bundle

Let Υ be the set of μ -integrable and smooth functions r defined from χ to \mathbb{R} . Let ϕ be a smooth and bijective u_0 -mapping $\phi : \mathbb{R} \rightarrow (0, \infty)$ such that $\forall (\theta, x) \in M \times (0, \infty)$, $\phi^{(1)}(x) \neq 0$ and $M = \{p(\cdot; \theta); \theta = (\theta^1, \dots, \theta^n) \in \Theta \subseteq \mathbb{R}^n\}$ a generalized statistical manifold endowed with the Riemannian metric g defined by (2.3) and parametrized by $\theta = (\theta^1, \dots, \theta^n)$. To each point $\theta \in M$, we associate a linear space H_θ defined by

$$H_\theta = \{r \in \Upsilon : \mathbb{E}_\theta'[r] = 0, \mathbb{E}_\theta''[r^2] < +\infty\},$$

where \mathbb{E}_θ' and \mathbb{E}_θ'' are respectively defined by relations (2.1) and (2.2). Throughout this paper we assume that $\mathbb{E}_\theta''[(\partial_i f_\theta)^2] < \infty$ and for all $(\theta, \theta') \in M^2$, $r \in H_\theta$

$$E_\theta''[r^2] < +\infty \implies \mathbb{E}_\theta'' \left[r^2 \left(\frac{\phi'(f_\theta)}{\phi'(f_{\theta'})} \right)^2 \right] < +\infty.$$

For each $\theta \in M$ and $r, s \in H_\theta$ we set

$$(3.1) \quad \langle r, s \rangle_\theta := \mathbb{E}_\theta''[rs].$$

Proposition 3.1. *For all $\theta \in M$, $\langle \cdot, \cdot \rangle_\theta$ is an inner product and $(H_\theta, \langle \cdot, \cdot \rangle_\theta)$ is a Hilbert space.*

Proof. Let $\theta \in M$. H_θ is a vector space and the map $\langle \cdot, \cdot \rangle_\theta$ defined by (3.1) is a positive definite bilinear form, then it is an inner product on H_θ . Using the Walter's proof (see [18]) of completeness of the set of measurable and square integrable functions, one proves the completeness of H_θ . \square

Since the tangent vectors $\partial_i f_\theta(x)$, which span $\mathbb{T}_\theta M$, satisfy $\mathbb{E}_\theta'[\partial_i f_\theta] = 0$ and $\mathbb{E}_\theta''[(\partial_i f_\theta)^2] < +\infty$, they belong to H_θ . Indeed, the tangent space $\mathbb{T}_\theta M$ of M at θ is a linear subspace of H_θ , and the inner product defined in $\mathbb{T}_\theta M$ is compatible with that in H_θ . Let N_θ be the orthogonal complement of $\mathbb{T}_\theta M$ in H_θ . Then, H_θ is decomposed into the direct sum

$$H_\theta = \mathbb{T}_\theta M \oplus N_\theta.$$

The aggregate of all H_θ 's attached to every $\theta \in M$ with a suitable topology,

$$H = \cup_{\theta \in M} H_\theta,$$

is called the fiber-bundle with base space M . Since the fiber space is a Hilbert space, it is called a Hilbert bundle of M . It should be noted that H_θ and $H_{\theta'}$ are different Hilbert spaces when $\theta \neq \theta'$.

Hence it is convenient to establish a one-to-one correspondence between H_θ and $H_{\theta'}$, when θ and θ' are neighboring points in M . When the correspondence is affine, it is called an affine connection. Let us assume that a vector $r \in H_\theta$ at θ corresponds to $r + dr \in H_{\theta+d\theta}$ at a neighboring points $\theta + d\theta$, where d denotes infinitesimally small change.

Lemma 3.2. *Let $\theta \in M$ and $r \in H_\theta$. Then*

$$\mathbb{E}'_\theta[dr] = -\mathbb{E}''_\theta[\partial_i f_\theta r] d\theta^i + o(\|d\theta\|).$$

Proof. Let $\theta \in M$ and $r \in H_\theta$. Then $r + dr \in H_{\theta+d\theta}$. Set $\Phi_x(\theta) = \phi^{(1)}(f_\theta(x))$, $x \in \chi$. The function Φ_x is differentiable on Θ . Then by Taylor expansion of the function $\theta \mapsto \Phi_x(\theta + d\theta)$ we obtain

$$\Phi_x(\theta + d\theta) = \Phi_x(\theta) + d_\theta \Phi_x(d\theta) + o(\|d\theta\|)$$

where $d_\theta \Phi_x$ denotes the differential of Φ_x at θ . Thus

$$\begin{aligned} r + dr \in H_{\theta+d\theta} &\Rightarrow \mathbb{E}'_{\theta+d\theta}[r + dr] = 0 \\ &\Rightarrow \int_\chi [r(x) + dr(x)] \Phi_x(\theta + d\theta) d\mu(x) = 0 \\ &\Rightarrow \int_\chi [r(x) + dr(x)] (\Phi_x(\theta) + d_\theta \Phi_x(d\theta) + o(\|d\theta\|)) d\mu(x) = 0 \\ &\Rightarrow \mathbb{E}'_\theta[dr] + \mathbb{E}''_\theta[r \partial_i f_\theta] d\theta^i \\ (3.2) \quad &+ o(\|d\theta\|) \int_\chi [r(x) + dr(x)] d\mu(x) = 0. \end{aligned}$$

Using $\int_\chi |r(x) + dr(x)| d\mu(x) < \infty$ and (3.2) we deduce $\mathbb{E}'_\theta[dr] = -\mathbb{E}''_\theta[\partial_i f_\theta r] d\theta^i + o(\|d\theta\|)$. \square

Next, we construct the α -connections on Hilbert bundle.

3.2 α -connections on Hilbert bundles

Given $r \in H_\theta$, differentiating the identity $\mathbb{E}'_\theta[r] = 0$ with respect to θ , we have $\mathbb{E}'_\theta[\partial_i r] = -\mathbb{E}''_\theta[\partial_i f_\theta r]$, $\mathbb{E}'[r] = \mathbb{E}'_\theta[\partial_i f_\theta] = 0$. Set

$$dr = \frac{1 + \alpha}{2} \mathbb{E}'_\theta[\partial_i r] u_0 d\theta^i - \frac{1 - \alpha}{2} \left[\frac{\phi^{(2)}(f_\theta)}{\phi^{(1)}(f_\theta)} \partial_i f_\theta r - \mathbb{E}''_\theta[u_0 \partial_i f_\theta] r \right] d\theta^i.$$

We have

$$\begin{aligned}
\mathbb{E}'_{\theta}[dr] &= \mathbb{E}' \left[\frac{1+\alpha}{2} \mathbb{E}'_{\theta}[\partial_i r] u_0 d\theta^i - \frac{1-\alpha}{2} \frac{\phi^{(2)}(f_{\theta})}{\phi^{(1)}(f_{\theta})} \partial_i f_{\theta} r d\theta^i \right] \\
&= \frac{1+\alpha}{2} \mathbb{E}'_{\theta}[\partial_i r] d\theta^i - \frac{1-\alpha}{2} \mathbb{E}' \left[\frac{\phi^{(2)}(f_{\theta})}{\phi^{(1)}(f_{\theta})} \partial_i f_{\theta} r d\theta^i \right] \\
(3.3) \quad &= \frac{1+\alpha}{2} \mathbb{E}'_{\theta}[\partial_i r] d\theta^i - \frac{1-\alpha}{2} \mathbb{E}'' [\partial_i f_{\theta} r] d\theta^i \\
&= -\mathbb{E}'' [\partial_i f_{\theta} r] d\theta^i.
\end{aligned}$$

The α -connection is given by the following α -covariant derivative $\bar{\nabla}^{(\alpha)}$. Let r be a vector field, which attaches a vector $r(\cdot, \theta)$ to every point $\theta \in M$. Then, the rate of the intrinsic change of the vector $r(\cdot, \theta)$ as θ changes in the direction ∂_i is given by the α -covariant derivative:

$$(3.4) \quad \bar{\nabla}_{\partial_i}^{(\alpha)} r = \partial_i r - \frac{1+\alpha}{2} \mathbb{E}'_{\theta}[\partial_i r] u_0 + \frac{1-\alpha}{2} \left[\frac{\phi^{(2)}(f_{\theta})}{\phi^{(1)}(f_{\theta})} \partial_i f_{\theta} r - \mathbb{E}''[u_0 \partial_i f_{\theta}] r \right].$$

This is a generalization of the α -connection studied by Amari[1]. We have $\mathbb{E}'_{\theta}[\bar{\nabla}_{\partial_i}^{(\alpha)} r] = 0$ because of the identity $\partial_i \mathbb{E}'_{\theta}[r] = \mathbb{E}'_{\theta}[\partial_i r] + \mathbb{E}''_{\theta}[\partial_i f_{\theta} r]$. The α -covariant derivative in the direction $A = A^i \partial_i \in \mathbb{T}_{\theta} M$ is given by

$$\bar{\nabla}_A^{(\alpha)} r = A^i \bar{\nabla}_{\partial_i}^{(\alpha)} r.$$

The 1-connection is called the exponential connection and the -1 -connection is called the mixture connection.

For each point $\theta \in M$, the tangent space $\mathbb{T}_{\theta} M$ is a subset of the Hilbert space H_{θ} . Hence the tangent bundle of M ,

$$\mathbb{T}M = \cup_{\theta \in M} \mathbb{T}_{\theta} M,$$

is a subset of H . We can define an affine connection in $\mathbb{T}M$ by introducing an affine correspondence between $\mathbb{T}_{\theta} M$ and $\mathbb{T}_{\theta'} M$ for neighboring points θ and θ' .

An affine connection given such that $r \in H_{\theta}$ corresponds to $r + dr \in H_{\theta+d\theta}$, induces an affine connection in $\mathbb{T}_{\theta} M$ such that $r \in \mathbb{T}_{\theta} M \subset H_{\theta}$ corresponds to the orthogonal projection of $r + dr \in H_{\theta+d\theta}$ onto $\mathbb{T}_{\theta+d\theta} M$.

3.3 Parallel transport on Hilbert bundles

We start this subsection by the following definition.

Definition 3.1. Let $c = \{c(t), t \in [0, 1]\}$ be a curve in M . A vector field $r(\cdot, t) \in H_{c(t)}$ defined along the curve c is said to be α -parallel, when

$$(3.5) \quad \bar{\nabla}_{\dot{c}}^{(\alpha)} r = \dot{r} - \frac{1+\alpha}{2} \mathbb{E}'_c[\dot{r}] u_0 + \frac{1-\alpha}{2} \left[\frac{\phi^{(2)}(f_c)}{\phi^{(1)}(f_c)} \dot{f}_c r - \mathbb{E}''_c[u_0 \dot{f}_c] r \right] = 0,$$

where \dot{r} denotes $\partial r / \partial t$, etc.

Definition 3.2. A vector $r_1(\cdot) \in H_{\theta_1}$ is the α -parallel transport of $r_0(\cdot) \in H_{\theta_0}$ along a curve $c = \{c(t), t \in [0, 1]\}$ connecting $\theta_0 = c(0)$ and $\theta_1 = c(1)$, when $r_0(\cdot) = r(\cdot, 0)$ and $r_1(\cdot) = r(\cdot, 1)$ in the solution $r(\cdot, \cdot)$ of (3.5).

Generally, the parallel transport along a curve c connecting θ to θ' depends on c .

Theorem 3.3. (see [12]) For an affine connection, parallel transport is independent of the path if and only if the curvature tensor vanishes.

Now, we investigate the e - and m -parallel transport operators from H_θ to $H_{\theta'}$, for $(\theta, \theta') \in M^2$. Then we can prove the following important theorem.

Theorem 3.4. Let ${}^{(e)}\pi_\theta^{\theta'}$ and ${}^{(m)}\pi_\theta^{\theta'}$ be the e - and m -parallel transport operators from H_θ to $H_{\theta'}$. Then

$$\begin{aligned} {}^{(e)}\pi_\theta^{\theta'} r(x) &= r(x) - \mathbb{E}'_{\theta'}[r]u_0(x), \\ {}^{(m)}\pi_\theta^{\theta'} r(x) &= \frac{r(x)\phi^{(1)}(f_\theta(x)) \int_{\mathcal{X}} u_0\phi^{(1)}(f_{\theta'})d\mu}{\phi^{(1)}(f_{\theta'}(x)) \int_{\mathcal{X}} u_0\phi^{(1)}(f_\theta)d\mu}. \end{aligned}$$

Proof. Let $c = \{c(t), t \in [0, 1]\}$ be a curve connecting two points $\theta = c(0)$ and $\theta' = c(1)$. Let $r^{(\alpha)}(x, t)$ be an α -parallel transport vector defined along the curve c . Then, it satisfies (3.5). When $\alpha = 1$, it is reduced to

$$(3.6) \quad \frac{\dot{r}^{(e)}(x, t)}{u_0(x)} = \mathbb{E}'_{c(t)} \left[\dot{r}^{(e)}(\cdot, t) \right].$$

Since the right-hand side

$$\mathbb{E}'_{c(t)} \left[\dot{r}^{(e)}(\cdot, t) \right] = \frac{\int_{\mathcal{X}} \dot{r}^{(e)}(x, t)\phi^{(1)}(f_{c(t)}(x)) d\mu(x)}{\int_{\mathcal{X}} u_0(x)\phi^{(1)}(f_{c(t)}(x)) d\mu(x)}$$

of (3.6) does not depend on x , its solution (with the initial condition $r(x) = r^{(e)}(x, 0)$) is given by

$$r^{(e)}(x, t) = r(x) + a(t)u_0(x)$$

where

$$a(t) = -\mathbb{E}'_{c(t)}[r].$$

Then

$${}^{(e)}\pi_\theta^{\theta'} r(x) = r(x) - \mathbb{E}'_{\theta'}[r]u_0(x).$$

When $\alpha = -1$, (3.5) is reduced to

$$\dot{r}^{(m)}(x, t) + \left\{ \frac{\phi^{(2)}(f_{c(t)}(x))}{\phi^{(1)}(f_{c(t)}(x))} \dot{f}_{c(t)}(x) - \mathbb{E}''_{c(t)}[u_0 \dot{f}_{c(t)}] \right\} r^{(m)}(x, t) = 0.$$

The solution is

$$r^{(m)}(x, t) = k(x) \frac{\int_{\mathcal{X}} u_0\phi^{(1)}(f_{c(t)}(x))d\mu}{\phi^{(1)}(f_{c(t)}(x))}$$

where $k(x) = \frac{r(x)\phi^{(1)}(f_\theta(x))}{\int_{\mathcal{X}} u_0\phi^{(1)}(f_\theta(x))d\mu(x)}$. Then

$${}^{(m)}\pi_{\theta'}^{\theta'} r(x) = \frac{r(x)\phi^{(1)}(f_\theta(x)) \int_{\mathcal{X}} u_0\phi^{(1)}(f_{\theta'})d\mu}{\phi^{(1)}(f_{\theta'}(x)) \int_{\mathcal{X}} u_0\phi^{(1)}(f_\theta)d\mu}.$$

□

As consequence of the two previous theorems, we get the following result.

Corollary 3.5. *The exponential $\bar{\nabla}^{(1)}$ and mixture connection $\bar{\nabla}^{(-1)}$ are curvature free.*

Lemma 3.6. *We assume that $\forall x \in \mathcal{X}$, $\phi^{(2)}(x) > 0$. If there exists a real constant k such that*

$$(3.7) \quad \phi^{(2)}(f_\theta) = \phi^{(1)}(f_\theta) \int_{\mathcal{X}} u_0\phi^{(1)}(f_\theta) d\mu + k,$$

then

$$(3.8) \quad E_{\theta}''' [\partial_i f_\theta \partial_j f_\theta \partial_k f_\theta] = \mathbb{E}_{\theta}'' \left[\partial_i f_\theta \partial_k f_\theta \left(\frac{\phi^{(2)}(f_\theta)}{\phi^{(1)}(f_\theta)} \partial_j f_\theta + \mathbb{E}_{\theta}''(u_0 \partial_j f_\theta) \right) \right].$$

Proof. Let ϕ be a smooth and bijective u_0 -mapping such that $\forall(\theta, x) \in M \times (0, \infty)$, $\phi^{(1)}(x) \neq 0$, $\phi^{(2)}(x) > 0$ and ϕ satisfies (3.7). Let $(i, j, k) \in \{1, 2, \dots, n\}^3$.

$$\begin{aligned} (3.7) \Rightarrow \partial_j f_\theta \phi^{(3)}(f_\theta) &= \partial_j f_\theta \phi^{(2)}(f_\theta) \int_{\mathcal{X}} u_0 \phi^{(1)}(f_\theta) d\mu \\ &\quad + \phi^{(1)}(f_\theta) \int_{\mathcal{X}} u_0 \partial_j f_\theta \phi^{(2)}(f_\theta) d\mu \\ \Rightarrow \frac{\partial_j f_\theta \phi^{(3)}(f_\theta)}{\phi^{(2)}(f_\theta)} &= \partial_j f_\theta \int_{\mathcal{X}} u_0 \phi^{(1)}(f_\theta) d\mu \\ &\quad + \frac{\phi^{(1)}(f_\theta)}{\phi^{(2)}(f_\theta)} \int_{\mathcal{X}} u_0 \partial_j f_\theta \phi^{(2)}(f_\theta) d\mu \\ \Rightarrow \frac{\partial_j f_\theta \phi^{(3)}(f_\theta)}{\phi^{(2)}(f_\theta)} &= \frac{\partial_j f_\theta \phi^{(2)}(f_\theta)}{\phi^{(1)}(f_\theta)} \\ &\quad + \frac{\phi^{(1)}(f_\theta)}{\phi^{(2)}(f_\theta)} \int_{\mathcal{X}} u_0 \partial_j f_\theta \phi^{(2)}(f_\theta) d\mu \\ \Rightarrow \partial_i f_\theta \partial_k f_\theta \frac{\partial_j f_\theta \phi^{(3)}(f_\theta)}{\phi^{(2)}(f_\theta)} &= \partial_i f_\theta \partial_k f_\theta \left[\frac{\partial_j f_\theta \phi^{(2)}(f_\theta)}{\phi^{(1)}(f_\theta)} \right. \\ (3.9) \quad &\quad \left. + \frac{\phi^{(1)}(f_\theta)}{\phi^{(2)}(f_\theta)} \int_{\mathcal{X}} u_0 \partial_j f_\theta \phi^{(2)}(f_\theta) d\mu \right]. \end{aligned}$$

By taking the expectation of (3.9), we get the relation (3.8). □

In the following theorem, we show that with a sufficient and necessary condition on ϕ , the restriction to $\mathbb{T}M$ of the family of α -connections (3.4) of H is exactly the family of α -connections (2.4) of $\mathbb{T}M$ given by Rui et al.[17]. Since the geometry of M is indeed that of $\mathbb{T}M$, our α -geometry of H is an extension of that of M .

Theorem 3.7. For all i, j and k in $\{1, 2, \dots, n\}$ and $\theta \in M$,

$$g\left(\bar{\nabla}_{\partial_i}^{(\alpha)} \partial_j, \partial_k\right) = \Gamma_{ij,k}^{(\alpha)},$$

if and only if ϕ satisfies

$$\mathbb{E}_\theta''' [\partial_i f_\theta \partial_j f_\theta \partial_k f_\theta] = \mathbb{E}_\theta'' \left[\partial_i f_\theta \partial_k f_\theta \left(\frac{\phi^{(2)}(f_\theta)}{\phi^{(1)}(f_\theta)} \partial_j f_\theta + \mathbb{E}_\theta''(u_0 \partial_j f_\theta) \right) \right],$$

where $\Gamma_{ij,k}^{(\alpha)}$ is defined by relation (2.4).

Proof. Let i, j and k in $\{1, 2, \dots, n\}$. We have

$$\begin{aligned} g\left(\bar{\nabla}_{\partial_i}^{(\alpha)} \partial_j, \partial_k\right) &= \left\langle \bar{\nabla}_{\partial_i f_\theta}^{(\alpha)} \partial_j f_\theta, \partial_k f_\theta \right\rangle_\theta \\ &= \langle \partial_i \partial_j f_\theta, \partial_k f_\theta \rangle_\theta - \frac{1+\alpha}{2} \mathbb{E}_\theta' [\partial_i f_\theta \partial_j f_\theta] \langle u_0, \partial_k f_\theta \rangle_\theta \\ &\quad + \frac{1-\alpha}{2} \left\langle \frac{\phi^{(2)}(f_\theta)}{\phi^{(1)}(f_\theta)} \partial_i f_\theta \partial_j f_\theta, \partial_k f_\theta \right\rangle_\theta \\ &\quad - \frac{1-\alpha}{2} [\mathbb{E}_\theta'' [u_0 \partial_i f_\theta] \langle \partial_j f_\theta, \partial_k f_\theta \rangle_\theta] \\ &= \mathbb{E}_\theta'' [\partial_i \partial_j f_\theta \partial_k f_\theta] - \frac{1+\alpha}{2} \mathbb{E}_\theta' [\partial_i f_\theta \partial_j f_\theta] \mathbb{E}_\theta'' [u_0 \partial_k f_\theta] \\ &\quad + \frac{1-\alpha}{2} \mathbb{E}_\theta'' \left[\frac{\phi^{(2)}(f_\theta)}{\phi^{(1)}(f_\theta)} \partial_i f_\theta \partial_j f_\theta \partial_k f_\theta \right] \\ &\quad - \frac{1-\alpha}{2} \{ \mathbb{E}_\theta'' [u_0 \partial_i f_\theta] \mathbb{E}_\theta'' [\partial_j f_\theta \partial_k f_\theta] \}. \end{aligned}$$

We know that $\mathbb{E}_\theta'' [\partial_i \partial_j f_\theta \partial_k f_\theta] = \frac{1+\alpha}{2} \mathbb{E}_\theta'' [\partial_i \partial_j f_\theta \partial_k f_\theta] + \frac{1-\alpha}{2} \mathbb{E}_\theta'' [\partial_i \partial_j f_\theta \partial_k f_\theta]$,

$$\begin{aligned} \mathbb{E}_\theta''' [\partial_i f_\theta \partial_j f_\theta \partial_k f_\theta] &= \mathbb{E}_\theta'' \left[\frac{\phi^{(2)}(f_\theta)}{\phi^{(1)}(f_\theta)} \partial_i f_\theta \partial_j f_\theta \partial_k f_\theta \right] \\ &\quad + \mathbb{E}_\theta'' [\partial_i f_\theta \partial_k f_\theta] \mathbb{E}_\theta'' [u_0 \partial_j f_\theta]. \end{aligned}$$

Then one has $g\left(\bar{\nabla}_{\partial_i}^{(\alpha)} \partial_j, \partial_k\right) = \frac{1+\alpha}{2} \Gamma_{ij,k}^{(1)} + \frac{1-\alpha}{2} \Gamma_{ij,k}^{(-1)} = \Gamma_{ij,k}^{(\alpha)}$. \square

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