# Hypersurfaces with harmonic mean curvature vector in Euclidean space of arbitrary dimension

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**Abstract.** We prove that hypersurfaces in $\mathbb{E}^{n+1}$ satisfying $\triangle\vec{H} = \lambda\vec{H}$ whose second fundamental form has constant norm and with at most four distinct principal curvatures has constant mean curvature and constant scalar curvature. In particular, every such null 2-type hypersurface in Euclidean space $\mathbb{E}^{n+1}$ with at most four distinct principal curvatures has constant mean curvature and constant scalar curvature. Also, we obtain that every such biharmonic hypersurface in $\mathbb{E}^{n+1}$ with at most four distinct principal curvatures must be minimal and has constant scalar curvature. Furthermore, the mean curvature $H$ and constant norm of second fundamental form $\beta$ of every such nonminimal hypersurface satisfy $H^2 \leq \frac{\beta}{n}$ and equality holds if and only if $M$ is congruent to $\mathbb{S}^n(\frac{\beta}{n})$, where $\lambda = \beta$.

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## 1 Introduction

Let $M^n$ be an $n$-dimensional, connected submanifold of Euclidean space $\mathbb{E}^m$. Then, it is well known that

$$\triangle\vec{x} = -n\vec{H}, \tag{1.1}$$

where $\vec{x}$, $\vec{H}$ and $\triangle$ are the position vector field, the mean curvature vector field and the Laplace operator on $M^n$, respectively.

A submanifold $M^n$ in $\mathbb{E}^m$ is said to be of finite type [2, 3, 8] if the position vector $\vec{x}$ of $M^n$ can be expressed as:

$$\vec{x} = \vec{x}_0 + \vec{x}_1 + \cdots + \vec{x}_k, \tag{1.2}$$

where $\vec{x}_0$ is a constant vector and $\vec{x}_1,\ldots, \vec{x}_k$ are non-constant maps satisfying $\triangle\vec{x}_q = \lambda_q\vec{x}_q$, $q = 1,\ldots,k$. If all eigenvalues $\lambda_q$, $q = 1,\ldots,k$ are mutually distinct, then the

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submanifold is said to be of $k$-type. A submanifold is called null $k$-type if one of $\lambda_q$, $q = 1, \ldots, k$ is zero. By taking $\vec{x}_0$ to be the origin, we have the following simple spectral decomposition of $\vec{x}$ for a null 2-type submanifold $M$:

$$\vec{x} = \vec{x}_1 + \vec{x}_2, \quad \triangle\vec{x}_1 = 0, \quad \triangle\vec{x}_2 = \lambda\vec{x}_2, \tag{1.3}$$

where $\lambda$ is a non-zero constant. From (1.1) and (1.3), we get

$$\triangle\vec{H} = \lambda\vec{H}. \tag{1.4}$$

In particular, a submanifold is called biharmonic if $\triangle\vec{H} = 0$.

The study of biharmonic submanifolds in Euclidean space was initiated by Chen in mid 1980s. In particular, he proved that biharmonic surfaces in Euclidean 3-spaces are minimal. In 1991, Chen posed the following well-known conjecture [2]:

*The only biharmonic submanifolds of Euclidean spaces are the minimal ones.*

Chen's conjecture was verified and found true for submanifolds of some Euclidean spaces (Please see [2, 3, 6, 10, 11, 12, 19, 20, 21, 23]).

The study of submanifolds in Euclidean spaces satisfying $\triangle\vec{H} = \lambda\vec{H}$ was initiated by Chen in 1998 and arose in the context of his theory of submanifolds of finite type.

In 1991, Chen proposed the following interesting problem [2]:

*"Determine all submanifolds of Euclidean spaces which are of null 2-type".*

Chen proved that null 2-type surfaces in Euclidean space $\mathbb{E}^3$ are circular cylinders [4]. In [15], it was proved that a null 2-type Euclidean hypersurface in $\mathbb{E}^{n+1}$ with at most two distinct principal curvatures is a spherical cylinder $\mathbb{S}^p \times \mathbb{R}^{n-p}$. Later, Chen proved that a surface $M$ in the Euclidean space $\mathbb{E}^4$ is of null 2-type with parallel normalized mean curvature vector if and only if $M$ is an open portion of a circular cylinder in a hyperplane of $\mathbb{E}^4$; and the only null 2-type surfaces in $\mathbb{E}^4$ with constant mean curvature are open portion of helical cylinders [5].

In 1995, Hasanis and Vlachos [22] proved that null 2-type hypersurfaces in $\mathbb{E}^4$ have constant mean curvature. Chen and Garray proved that $\delta(2)$-ideal null 2-type hypersurfaces in Euclidean space are spherical cylinders [7].

In [13], Dursun classified 3-dimensional null 2-type submanifolds of the Euclidean space $\mathbb{E}^5$ with parallel normalized mean curvature vector. In [14], it was proved that a 3-dimensional submanifold $M$ of the Euclidean space $\mathbb{E}^5$ such that $M$ is not of 1-type is an open portion of a 3-dimensional helical cylinder if and only if $M$ is flat and of null 2-type with constant mean curvature and non-parallel mean curvature vector. It was proved that every null 2-type hypersurface with at most three distinct principal curvatures in a Euclidean space has constant mean curvature [16]. Recently, it was proved that null 2-type hypersurfaces with at most four distinct principal curvatures and whose second fundamental form has constant norm in $\mathbb{E}^6$ has constant mean curvature and constant scalar curvature [18]. For more work in this field, please see [3, 9].

In view of above development, we study biharmonic and null 2-type hypersurfaces in $\mathbb{E}^{n+1}$ whose second fundamental form is of constant norm. We prove that:

**Theorem 1.1.** *Let $M$ be a null 2-type hypersurface in $\mathbb{E}^{n+1}$ with four distinct principal curvatures and whose second fundamental form is of constant norm. Then, it has constant mean curvature and constant scalar curvature.*

**Theorem 1.2.** *Let $M$ be a null 2-type hypersurface in $\mathbb{E}^{n+1}$ with at most four distinct principal curvatures and whose second fundamental form is of constant norm. Then, it has constant mean curvature and constant scalar curvature.*

Also, we prove that:

**Theorem 1.3.** *Let $M$ be a biharmonic hypersurface in $\mathbb{E}^{n+1}$ with four distinct principal curvatures and whose second fundamental form is of constant norm. Then, it has constant mean curvature and constant scalar curvature.*

**Theorem 1.4.** *Let $M$ be a biharmonic hypersurface in $\mathbb{E}^{n+1}$ with at most four distinct principal curvatures and whose second fundamental form is of constant norm. Then, it must be minimal and has constant scalar curvature.*

As, Chen proved that a submanifold $M^n$ of $\mathbb{E}^m$ satisfies (1.4) if and only if it is biharmonic or is of 1-type or is of null 2-type [5]. Also, Takahashi proved that only 1-type hypersurfaces of Euclidean spaces are the minimal hypersurfaces and are open parts of hypersphere [24]. Therefore, combining Theorem 1.2, Theorem 1.4 and results obtained in [5, 24], we get

**Theorem 1.5.** *Let $M$ be a hypersurface satisfying $\triangle\vec{H} = \lambda\vec{H}$ in $\mathbb{E}^{n+1}$ with at most four distinct principal curvatures and whose second fundamental form is of constant norm. Then, it has constant mean curvature and constant scalar curvature.*

The following inequality is known in the literature as Cauchy's inequality.
**Theorem A.**[1] *If $\bar{a} = (a_1, \ldots, a_n)$ and $\bar{b} = (b_1, \ldots, b_n)$ are sequences of real numbers, then*

$$(\sum_{k=1}^{n} a_k b_k)^2 \leq \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2, \tag{1.5}$$

*with equality if and only if the sequences $\bar{a}$ and $\bar{b}$ are proportional, i.e., there is a $r \in \mathbb{R}$ such that $a_k = rb_k$ for each $k \in \{1, \ldots, n\}$.*

Using Theorem 1.5 and Cauchy's inequality, we give the following classification:

**Theorem 1.6.** *Let $M$ be a hypersurface satisfying $\triangle\vec{H} = \lambda\vec{H}$ in $\mathbb{E}^{n+1}$ with at most four distinct principal curvatures. Then, every such nonminimal hypersurface $M$ in the Euclidean space $\mathbb{E}^{n+1}$ with constant norm of second fundamental form $\beta$ satisfy $H^2 \leq \frac{\beta}{n}$ and equality holds if and only if $M$ is congruent to $\mathbb{S}^n(\frac{\lambda}{n})$.*

# 2 Preliminaries

Let $(M, g)$ be a hypersurface isometrically immersed in an $(n+1)$-dimensional Euclidean space $(\mathbb{E}^{n+1}, \overline{g})$ and $g = \overline{g}_{|M}$.

Let $\overline{\nabla}$ and $\nabla$ denote linear connections on $\mathbb{E}^{n+1}$ and $M$, respectively. Then, the Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall \quad X, Y \in \Gamma(TM), \tag{2.1}$$

$$\overline{\nabla}_X \xi = -\mathcal{A}_\xi X, \tag{2.2}$$

where $\xi$ is the unit normal vector to $M$, $h$ is the second fundamental form and $\mathcal{A}$ is the shape operator. It is well known that the second fundamental form $h$ and shape operator $\mathcal{A}$ are related by

$$\overline{g}(h(X,Y),\xi) = g(\mathcal{A}_\xi X, Y). \tag{2.3}$$

The mean curvature is given by

$$H = \frac{1}{n}\operatorname{trace}\mathcal{A}. \tag{2.4}$$

The Gauss and Codazzi equations are given by

$$R(X,Y)Z = g(\mathcal{A}Y,Z)\mathcal{A}X - g(\mathcal{A}X,Z)\mathcal{A}Y, \tag{2.5}$$

$$(\nabla_X\mathcal{A})Y = (\nabla_Y\mathcal{A})X, \tag{2.6}$$

respectively, where $R$ is the curvature tensor and

$$(\nabla_X\mathcal{A})Y = \nabla_X\mathcal{A}Y - \mathcal{A}(\nabla_X Y) \tag{2.7}$$

for all $X, Y, Z \in \Gamma(TM)$.

By comparing the tangential and normal components of (1.4), the necessary and sufficient conditions for $M$ to be null 2-type in $\mathbb{E}^{n+1}$ are

$$\triangle H + H\operatorname{trace}(\mathcal{A}^2) = \lambda H, \tag{2.8}$$

$$\mathcal{A}(\operatorname{grad} H) + \frac{n}{2}H\operatorname{grad} H = 0, \tag{2.9}$$

where $H$ denotes the mean curvature. Also, the Laplace operator $\triangle$ of a scalar valued function $f$ is given by [3]

$$\triangle f = -\sum_{i=1}^{n}(e_ie_if - (\nabla_{e_i}e_i)f), \tag{2.10}$$

where $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal local tangent frame on $M$.

## 3 Null 2-type hypersurfaces in $\mathbb{E}^{n+1}$ with four distinct principal curvatures

In this section we study null 2-type hypersurface $M$ in $\mathbb{E}^{n+1}$ with four distinct principal curvatures. We assume that the mean curvature is not constant and $\operatorname{grad} H \neq 0$. Now, assuming non constant mean curvature implies the existence of an open connected subset $U$ of $M$ with $\operatorname{grad}_x H \neq 0$, for all $x \in U$. From (2.9), it is easy to see that $\operatorname{grad} H$ is an eigenvector of the shape operator $\mathcal{A}$ with the corresponding principal curvature $-\frac{n}{2}H$.

We denote by $A, B$, the following sets

$$A = \{1, 2, \ldots, n\}, \quad B = \{2, 3, \ldots, n\}.$$

The shape operator $\mathcal{A}$ of hypersurface $M$ in $\mathbb{E}^{n+1}$ will take the following form with respect to a suitable orthonormal frame $\{e_1, e_2, \ldots, e_n\}$

$$\mathcal{A}_H e_i = \lambda_i e_i, \quad i \in A, \tag{3.1}$$

where $\lambda_i$ is eigenvalue corresponding to eigenvector $e_i$ of the shape operator.

Without losing generality, we choose $e_1$ in the direction of $\operatorname{grad} H$ and therefore, we get the corresponding eigenvalue $\lambda_1 = -\frac{n}{2}H$. The $\operatorname{grad} H$ can be expressed as

$$\operatorname{grad} H = \sum_{i=1}^{n} e_i(H) e_i. \tag{3.2}$$

As we have taken $e_1$ parallel to $\operatorname{grad} H$, consequently

$$e_1(H) \neq 0, \quad e_i(H) = 0, \quad i \in B. \tag{3.3}$$

We express

$$\nabla_{e_i} e_j = \sum_{m=1}^{n} \omega_{ij}^m e_m, \quad i, j \in A. \tag{3.4}$$

Using (3.4) and the compatibility conditions $(\nabla_{e_k} g)(e_i, e_i) = 0$ and $(\nabla_{e_k} g)(e_i, e_j) = 0$, we obtain

$$\omega_{ki}^i = 0, \quad \omega_{ki}^j + \omega_{kj}^i = 0, \tag{3.5}$$

for $i \neq j$ and $i, j, k \in A$.

Taking $X = e_i, Y = e_j$ in (2.7) and using (3.1) and (3.4), we get

$$(\nabla_{e_i} \mathcal{A}) e_j = e_i(\lambda_j) e_j + \sum_{k=1}^{n} \omega_{ij}^k e_k (\lambda_j - \lambda_k).$$

Putting the value of $(\nabla_{e_i} \mathcal{A}) e_j$ in (2.6), we find

$$e_i(\lambda_j) e_j + \sum_{k=1}^{n} \omega_{ij}^k e_k (\lambda_j - \lambda_k) = e_j(\lambda_i) e_i + \sum_{k=1}^{n} \omega_{ji}^k e_k (\lambda_i - \lambda_k),$$

whereby for $i \neq j = k$ and $i \neq j \neq k$, we obtain

$$e_i(\lambda_j) = (\lambda_i - \lambda_j) \omega_{ji}^j = (\lambda_j - \lambda_i) \omega_{jj}^i, \quad \text{and} \tag{3.6}$$

$$(\lambda_i - \lambda_j) \omega_{ki}^j = (\lambda_k - \lambda_j) \omega_{ik}^j, \tag{3.7}$$

respectively, for $i, j, k \in A$.

Now, we show that $\lambda_j \neq \lambda_1$ for $j \in B$. In fact, if $\lambda_j = \lambda_1 = -\frac{n}{2}H$ for $j \neq 1$, from (3.6), we find

$$e_1(\lambda_j) = (\lambda_1 - \lambda_j)\omega_{j1}^j = 0, \quad \text{or} \quad -\frac{n}{2}e_1(H) = 0, \tag{3.8}$$

which contradicts the first expression of (3.3). Therefore, $\lambda_1 = -\frac{n}{2}H$ has multiplicity one.

Since $M$ has four distinct principal curvatures. We can assume that $\lambda_1 = -\frac{n}{2}H$, $\beta_1, \beta_2$ and $\beta_3$ are four distinct principal curvatures of the hypersurface $M$ with multiplicity 1, $p, q$ and $r$ respectively, such that

$$\lambda_2 = \lambda_3 = \cdots = \lambda_{p+1} = \beta_1,$$
$$\lambda_{p+2} = \lambda_{p+3} = \cdots = \lambda_{p+q+1} = \beta_2,$$
$$\lambda_{p+q+2} = \lambda_{p+q+3} = \cdots = \lambda_{p+q+r+1} = \beta_3,$$

and $p + q + r + 1 = n$.

Using (2.4) and (3.1), we obtain that

$$\sum_{j=2}^{n} \lambda_j = p\beta_1 + q\beta_2 + r\beta_3 = \frac{3n}{2}H. \tag{3.9}$$

We denote by $C_1, C_2$ and $C_3$, the following sets

$$C_1 = \{2, 3, \ldots, p{+}1\}, \quad C_2 = \{p{+}2, p{+}3, \ldots, p{+}q{+}1\}, \quad C_3 = \{p{+}q{+}2, p{+}q{+}3, \ldots, n\}.$$

Putting $i = 1$ in (3.6) and using (3.5), we have

$$\omega_{jj}^1 = \frac{e_1(\beta_{i_1})}{\beta_{i_1} - \lambda_1}, \quad j \in C_{i_1}, \quad i_1 = 1, 2, 3. \tag{3.10}$$

Taking $i \in C_{i_1}$ in (3.6), we find

(3.11)

Taking $i, j \in C_1$ and $i \neq j$ in (3.6), we get

$$e_j(\beta_1) = 0, \quad j \in C_1, \quad p > 1. \tag{3.12}$$

Taking $i, j \in C_2$ and $i \neq j$ in (3.6), we find

$$e_j(\beta_2) = 0, \quad j \in C_2, \quad q > 1. \tag{3.13}$$

Taking $i, j \in C_3$ and $i \neq j$ in (3.6), we obtain

$$e_j(\beta_3) = 0, \quad j \in C_3, \quad r > 1. \tag{3.14}$$

Using (3.3), (3.4) and the fact that $[e_i\, e_j](H) = 0 = \nabla_{e_i}e_j(H) - \nabla_{e_j}e_i(H) = \omega_{ij}^1 e_1(H) - \omega_{ji}^1 e_1(H)$, for $i \neq j$, we find

$$\omega_{ij}^1 = \omega_{ji}^1, \quad i, j \in B. \tag{3.15}$$

Putting $i \neq 1, j = 1$ in (3.6) and using (3.3) and (3.5), we find

$$\omega_{1i}^1 = 0, \quad i \in A. \tag{3.16}$$

Putting $i = 1$ in (3.7), we obtain

$$\omega_{k1}^j = 0, \quad j \neq k \quad \text{and} \quad j, k \in C_{i_1}, \quad i_1 = 1, 2, 3. \tag{3.17}$$

Taking $i \in C_{i_1}$ in (3.7), we have

$$\omega_{ki}^j = 0, \quad j \neq k \quad \text{and} \quad j, k \in C_{i_2}, \quad i_1 \neq i_2, \quad i_1, i_2 = 1, 2, 3. \tag{3.18}$$

Putting $j = 1$ in (3.7) and using (3.15), we get

$$\omega_{ki}^1 = \omega_{ik}^1 = 0, \quad i \in C_{i_1}, \quad k \in C_{i_2}, \quad i_1 \neq i_2, \quad i_1, i_2 = 1, 2, 3. \tag{3.19}$$

Putting $i = 1$ in (3.7) and using (3.19) and (3.5), we find

$$\omega_{1k}^j = \omega_{k1}^j = 0, \quad j \in C_{i_1}, \quad k \in C_{i_2}, \quad i_1 \neq i_2, \quad i_1, i_2 = 1, 2, 3. \tag{3.20}$$

Combining (3.19) and (3.17), we obtain

$$\omega_{ji}^1 = \omega_{ij}^1 = 0, \quad i \neq j, \quad i, j \in B. \tag{3.21}$$

Now, using (3.4) and (3.16)$\sim$(3.21), we have:

**Lemma 3.1.** *Let $M$ be a null 2-type hypersurface in Euclidean space $\mathbb{E}^{n+1}$ of non-constant mean curvature with four distinct principal curvatures, and having the shape operator given by (3.1) with respect to a suitable orthonormal frame $\{e_1, e_2, \ldots, e_n\}$. Then,*

$$\nabla_{e_1} e_1 = 0, \quad \nabla_{e_1} e_i = \sum_{C_{i_1}} \omega_{1i}^m e_m \quad \forall \quad i \in C_{i_1}, \quad m \neq i,$$

$$\nabla_{e_i} e_1 = -\omega_{ii}^1 e_i \quad \forall \quad i \in B, \quad \nabla_{e_i} e_i = \sum_{A} \omega_{ii}^m e_m \quad \forall \quad i \in B, \quad m \neq i,$$

$$\nabla_{e_i} e_j = \sum_{C_{i_1}} \omega_{ij}^m e_m \quad \forall \quad i, j \in C_{i_1}, \quad i \neq j, \quad j \neq m,$$

$$\nabla_{e_i} e_j = \omega_{ij}^i e_i + \sum_{B \setminus C_{i_1}} \omega_{ij}^m e_m \quad \forall \quad i \in C_{i_1}, \quad j \in B \setminus C_{i_1}, \quad m \neq j,$$

*where $\sum_{C_{i_1}}$ denotes the summation taken over all the element of $C_{i_1}$ for $i_1 = 1, 2, 3$, and $\omega_{ij}^i$ satisfy (3.5) and (3.6).*

Next, using Lemma 3.1, and equations (2.10), (3.3) and (3.28) in (2.8), we obtain

$$-e_1 e_1(H) + \sum_{j=2}^{n} \omega_{jj}^1 e_1(H) + \beta H = \lambda H. \tag{3.22}$$

Using (3.3), Lemma 3.1, and the fact that $[e_i\ e_1](H) = 0 = \nabla_{e_i} e_1(H) - \nabla_{e_1} e_i(H)$, we find

$$e_i e_1(H) = 0, \quad i \in B. \tag{3.23}$$

Using (3.23) $[e_i\ e_1](e_1(H)) = 0 = \nabla_{e_i} e_1(e_1(H)) - \nabla_{e_1} e_i(e_1(H))$, we obtain

$$e_i e_1 e_1(H) = 0, \quad i \in B. \tag{3.24}$$

Evaluating $g(R(e_1, e_i)e_1, e_i)$, $g(R(e_1, e_i)e_i, e_j)$ and $g(R(e_i, e_j)e_i, e_1)$, and using Lemma 3.1 and Gauss equation (2.5), we get

$$e_1(\omega_{ii}^1) - (\omega_{ii}^1)^2 = \lambda_1 \lambda_i, \quad i \in B, \tag{3.25}$$

$$e_1(\omega_{ii}^j) - \omega_{ii}^j \omega_{ii}^1 = 0, \quad i \in C_{i_1}, \quad j \in B \setminus C_{i_1}, \quad i_1 = 1, 2, 3, \tag{3.26}$$

and

$$e_j(\omega_{ii}^1) + \omega_{ii}^j \omega_{jj}^1 - \omega_{ii}^j \omega_{ii}^1 = 0, \quad i \in C_{i_1}, \quad j \in B \setminus C_{i_1}, \quad i_1 = 1, 2, 3, \tag{3.27}$$

respectively.

We denote by $\beta$ the squared norm of the second fundamental form $h$. Then, from (3.1), we find

$$\beta = \frac{n^2 H^2}{4} + p\beta_1^2 + q\beta_2^2 + r\beta_3^2 = \frac{n^2 H^2}{4} + \sum_{j=2}^{n} \lambda_j^2. \tag{3.28}$$

Evaluating the scalar curvature of the hypersurface and using (2.5) and (3.1), we get

$$\rho = n^2 H^2 - \beta, \tag{3.29}$$

where $\rho$ denotes the scalar curvature.

# 4 Null 2-type hypersurfaces in $\mathbb{E}^{n+1}$ with constant norm of second fundamental form

In this section, we study null 2-type hypersurface $M$ in $\mathbb{E}^{n+1}$ with constant norm of second fundamental form. Since equations (3.12), (3.13) (3.14) are dependent upon the multiplicities $p, q, r$ of the principal curvatures $\beta_1, \beta_2, \beta_3$, respectively, we consider the following cases:

**Case I**: Let $p = 1$, $q = 1$ and $r = n - 3$, **Case II**: Let $p = 1$, $q > 1$ and $r > 1$, **Case III**: Let $p > 1$, $q > 1$ and $r > 1$.

In all the above cases, we show that

$$\omega_{ii}^j = 0, \quad j \in C_{i_1}, \quad i \in C_{i_2}, \quad i_1 \neq i_2, \quad i_1, i_2 = 1, 2, 3. \tag{4.1}$$

**Case I**: Let $p = 1$, $q = 1$ and $r = n - 3$. In this case, we get $C_1 = \{2\}, C_2 = \{3\}$ and $C_3 = \{4, 5, \ldots, n\}$.

Now, we prove:

**Lemma 4.1.** *Let $M$ be a null 2-type hypersurface in Euclidean space $\mathbb{E}^{n+1}$ with four distinct principal curvatures, and having the shape operator given by* (3.1) *with respect to a suitable orthonormal frame $\{e_1, e_2, \ldots, e_n\}$. If the second fundamental form is of constant norm and $p = 1$, $q = 1$ and $r = n - 3$, then*

$$\omega_{22}^{j} = \omega_{33}^{j} = 0, \quad \text{for} \quad j \in C_3. \tag{4.2}$$

**Proof.** Equation (3.9) can be written as

$$\beta_1 + \beta_2 + (n-3)\beta_3 = \frac{3nH}{2}. \tag{4.3}$$

Differentiating (4.3) with respect to $e_j$ and using (3.3) and (3.14), we get

$$e_j(\beta_1) + e_j(\beta_2) = 0, \quad j \in C_3. \tag{4.4}$$

Differentiating (3.28) with respect to $e_j$ and using (4.4) and (3.14), we find

$$(\beta_2 - \beta_1)e_j(\beta_2) = 0, \quad \text{for} \quad j \in C_3. \tag{4.5}$$

From (4.5) and (4.4), we get $e_j(\beta_2) = e_j(\beta_1) = 0$, which using (3.6) completes the proof of the Lemma.

Next, we have

**Lemma 4.2.** *Let $M$ be a null 2-type hypersurface with four distinct principal curvatures in Euclidean space $\mathbb{E}^{n+1}$, having the shape operator given by* (3.1) *with respect to a suitable orthonormal frame $\{e_1, e_2, \ldots, e_n\}$. If the second fundamental form is of constant norm and $p = 1$, $q = 1$ and $r = n - 3$, then*

$$e_i(\omega_{ii}^{1}) = -\sum_{i \neq j, j=2}^{n} \omega_{jj}^{i}[\omega_{jj}^{1} - \omega_{ii}^{1}], \tag{4.6}$$

$$e_i(\omega_{ii}^{1}) = -\frac{1}{(\lambda_i - \lambda_1)} \sum_{i \neq j, j=2}^{n} \omega_{jj}^{i}(\omega_{jj}^{1} - \omega_{ii}^{1})(2\lambda_j - \lambda_i - \lambda_1), \tag{4.7}$$

$$\sum_{j \neq i, j=2}^{n} \omega_{jj}^{i}[(\omega_{jj}^{1} - \omega_{ii}^{1})(\lambda_j - \lambda_i)] = 0, \tag{4.8}$$

$$\sum_{j \neq i, j=2}^{n} \omega_{jj}^{i}(\lambda_j - \lambda_i)^2 = 0, \tag{4.9}$$

*and*

$$\sum_{j \neq i, j=2}^{n} \omega_{jj}^{i}(\lambda_j - \lambda_i)[(\omega_{jj}^{1}(3\lambda_j - \lambda_i - 2\lambda_1) - 2\omega_{ii}^{1}(\lambda_i - \lambda_1)] = 0, \tag{4.10}$$

*for $i \in C_1, C_2$.*

**Proof.** Differentiating (3.22) with respect to $e_i$ and using (3.3), (3.23) and (3.24), we find

$$e_i(\omega_{ii}^1) + \sum_{i\neq j, j=2}^{n} e_i(\omega_{jj}^1) = 0, \tag{4.11}$$

whereby using (3.27), we find (4.6).

Differentiating (3.9) with respect to $e_i$ and using (3.3), we get

$$e_i(\lambda_i) + \sum_{j\neq i, j=2}^{n} e_i(\lambda_j) = 0, \quad i \in C_1, C_2. \tag{4.12}$$

Next, differentiating (4.12) with respect to $e_1$, we have

$$e_1 e_i(\lambda_i) + \sum_{i\neq j, j=2}^{n} e_1 e_i(\lambda_j) = 0. \tag{4.13}$$

From Lemma 3.1, we get that $\nabla_{e_1} e_i = 0$ for $i \in C_1, C_2$. Consequently, $e_i e_1 - e_1 e_i = \nabla_{e_i} e_1 - \nabla_{e_1} e_i = -\omega_{ii}^1 e_i$, for $i = 2, 3$. Therefore, equation (4.13) can be written as

$$e_i e_1(\lambda_i) + \omega_{ii}^1 e_i(\lambda_i) + \sum_{i\neq j, j=2}^{n} e_1 e_i(\lambda_j) = 0. \tag{4.14}$$

Using (3.6) and (4.12) in (4.14), we get

$$\begin{aligned} e_i(\omega_{ii}^1)(\lambda_i - \lambda_1) + \omega_{ii}^1 e_i(\lambda_i) - \sum_{i\neq j, j=2}^{n} \omega_{ii}^1 e_i(\lambda_j) \\ + \sum_{i\neq j, j=2}^{n} [e_1(\omega_{jj}^i)(\lambda_j - \lambda_i) + \omega_{jj}^i e_1(\lambda_j - \lambda_i)] = 0. \end{aligned} \tag{4.15}$$

Using (3.6), (3.26) and (4.12) in (4.15), we find

$$\begin{aligned} e_i(\omega_{ii}^1)(\lambda_i - \lambda_1) - 2\sum_{i\neq j, j=2}^{n} \omega_{ii}^1 e_i(\lambda_j) + \sum_{i\neq j, j=2}^{n} [\omega_{jj}^1 \omega_{jj}^i (\lambda_j - \lambda_i) \\ + \omega_{jj}^i(\omega_{jj}^1(\lambda_j - \lambda_1) - \omega_{ii}^1(\lambda_i - \lambda_1))] = 0. \end{aligned} \tag{4.16}$$

Using (3.6) in (4.16), we obtain (4.7).
Eliminating $e_i(\omega_{ii}^1)$ from (4.6) and (4.7), we obtain (4.8).

Next, differentiating (3.28) with respect to $e_i$, we get

$$\sum_{j=2}^{n} \lambda_j e_i(\lambda_j) = 0, \quad \text{or} \quad \lambda_i e_i(\lambda_i) + \sum_{j\neq i, j=2}^{n} \lambda_j e_i(\lambda_j) = 0, \quad i = 2, 3. \tag{4.17}$$

Using (4.12) and (3.6) in (4.17), we obtain (4.9).

Now, differentiating (4.9) with respect to $e_1$, we get

$$\sum_{j\neq i, j=2}^{n} e_1(\omega_{jj}^i)(\lambda_j - \lambda_i)^2 + 2\omega_{jj}^i(\lambda_j - \lambda_i)e_1(\lambda_j - \lambda_i) = 0. \tag{4.18}$$

Using (3.26) and (3.6) in (4.18), we find (4.10), which completes the proof of the Lemma.

Further, we have

**Lemma 4.3.** *Let $M$ be a null 2-type hypersurface with four distinct principal curvatures in Euclidean space $\mathbb{E}^{n+1}$, having the shape operator given by (3.1) with respect to suitable orthonormal frame $\{e_1, e_2, \ldots, e_n\}$. If the second fundamental form is of constant norm and $p = 1$, $q = 1$ and $r = n - 3$, then*

$$\omega_{33}^2 = \omega_{jj}^2 = 0, \quad \omega_{22}^3 = \omega_{jj}^3 = 0,$$

*for $j \in C_3$.*

**Proof.** Putting $i = 2, 3$ in (4.8), we find the following:

$$\omega_{33}^2[(\omega_{33}^1 - \omega_{22}^1)(\beta_2 - \beta_1)] + (n-3)\omega_{jj}^2[(\omega_{jj}^1 - \omega_{22}^1)(\beta_3 - \beta_1)] = 0, \tag{4.19}$$

and

$$\omega_{22}^3[(\omega_{33}^1 - \omega_{22}^1)(\beta_2 - \beta_1)] + (n-3)\omega_{jj}^3[(\omega_{jj}^1 - \omega_{33}^1)(\beta_3 - \beta_2)] = 0, \tag{4.20}$$

respectively.

Similarly, by putting $i = 2, 3$ in (4.9), we get

$$\omega_{33}^2(\beta_2 - \beta_1)^2 + (n-3)\omega_{jj}^2(\beta_3 - \beta_1)^2 = 0, \tag{4.21}$$

and

$$\omega_{22}^3(\beta_2 - \beta_1)^2 + (n-3)\omega_{jj}^3(\beta_3 - \beta_2)^2 = 0, \tag{4.22}$$

respectively.

Similarly, by putting $i = 2, 3$ in (4.10), we get

$$\begin{aligned} &\omega_{33}^2(\beta_2 - \beta_1)[\omega_{33}^1(3\beta_2 - \beta_1 - 2\lambda_1) - 2\omega_{22}^1(\beta_1 - \lambda_1)] \\ &+(n-3)\omega_{jj}^2(\beta_3 - \beta_1)[\omega_{jj}^1(3\beta_3 - \beta_1 - 2\lambda_1) - 2\omega_{22}^1(\beta_1 - \lambda_1)] = 0, \end{aligned} \tag{4.23}$$

and

$$\begin{aligned} &\omega_{22}^3(\beta_1 - \beta_2)[\omega_{22}^1(3\beta_1 - \beta_2 - 2\lambda_1) - 2\omega_{33}^1(\beta_2 - \lambda_1)] \\ &+(n-3)\omega_{jj}^3(\beta_3 - \beta_2)[\omega_{jj}^1(3\beta_3 - \beta_2 - 2\lambda_1) - 2\omega_{33}^1(\beta_2 - \lambda_1)] = 0, \end{aligned} \tag{4.24}$$

respectively.

We claim that $\omega_{33}^2$=0 and $\omega_{jj}^2$=0. In fact, if $\omega_{33}^2 \neq 0$ and $\omega_{jj}^2 \neq 0$, then the value of determinant formed by coefficients of $\omega_{33}^2$ and $\omega_{jj}^2$ in (4.19) and (4.21) and the value of determinant formed by coefficients of $\omega_{33}^2$ and $\omega_{jj}^2$ in (4.21) and (4.23) will be zero. Therefore, we obtain that

$$(\beta_3 - \beta_1)(\omega_{33}^1 - \omega_{22}^1) - (\omega_{jj}^1 - \omega_{22}^1)(\beta_2 - \beta_1) = 0, \tag{4.25}$$

and

$$\begin{aligned}&(\beta_3 - \beta_1)[\omega_{33}^1(3\beta_2 - \beta_1 - 2\lambda_1) - 2\omega_{22}^1(\beta_1 - \lambda_1)]\\ &-(\beta_2 - \beta_1)[\omega_{jj}^1(3\beta_3 - \beta_1 - 2\lambda_1) - 2\omega_{22}^1(\beta_1 - \lambda_1)] = 0,\end{aligned} \tag{4.26}$$

respectively.

Eliminating $\omega_{33}^1$, from (4.25) and (4.26), we get

$$\omega_{22}^1 = \omega_{jj}^1, \tag{4.27}$$

which is not possible as from (3.25), it gives $\lambda_2 = \lambda_j$, a contradiction. Therefore, $\omega_{33}^2 = 0$ and $\omega_{jj}^2 = 0$.

In an analogous manner, using (4.20), (4.22) and (4.24), we find that $\omega_{22}^3 = 0$ and $\omega_{jj}^3 = 0$, which completes the proof of the Lemma.

**Case II**: Let $p = 1$, $q > 1$ and $r > 1$. In this case, we have $C_1 = \{2\}$, $C_2 = \{3, 4, \ldots, q+2\}$ and $C_3 = \{q+3, q+4, \ldots, n\}$.

Now, we prove:

**Lemma 4.4.** *Let $M$ be a null 2-type hypersurface with four distinct principal curvatures in Euclidean space $\mathbb{E}^{n+1}$, having the shape operator given by (3.1) with respect to a suitable orthonormal frame $\{e_1, e_2, \ldots, e_n\}$. If the second fundamental form is of constant norm and $p = 1$, $q > 1$ and $r > 1$, then*

$$\omega_{22}^j = \omega_{kk}^j = \omega_{22}^k = \omega_{jj}^k = 0, \quad \textit{for} \quad j \in C_2, \quad k \in C_3. \tag{4.28}$$

**Proof.** Equation (3.9) can be written as

$$\beta_1 + q\beta_2 + (n-q-2)\beta_3 = \frac{3nH}{2}. \tag{4.29}$$

Differentiating (4.29) with respect to $e_j$ and using (3.3) and (3.13), we get

$$e_j(\beta_1) + (n-q-2)e_j(\beta_3) = 0, \quad j \in C_2. \tag{4.30}$$

Differentiating (3.28) with respect to $e_j$ and using (4.30) and (3.13), we find

$$(n-q-2)(\beta_3 - \beta_1)e_j(\beta_3) = 0, \quad j \in C_2. \tag{4.31}$$

From (4.30) and (4.31), we get

$$e_j(\beta_3) = e_j(\beta_1) = 0, \quad j \in C_2. \tag{4.32}$$

Differentiating (4.29) with respect to $e_k$ and using (3.3) and (3.14), we get

$$e_k(\beta_1) + qe_k(\beta_2) = 0, \quad k \in C_3. \tag{4.33}$$

Differentiating (3.28) with respect to $e_j$ and using (4.33) and (3.14), we find

$$q(\beta_2 - \beta_1)e_k(\beta_2) = 0, \quad k \in C_3. \tag{4.34}$$

From (4.33) and (4.34), we get

$$e_k(\beta_2) = e_k(\beta_1) = 0, \quad k \in C_3. \tag{4.35}$$

Using (3.6) in (4.32) and (4.35), we get (4.28), which completes the proof of the Lemma.

Next, we have

**Lemma 4.5.** *Let $M$ be a null 2-type hypersurface with four distinct principal curvatures in Euclidean space $\mathbb{E}^{n+1}$, having the shape operator given by (3.1) with respect to a suitable orthonormal frame $\{e_1, e_2, \ldots, e_n\}$. If the second fundamental form is of constant norm and $p = 1$, $q > 1$ and $r > 1$, then*

$$e_2(\omega_{22}^1) = -\sum_{j=3}^{n} \omega_{jj}^2[\omega_{jj}^1 - \omega_{22}^1], \tag{4.36}$$

$$e_2(\omega_{22}^1) = -\frac{1}{(\lambda_2 - \lambda_1)} \sum_{j=3}^{n} \omega_{jj}^2(\omega_{jj}^1 - \omega_{22}^1)(2\lambda_j - \lambda_2 - \lambda_1), \tag{4.37}$$

$$\sum_{j=3}^{n} \omega_{jj}^2[(\omega_{jj}^1 - \omega_{22}^1)(\lambda_j - \lambda_2)] = 0, \tag{4.38}$$

$$\sum_{j=3}^{n} \omega_{jj}^2(\lambda_j - \lambda_2)^2 = 0, \tag{4.39}$$

$$\sum_{j=3}^{n} \omega_{jj}^2(\lambda_j - \lambda_2)[(\omega_{jj}^1(3\lambda_j - \lambda_2 - 2\lambda_1) - 2\omega_{22}^1(\lambda_2 - \lambda_1)] = 0. \tag{4.40}$$

**Proof.** The proof follows from the proof of Lemma 4.2 by taking $i = 2$.

Further, we have

**Lemma 4.6.** *Let $M$ be a null 2-type hypersurface with four distinct principal curvatures in Euclidean space $\mathbb{E}^{n+1}$, having the shape operator given by (3.1) with respect to a suitable orthonormal frame $\{e_1, e_2, \ldots, e_n\}$. If the second fundamental form is of constant norm and $p = 1$, $q > 1$ and $r > 1$, then*

$$\omega_{jj}^2 = 0, \quad \omega_{kk}^2 = 0,$$

*for $j \in C_2$ and $k \in C_3$.*

**Proof.** From (4.38), (4.39) and (4.40), we find

$$q\omega_{jj}^2[(\omega_{jj}^1-\omega_{22}^1)(\beta_2-\beta_1)]+(n-q-2)\omega_{kk}^2[(\omega_{kk}^1-\omega_{22}^1)(\beta_3-\beta_1)]=0, \tag{4.41}$$

$$q\omega_{jj}^2(\beta_2-\beta_1)^2+(n-q-2)\omega_{kk}^2(\beta_3-\beta_1)^2=0, \tag{4.42}$$

and

$$\begin{aligned} q\omega_{jj}^2(\beta_2-\beta_1)[\omega_{jj}^1(3\beta_2-\beta_1-2\lambda_1)-2\omega_{22}^1(\beta_1-\lambda_1)] \\ +(n-q-2)\omega_{kk}^2(\beta_3-\beta_1)[\omega_{jj}^1(3\beta_3-\beta_1-2\lambda_1)-2\omega_{22}^1(\beta_1-\lambda_1)]=0, \end{aligned} \tag{4.43}$$

respectively, for $j\in C_2$ and $k\in C_3$.

We claim that $\omega_{jj}^2$=0 and $\omega_{kk}^2$=0. In fact, if $\omega_{jj}^2\neq 0$ and $\omega_{kk}^2\neq 0$, then the value of determinant formed by coefficients of $\omega_{jj}^2$ and $\omega_{kk}^2$ in (4.41) and (4.42) and the value of determinant formed by coefficients of $\omega_{jj}^2$ and $\omega_{kk}^2$ in (4.42) and (4.43) will be zero. Therefore, we obtain that

$$(\beta_3-\beta_1)(\omega_{jj}^1-\omega_{22}^1)-(\omega_{kk}^1-\omega_{22}^1)(\beta_2-\beta_1)=0, \tag{4.44}$$

and

$$\begin{aligned} &(\beta_3-\beta_1)[\omega_{jj}^1(3\beta_2-\beta_1-2\lambda_1)-2\omega_{22}^1(\beta_1-\lambda_1)] \\ &-(\beta_2-\beta_1)[\omega_{kk}^1(3\beta_3-\beta_1-2\lambda_1)-2\omega_{22}^1(\beta_1-\lambda_1)]=0, \end{aligned} \tag{4.45}$$

respectively.

Eliminating $\omega_{jj}^1$, from (4.44) and (4.45), we get

$$\omega_{22}^1=\omega_{kk}^1, \tag{4.46}$$

which is not possible as from (3.25), it gives $\lambda_2=\lambda_k$, a contradiction. Therefore, $\omega_{jj}^2=0$ and $\omega_{kk}^2=0$, and the Lemma is proved.

**Case III**: Let $p>1$, $q>1$ and $r>1$. In this case, we have

**Lemma 4.7.** *Let $M$ be a null 2-type hypersurface with four distinct principal curvatures in Euclidean space $\mathbb{E}^{n+1}$, having the shape operator given by (3.1) with respect to a suitable orthonormal frame $\{e_1,e_2,\dots,e_n\}$. If the second fundamental form is of constant norm and $p>1$, $q>1$ and $r>1$, then*

$$\omega_{ii}^j=0,\quad j\in C_{i_1},\quad i\in B\setminus C_{i_1},\quad i_1=1,2,3.$$

**Proof.** Differentiating (3.9) with respect to $e_j$ and using (3.3) and (3.12), we get

$$qe_j(\beta_2)+re_j(\beta_3)=0,\quad j\in C_1. \tag{4.47}$$

Differentiating (3.28) with respect to $e_i$ and using (4.47) and (3.12), we find

$$r(\beta_3-\beta_2)e_j(\beta_3)=0,\quad j\in C_1. \tag{4.48}$$

From (4.47) and (4.48), we get $e_j(\beta_3) = e_j(\beta_2) = 0$, which using (3.6) gives $\omega_{ii}^j = 0, \quad j \in C_1, \quad i \in B \setminus C_1$.

Similarly, we can show that $\omega_{ii}^j = 0, \quad j \in C_2, \quad i \in B \setminus C_2$ and $\omega_{ii}^j = 0, \quad j \in C_3, \quad i \in B \setminus C_3$, which completes the proof of the Lemma.

In view of Lemmas 4.1, 4.3, 4.4, 4.6 and 4.7, we conclude that in all the cases i.e. Case I, Case II and Case III of multiplicities of principal curvatures, we find (4.1).

Now, we have:

**Lemma 4.8.** *Let $M$ be a null 2-type hypersurface with four distinct principal curvatures in Euclidean space $\mathbb{E}^{n+1}$, having the shape operator given by* (3.1) *with respect to a suitable orthonormal frame $\{e_1, e_2, \ldots, e_n\}$. If the second fundamental form is of constant norm, then*

$$\omega_{ij}^k = \omega_{ji}^k = \omega_{ki}^j = \omega_{ik}^j = \omega_{jk}^i = \omega_{kj}^i = 0, \tag{4.49}$$

*for $i \in C_1$, $j \in C_2$ and $k \in C_3$.*

**Proof.** Evaluating $g(R(e_1, e_i)e_j, e_k)$ and $g(R(e_1, e_j)e_i, e_k)$, and using (2.5), (3.1), (4.1) and Lemma 3.1, we find

$$e_1(\omega_{ij}^k) + \sum_{m=p+q+2}^{n} \omega_{ij}^m \omega_{1m}^k - \sum_{m=p+2}^{p+q+1} \omega_{im}^k \omega_{1j}^m - \sum_{m=2}^{p+1} \omega_{mj}^k \omega_{1i}^m - \omega_{ij}^k \omega_{ii}^1 = 0, \tag{4.50}$$

and

$$e_1(\omega_{ji}^k) + \sum_{m=p+q+2}^{n} \omega_{ji}^m \omega_{1m}^k - \sum_{m=p+2}^{p+q+1} \omega_{mi}^k \omega_{1j}^m - \sum_{m=2}^{p+1} \omega_{jm}^k \omega_{1i}^m - \omega_{ji}^k \omega_{jj}^1 = 0, \tag{4.51}$$

respectively.

Taking $i \in C_1$, $j \in C_2$ and $k \in C_3$ in (3.7), we get

$$(\beta_1 - \beta_3)\omega_{ji}^k = (\beta_2 - \beta_3)\omega_{ij}^k \quad \text{or} \quad \omega_{ji}^k = \alpha\omega_{ij}^k, \tag{4.52}$$

where $\alpha = \frac{(\beta_2 - \beta_3)}{(\beta_1 - \beta_3)}$. Also, differentiating $\alpha$ with respect to $e_1$ and using (3.10), we find

$$e_1(\alpha) = \frac{[\omega_{jj}^1(\beta_2 - \lambda_1) - \omega_{kk}^1(\beta_3 - \lambda_1)] - \alpha[\omega_{ii}^1(\beta_1 - \lambda_1) - \omega_{kk}^1(\beta_3 - \lambda_1)]}{(\beta_1 - \beta_3)}, \tag{4.53}$$

for $i \in C_1$, $j \in C_2$ and $k \in C_3$.

Similarly, for $i \in C_1$, $j \in C_2$ and $m \in C_3$ in (3.7), we find

$$(\beta_1 - \beta_3)\omega_{ji}^m = (\beta_2 - \beta_3)\omega_{ij}^m. \tag{4.54}$$

Multiplying with $\omega_{1m}^k$ on both sides in (4.54) and taking summation over $m$ from $p + q + 2$ to $n$, we get

$$(\beta_1 - \beta_3) \sum_{m=p+q+2}^{n} \omega_{ji}^m \omega_{1m}^k = (\beta_2 - \beta_3) \sum_{m=p+q+2}^{n} \omega_{ij}^m \omega_{1m}^k, \tag{4.55}$$

for $i \in C_1$, $j \in C_2$ and $m \in C_3$.

Also, taking $i \in C_1$, $m \in C_2$ and $k \in C_3$ in (3.7), we obtain

$$(\beta_1 - \beta_3)\omega_{mi}^k = (\beta_2 - \beta_3)\omega_{im}^k,$$

or

$$(\beta_1 - \beta_3)\sum_{m=p+2}^{p+q+1} \omega_{mi}^k \omega_{1j}^m = (\beta_2 - \beta_3)\sum_{m=p+2}^{p+q+1} \omega_{im}^k \omega_{1j}^m. \tag{4.56}$$

Taking $m \in C_1$, $j \in C_2$ and $k \in C_3$ in (3.7), we have

$$(\beta_1 - \beta_3)\omega_{jm}^k = (\beta_2 - \beta_3)\omega_{mj}^k,$$

or

$$(\beta_1 - \beta_3)\sum_{m=2}^{p+1} \omega_{jm}^k \omega_{1i}^m = (\beta_2 - \beta_3)\sum_{m=2}^{p+1} \omega_{mj}^k \omega_{1i}^m. \tag{4.57}$$

Differentiating (4.52) with respect to $e_1$, we get

$$e_1(\omega_{ji}^k) = \alpha e_1(\omega_{ij}^k) + e_1(\alpha)\omega_{ij}^k. \tag{4.58}$$

Using (4.50), (4.51), (4.52), (4.55), (4.56), (4.57) and (4.53) in (4.58), we obtain

$$\omega_{ij}^k(\omega_{jj}^1 - \omega_{kk}^1) = \omega_{ji}^k(\omega_{ii}^1 - \omega_{kk}^1). \tag{4.59}$$

Substituting the value of $\omega_{ij}^k$ from (4.52) in (4.59), we find

$$\omega_{ji}^k[(\beta_1 - \beta_3)(\omega_{jj}^1 - \omega_{kk}^1) - (\omega_{ii}^1 - \omega_{kk}^1)(\beta_2 - \beta_3)] = 0. \tag{4.60}$$

From (4.25), we have seen that assuming

$$(\beta_1 - \beta_3)(\omega_{jj}^1 - \omega_{kk}^1) - (\omega_{ii}^1 - \omega_{kk}^1)(\beta_2 - \beta_3) = 0,$$

leads to contradiction, therefore from (4.60), we get $\omega_{ji}^k = 0$ and (4.52) gives $\omega_{ij}^k = 0$. Also, from (3.5), we get $\omega_{jk}^i = -\omega_{ji}^k$ and $\omega_{ij}^k = -\omega_{ik}^j$. Consequently, we obtain $\omega_{jk}^i = 0$ and $\omega_{ik}^j = 0$, which together with (3.7) gives $\omega_{kj}^i = 0$ and $\omega_{ki}^j = 0$, whereby proving the Lemma.

# 5 Proof of Theorem 1.1

Let (4.49) hold. Then evaluating $g(R(e_i, e_j)e_i, e_j)$, $g(R(e_i, e_k)e_i, e_k)$, and $g(R(e_j, e_k)e_j, e_k)$ and keeping in view (4.1), (4.49) and Lemma 3.1, we find

$$\omega_{ii}^1\omega_{jj}^1 = -\lambda_i\lambda_j, \quad \omega_{ii}^1\omega_{kk}^1 = -\lambda_i\lambda_k, \quad \omega_{jj}^1\omega_{kk}^1 = -\lambda_j\lambda_k, \tag{5.1}$$

respectively, for $i \in C_1, j \in C_2$ and $k \in C_3$.

From (5.1), we get

$$(\omega^1_{ii})^2 + (\lambda_i)^2 = 0, \quad (\omega^1_{jj})^2 + (\lambda_j)^2 = 0, \quad (\omega^1_{kk})^2 + \lambda_k^2 = 0. \tag{5.2}$$

From (5.2), we find $\lambda_i = \lambda_j = \lambda_k = 0$, which is a contradiction to the fact that there are four distinct principal curvatures. Therefore, $H$ must be constant. Also, using this fact in (3.29), we get that scalar curvature is constant, whereby proving Theorem 1.1.

The case of at most three distinct principal curvatures for null 2-type hypersurfaces in Euclidean space of arbitrary dimension has already been treated in [16], also with the conclusion that $H$ must be constant. Now, if second fundamental form is of constant norm, then from (3.29), we find that scalar curvature is also constant. Therefore, Theorem 1.2 is obtained as a consequence of Theorem 1.1 and [16].

# 6 Biharmonic hypersurfaces in $\mathbb{E}^{n+1}$ with constant norm of second fundamental form

Let $M$ be a biharmonic hypersurface in $\mathbb{E}^{n+1}$ with four distinct principal curvatures and whose second fundamental form has constant norm. From biharmonic equation $\triangle\vec{H} = 0$, the necessary and sufficient conditions for $M$ to be biharmonic in $\mathbb{E}^{n+1}$ are

$$\triangle H + H \,\mathrm{trace}\, \mathcal{A}^2 = 0, \tag{6.1}$$

and

$$\mathcal{A}(\mathrm{grad}\, H) + \frac{n}{2} H \,\mathrm{grad}\, H = 0. \tag{6.2}$$

Now, proceeding in similar way as of section 3, section 4 and section 5, by taking $\lambda = 0$, we obtain that $H$ is constant. Using this fact in (6.1), we find that $H = 0$. Therefore, from (3.29), we get that the scalar curvature is also constant, which completes the proof of Theorem 1.3.

The cases of two or three distinct principal curvatures for biharmonic hypersurfaces in Euclidean spaces of arbitrary dimension has already been treated in [12, 20], also with the conclusion that $H$ must be minimal. Now, if the second fundamental form is of constant norm, then from (3.29), we find that scalar curvature is also constant. Therefore, Theorem 1.4 is a consequence of Theorem 1.3 and [12, 20].

# 7 Nonminimal hypersurfaces in $\mathbb{E}^{n+1}$ with $\triangle\vec{H} = \lambda\vec{H}$

In this section, we study nonminimal hypersurfaces in $\mathbb{E}^{n+1}$ satisfying $\triangle\vec{H} = \lambda\vec{H}$ with constant norm of second fundamental forms. We give proof of Theorem 1.6.

**Proof of Theorem 1.6.** According to Theorem 1.5, the mean curvature $H$ is constant for a hypersurface satisfying $\triangle\vec{H} = \lambda\vec{H}$ in $\mathbb{E}^{n+1}$ with constant norm of second fundamental form $\beta$ having four distinct principal curvatures. Since $M$ is nonminimal hence $H \neq 0$. Therefore, from (2.8), we find $\lambda = \beta$. Then, using (2.4) and Cauchy's inequality (1.5), we have

$$n^2H^2 = (\sum_{i=1}^{n} \lambda_i)^2 \leq n(\sum_{i=1}^{n} \lambda_i^2) = n\beta, \tag{7.1}$$

which shows $H^2 \leq \frac{\beta}{n}$. When $H^2 = \frac{\beta}{n}$, all equalities in (7.1) hold and which by use of (2.4), gives

$$\lambda_1 = \lambda_2 = \cdots = \lambda_n = \pm\sqrt{\frac{\beta}{n}}.$$

Therefore, we conclude that $M$ is congruent to $\mathbb{S}^n(\frac{\beta}{n})$.

Conversely, we know that if $M$ is congruent to $\mathbb{S}^n(\frac{\beta}{n})$, then shape operator $\mathcal{A}_H = \pm\sqrt{\frac{\beta}{n}}I$ with $I$ identity operator. Therefore, from (2.4), we have $H^2 = \frac{\beta}{n}$. Whereby completing the proof. $\square$

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