Lie algebroid structures on the 1-jet bundle of a Jacobi manifold

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Abstract. This work consists in proving that the bundle of jets of order 1 of a Jacobi manifold is a Lie algebroid.

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1 Introduction

The Lie algebroid bracket on the 1-jet bundle of a Jacobi manifold was built without proof by Y. Kerbrat and Z. Souici-Benhammadi in [4]. This construction shows that the Lie algebroid of a contact groupoid is isomorphic to the Lie algebroid of a Jacobi manifold.

In this paper we prove that the bundle of jets of order 1 of a Jacobi manifold is a Lie algebroid. We start by recalling the notions of Poisson and Jacobi manifolds, and of Lie algebroid [1],[3],[5]; an example of Poisson manifold will be given from [3]. The latter will be used for proving our main result.

We will describe from the Poissonnification of a Jacobi manifold (see [2, 5]), which allows us to prove that the structure constructed in [4] on the 1-jet bundle of a Jacobi manifold $J^1(M, \mathbb{R})$, is a Lie algebroid structure.

2 Preliminaries

2.1 Jacobi manifolds

Let M be a connected paracompact smooth differentiable manifold of dimension n. As well, let E be a vector field on M and let Λ be a contravariant skew-symmetric 2-tensor field. We shall further denote $N = C^\infty(M, \mathbb{R})$.

From the couple (E, Λ), we associate the mapping $N \to \chi(M)$ given by:

$$f \to J_f = f.E + \Lambda^\# df,$$

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where $\Lambda^{\#} : \Omega^1(M) \to \chi(M)$ is the associated linear application of Λ and where the *skew-symmetric bracket* on N is given by:

$$\{f, g\} = i_\Lambda df \wedge dg + f.E(g) - g.E(f)/$$

Proposition 2.1. *The following equivalence holds true:*

$$[J_f, J_g] = J_{[f,g]} \Longleftrightarrow \{f, g, \{h\}\} + \{\{g, h\}, f\} + \{h, \{f, g\}\} = 0. \tag{2.1}$$

Definition 2.1. A triple (M, Λ, E) satisfying one of the two equivalent conditions from (2.1), will be called a *Jacobi manifold.*

Remark 2.2. 1) We can also define a Jacobi manifold M as a manifold equipped with a bivector Λ and a vector field E, such that:

$$L_E\Lambda = [E, \Lambda] = 0 \text{ and} [\Lambda, \Lambda] = 2E \wedge \Lambda,$$

where $[.,.]$ is the Schouten bracket (see [1],[6]).
2) If $E = 0$, then the manifold M is a Poisson manifold.

2.2 Lie algebroids

Definition 2.3. A Lie algebroid on the differentiable manifold M is a triplet $(E, [.,.], \rho)$, where $E \to M$ is a smooth vector bunde over M with a Lie algebra structure $[.,.]$ on the module $\underline{E}$ of smooth global sections of E, and a morphism of vector bundles $\rho : E \to TM$, such that for $s, s' \in Sect(M, E)$ and $f \in C^\infty(M, \mathbb{R})$, we have

1) $[\rho \circ s, \rho \circ s'] = \rho \circ [s, s']$;
2) $[s, f.s'] = f.[s, s'] + (\rho \circ s)(f).s'$.

2.3 The algebroid of a Poisson manifold ([3])

Theorem 2.2. *Let (P, Λ) be a Poisson manifold. Then T^*P is canonically provided with a Lie algebroid structure, where the morphism of vector bundles of T^*P into TP is $\Lambda^{\#}$, and where the bracket of two differential forms on P is given by:*

$$\{\omega_1, \omega_2\} = i_{\Lambda^{\#}\omega_1} d\omega_2 - i_{\Lambda^{\#}\omega_2} d\omega_1 + di_\Lambda(\omega_1 \wedge \omega_2)$$

2.4 The Poissonnification of a Jacobi manifold

Let (M, Λ, E) be a Jacobi manifold, and let $P = \mathbb{R}_+^* \times M =]0, +\infty[\times M$, and $(x^0, x) \in P$. We define on P a 2- contravariant skew-symmetric tensor $\widetilde{\Lambda}$, by:

$$\widetilde{\Lambda}_{(x^0,x)} = \frac{\partial}{\partial x^0} \wedge E_x + \frac{1}{x^0}.\Lambda_x \tag{2.2}$$

or, in local coordinates,

$$\widetilde{\Lambda}^{0i} = -\widetilde{\Lambda}^{i\,0} = E^i; \quad \widetilde{\Lambda}^{ij} = \frac{1}{x^0}\Lambda^{ij}. \tag{2.3}$$

From [2] and [5], we have:

Proposition 2.3. $(P, \widetilde{\Lambda})$ *is a Poisson manifold.*

Definition 2.4. $(P, \widetilde{\Lambda})$ is called the Poissonification of the Jacobi manifold (M, Λ, E).

3 The Lie algebroid structure on the 1-jet bundle of a Jacobi manifold

Let (M, Λ, E) be a Jacobi manifold and let $(P, \widetilde{\Lambda})$ be its Poissonnification. We denote by $J^1(M, \mathbb{R}) = \mathbb{R} \times T^*M$ the vector bundle over TM of 1- jets of real functions on M.

In the following we prove that the Lie algebroid structure on the 1-jet bundle $J^1(M, \mathbb{R})$ stated in [4] is indeed a Lie algebroid structure.

We shall further denote by $\underline{J^1(M, \mathbb{R})}$ the module of sections $C^\infty(M, \mathbb{R}) \times \Omega^1(M)$ of $J^1(M, \mathbb{R})$. Let $\omega \in \underline{J^1(M, \mathbb{R})}$. Then $\omega = (\omega_0, \omega_1)$, where $\omega_0 \in C^\infty(M, \mathbb{R})$ and $\omega_1 \in \Omega^1(M)$. We define $\phi : \underline{J^1(M, \mathbb{R})} \to \Omega^1(P)$ by:

$$\omega = (\omega_0, \omega_1) \to \phi(\omega) = \widetilde{\omega} = \omega_0.dx^0 + x^0.\omega_1, \tag{3.1}$$

where $x^0 : P \to \mathbb{R}$ is the natural coordinate mapping over the $\mathbb{R}_+^*$ factor.

From the Jacobi structure, we trivially deduce a vector bundle morphism $\rho : J^1(M, \mathbb{R}) \to TM$, or $\rho : \underline{J^1(M, \mathbb{R})} \to \chi(M)$, given such that for any $\omega = (\omega_0, \omega_1) \in \underline{J^1(M, \mathbb{R})}$, we have:

$$\rho \circ \omega = \omega_0.E + \Lambda^\# \omega_1. \tag{3.2}$$

We denote $\rho \circ \omega$ by $\rho(\omega)$, and we define [4] a bracket on $\underline{J^1(M, \mathbb{R})}$ denoted by $[\omega, \omega']$, given by

$$\omega \in \underline{J^1(M, \mathbb{R})} \Rightarrow \omega = (\omega_0, \omega_1),$$
$$\omega' \in \underline{J^1(M, \mathbb{R})} \Rightarrow \omega' = ({\omega_0}', {\omega_1}').$$

Proposition 3.1. *For any pair of sections ω, ω' of $J^1(M, \mathbb{R})$, there exists a single section $[\omega, \omega']$ of $J^1(M, \mathbb{R})$ such that:*

$$\phi([\omega, \omega']) = \{\phi(\omega), \phi(\omega')\}$$

where ϕ is given by formula (3.1)*. Moreover $[\omega, \omega'] = ([\omega, \omega']_0, [\omega, \omega']_1)$, with:*

$$[\omega, \omega']_0 = -i_\Lambda(\omega_1 \wedge {\omega_1}') + i_{\rho(\omega)} d{\omega_0}' - i_{\rho(\omega')} d\omega_0, \tag{3.3}$$

$$\begin{aligned}[\omega, \omega']_1 = {} & i_{\rho(\omega)} d{\omega_1}' - i_{\rho(\omega')} d\omega_1 + i_E\omega_1(d{\omega_0}' - {\omega_1}') - i_E{\omega_1}'(d\omega_0 - \omega_1) \\ & + d(\omega_0.i_E{\omega_1}' - {\omega_0}'.i_E\omega_1 + i_\Lambda(\omega_1 \wedge {\omega_1}')).\end{aligned} \tag{3.4}$$

Proof. For $P = \mathbb{R}_+^* \times M =]0, +\infty[\times M$ and let $(x^0, x^i) = (x^\alpha)$ be the coordinates of P. Then, the Poisson tensor $\widetilde{\Lambda}$ is given by the following formulas:

$$i_{\widetilde{\Lambda}}(dx^0 \wedge \alpha) = i_E \alpha$$

where $\alpha \in \Omega^1(M)$, and

$$i_{\widetilde{\Lambda}}(\alpha_1 \wedge \alpha_2) = \frac{1}{x^0}.i_\Lambda(\alpha_1 \wedge \alpha_2),$$

where

$$i_\Lambda(\alpha_1 \wedge \alpha_2) = i_{\Lambda^\#\alpha_1}\alpha_2 \text{ and } \alpha_1, \alpha_2 \in \Omega^1(M).$$

We put

$$\begin{aligned}\widetilde{\omega} &= \phi(\omega) = \omega_0.dx^0 + x^0.\omega_1\\ \widetilde{\omega}' &= \phi(\omega') = \omega_0{}'.dx^0 + x^0.\omega_1{}'.\end{aligned}$$

Then

$$\widetilde{\Lambda}^{\neq}\widetilde{\omega} = -x^0.i_E\omega_1.\frac{1}{x^0} + \rho(\omega), \tag{3.5}$$

where $\rho(\omega) = \omega_0.E + \Lambda^{\#}\omega_1$ is independent of x^0, since

$$\{\, \widetilde{\omega}, \widetilde{\omega}' \} = i_{\Lambda^{\#}\ \widetilde{\omega}} d\widetilde{\omega}' - i_{\Lambda^{\#}\widetilde{\omega}'} d\,\widetilde{\omega} + di_{\Lambda}(\widetilde{\omega} \wedge \widetilde{\omega}').$$

We search a section ω'' of $J^1(M,\mathbb{R})$ such that

$$\begin{aligned}&\widetilde{\omega}'' = \{\, \widetilde{\omega}, \widetilde{\omega}' \,\}\\ &\widetilde{\omega}'' = \phi(\omega'') = \phi([\omega,\omega']) = \{\, \widetilde{\omega}, \widetilde{\omega}' \,\}.\end{aligned}$$

So we have to search for $\omega_0{}''$ and $\omega_1{}''$ such that

$$\begin{aligned}&\widetilde{\omega}'' = \omega_0{}''.dx^0 + x^0.\omega_1{}''\\ &\widetilde{\omega}' = \phi(\omega') = \omega_0{}'.dx^0 + x^0.\omega_1{}' \Longrightarrow d\widetilde{\omega}' = dx^0 \wedge (\omega_1{}' - d\omega_0{}') + x^0.d\omega_1{}'.\end{aligned}$$

We then derive the formulas:

$$\begin{aligned}&i_{\widetilde{\Lambda}^{\neq}\widetilde{\omega}} d\widetilde{\omega}' = -x^0.i_E\omega_1(\omega_1{}' - d\omega_0{}') - i_{\rho(\omega)}(\omega_1{}' - d\omega_0{}').dx^0 + x^0.i_{\rho(\omega)}d\omega_1{}'\\ &i_{\widetilde{\Lambda}}(\widetilde{\omega} \wedge \widetilde{\omega}') = x^0.(i_E\omega_1{}' - \omega_0{}'.i_E\omega_1 + i_\Lambda(\omega_1 \wedge \omega_1{}')).\end{aligned}$$

By reporting in the Lie algebroid the bracket of the Poisson manifold $(P,\widetilde{\Lambda})$, we get:

$$\begin{aligned}\{\widetilde{\omega}, \widetilde{\omega}' \,\} = \;& [i_{\rho(\omega)}(d\omega_0{}' - \omega_1{}') - i_{\rho(\omega')}(d\omega_0 - \omega_1) + i_\Lambda(\omega_1 \wedge \omega_1{}') + \omega_0.i_E\omega_1{}'\\ & -\omega_0'.i_E\omega_1].dx^0 + x^0.[i_{\rho(\omega)}d\omega_1{}' + i_E\omega_1(d\omega_0{}' - \omega_1{}') - i_{\rho(\omega')}d\omega_1\\ & -i_E\omega_1{}'.(d\omega_0 - \omega_1) + d(\omega_0.i_E\omega_1{}' - \omega_0{}'.i_E\omega_1) + di_\Lambda(\omega_1 \wedge \omega_1{}')].\end{aligned}$$

Then (3.3) and (3.4) lead to:

$$\begin{aligned}[\omega,\omega']_0 = \;& -i_\Lambda(\omega_1 \wedge \omega_1{}') + i_{\rho(\omega)}d\omega_0{}' - i_{\rho(\omega')}d\omega_0,\\ [\omega,\omega']_1 = \;& i_{\rho(\omega)}d\omega_1{}' - i_{\rho(\omega')}d\omega_1 + i_E\omega_1(d\omega_0{}' - \omega_1{}') - i_E\omega_1{}'(d\omega_0 - \omega_1)\\ & +d(\omega_0.i_E\omega_1{}' - \omega_0{}'.i_E\omega_1 + i_\Lambda(\omega_1 \wedge \omega_1{}')),\end{aligned}$$

which proves the claim. □

Theorem 3.2. *Let* $(J^1(M,\mathbb{R}) \to TM, [.,.], \rho)$ *be a Lie algebroid, where* $[.,.]$ *is defined by formulas* (3.3), (3.4) *and let* ρ *be given by formula* (3.2) *in*

$$\begin{tikzcd} J^1(M,\mathbb{R}) \arrow[rr, "\rho"] \arrow[dr] & & TM \arrow[dl] \\ & M & \end{tikzcd}$$

Proof. 1) It is obvious that the bracket $[.,.]$ is bilinear and skew-symmetric, and it verifies Jacobi identity since ϕ is injective.

2) We check that ρ defines a Lie algebra morphism. Let $\omega, \omega' \in \underline{J^1(M, \mathbb{R})}$. Using the formula (3.5) on the Poissonnification $(P, \widetilde{\Lambda})$ of the Jacobi manifold (M, Λ, E), we get:

$$\begin{aligned}
\widetilde{\Lambda}^{\neq}\ \phi([\omega, \omega']) &= -x^0.i_E[\omega, \omega']_1.\frac{\partial}{\partial x^0} + \rho([\omega, \omega']) \\
&= \widetilde{\Lambda}^{\neq}\{\phi(\omega), \phi(\omega')\} \\
&= [\widetilde{\Lambda}^{\neq}\phi(\omega), \widetilde{\Lambda}^{\neq}\phi(\omega')] \\
&= [-x^0.i_E\omega_1.\frac{\partial}{\partial x^0} + \rho(\omega), -x^0.i_E\omega_1{}'.\frac{\partial}{\partial x^0} + \rho(\omega')]
\end{aligned}$$

Since

$$\begin{aligned}
[\omega, \omega']_1 = \quad & \omega_1{}'' = i_{\rho(\omega)}d\omega_1{}' - i_{\rho(\omega')}d\omega_1 + i_E\omega_1(d\omega_0{}' - \omega_1{}') \\
& -i_E\omega_1{}'(d\omega_0 - \omega_1) + d(\omega_0.i_E\omega_1{}' - \omega_0{}'.i_E\omega_1 + i_\Lambda(\omega_1 \wedge \omega_1{}')),
\end{aligned}$$

from (3.4) and

$$[\rho(\omega'), x^0.i_E\omega_1.\frac{\partial}{\partial x^0}] = i_{\rho(\omega')}di_E\omega_1.x^0.\frac{\partial}{\partial x^0},$$

we infer:

$$\rho([\omega, \omega']) = [\rho(\omega), \rho(\omega')].$$

3) Let's check that we have:

$$[\omega, f.\omega'] = f.[\omega, \omega'] + i_{\rho(\omega)}df.\omega',$$

for $\omega, \omega' \in J^1(M, \mathbb{R})$ and $f \in C^\infty(M, \mathbb{R})$. We place ourselves again on the Poissonnification $(P, \widetilde{\Lambda})$,

$$\begin{aligned}
\rho([\omega, f.\omega']) &= \{\phi(\omega), \phi(f.\omega')\} \\
&= \{\phi(\omega), f.\phi(\omega')\} \\
&= f.\{\phi(\omega), \phi(\omega')\} + i_{\widetilde{\Lambda}^{\neq}\phi(\omega)}df.\phi(\omega') \\
&= f.\phi([\omega, \omega']) + i_{\rho(\omega)}df.\phi(\omega') \\
&= \phi(f.[\omega, \omega']) + i_{\rho(\omega)}df.\omega')
\end{aligned}$$

From the injectivity of ϕ, we deduce:

$$[\omega, f.\omega'] = f.[\omega, \omega']) + i_{\rho(\omega)}df.\omega'.$$

we therefore conclude that $(J^1(M, \mathbb{R}) \to TM, [.,.], \rho)$ is a Lie algebroid, and we call it the algebroid of the Jacobi manifold (M, Λ, E). □

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